Normal distribution

The most widely used of all the continuous probability distributions is the *normal distribution* (also known as Gaussian). It serves as a popular model for measurement errors, particle displacements under Brownian motion, stock market fluctuations, human intelligence and many other things. It is also used as an approximation for Binomial (for large n) and Gamma (for large α) distributions.

The normal density follows the well-known symmetric bell-shaped curve. The curve is centered at the mean value μ and its spread is, of course, measured by the standard deviation σ. These two parameters, μ and σ^2, completely determine the shape and center of the normal density function.

Definition.

The normal random variable X has the PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right], \qquad \text{for } -\infty < x < \infty$$

It will be denoted as $X \sim \mathcal{N}(\mu, \sigma^2)$

The normal random variable with $\mu = 0$ and $\sigma = 1$ is said to have the *standard normal distribution* and will be called Z. Its density becomes $f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$. Direct integration would show that $\mathbb{E}(Z) = 0$ and $V(Z) = 1$.

Usefulness of Z

We are able to transform the observations of any normal random variable X to a new set of observations of a standard normal random variable Z. This can be done by means of the transformation

$$Z = \frac{X-\mu}{\sigma}.$$

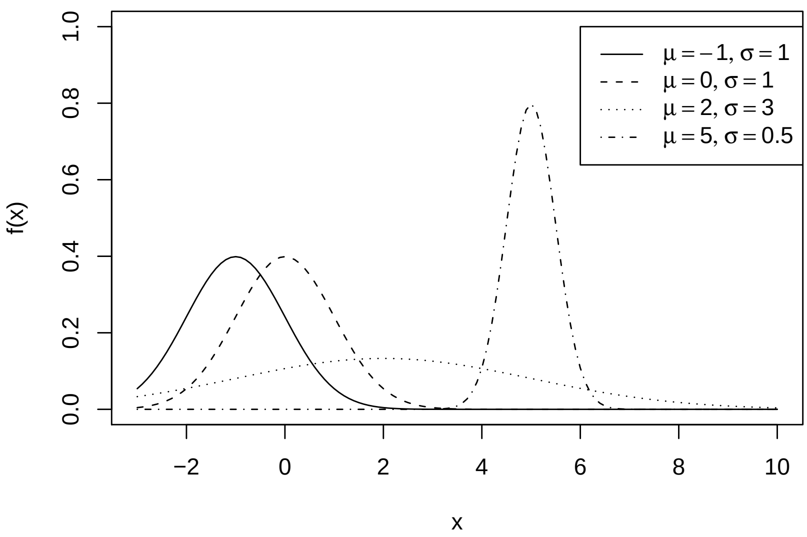


Figure 4.5: Normal densities

Example 1

Popular (and controversial) IQ scores are scaled to have the mean $\mu = 100$ and standard deviation $\sigma = 15$. Then, if a person has an IQ of 115, it can be transformed into Z-score as $z = (115 - 100)/15 = 1$ and expressed as "one standard deviation above the mean". A lot of standardized test scores (like SAT) follow the same principle. □

The values of the CDF of Z can be obtained from Table A. Namely,

$$F(z) = \begin{cases} 0.5 + \mathtt{TA}(z), & z \geq 0 \\ 0.5 - \mathtt{TA}(|z|), & z < 0 \end{cases}$$

where $\mathtt{TA}(z) = P(0 < Z < z)$ denotes table area of z. The second equation follows from the symmetry of the Z distribution.

Table A allows us to calculate probabilities and percentiles associated with normal random variables, as the direct integration of normal density is not possible.

Example 2.

If Z denotes a standard normal variable, find

(a) $P(Z \leq 1)$ (b) $P(Z > 1)$ (c) $P(Z < -1.5)$ (d) $P(-1.5 \leq Z \leq 0.5)$.

(e) Find a number, say z_0, such that $P(0 \leq Z \leq z_0) = 0.49$

Solution. This example provides practice in using Normal probability Table. We see that

a) $P(Z \leq 1) = P(Z \leq 0) + P(0 \leq Z \leq 1) = 0.5 + 0.3413 = 0.8413.$

b) $P(Z > 1) = 0.5 - P(0 \leq Z \leq 1) = 0.5 - 0.3413 = 0.1587$

c) $P(Z < -1.5) = P(Z > 1.5) = 0.5 - P(0 \leq Z \leq 1.5) = 0.5 - 0.4332 = 0.0668.$

d) $P(-1.5 \leq Z \leq 0.5) = P(-1.5 \leq Z \leq 0) + P(0 \leq Z \leq 0.5)$

$= P(0 \leq Z \leq 1.5) + P(0 \leq Z \leq 0.5) = 0.4332 + 0.1915 = 0.6247.$

e) To find the value of z_0 we must look for the given probability of 0.49 on the area side of Normal probability Table. The closest we can come is at 0.4901, which corresponds to a Z value of 2.33. Hence $z_0 = 2.33$.

□

Example

For $X \sim \mathcal{N}(50, 10^2)$, find the probability that X is between 45 and 62.

Solution. The Z- values corresponding to X = 45 and X = 62 are

$$Z_1 = \frac{45 - 50}{10} = -0.5 \qquad \text{and} \qquad Z_2 = \frac{62 - 50}{10} = 1.2.$$

Therefore, $P(45 \leq X \leq 62) = P(-0.5 \leq Z \leq 1.2) = \mathtt{TA}(1.2) + \mathtt{TA}(0.5) = 0.3849 + 0.1915 = 0.5764$ □

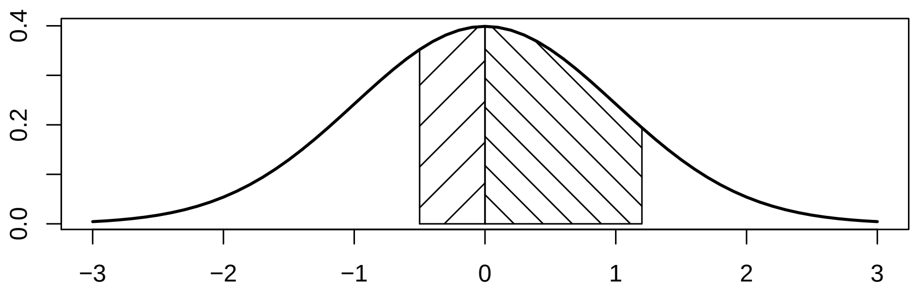


Figure : Splitting a normal area into two Table Areas

Example .

Given a random variable X having a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X is greater than 362.

Solution. To find $P(X > 362)$, we need to evaluate the area under the normal curve to the right of $x = 362$. This can be done by transforming $x = 362$ to the corresponding Z-value. We get

$$z = \frac{x - \mu}{\sigma} = \frac{362 - 300}{50} = 1.24 \text{Hence } P(X >$$

$362) = P(Z > 1.24) = P(Z < -1.24) = 0.5 - \mathtt{TA}(1.24) = 0.1075.$ □

Example .

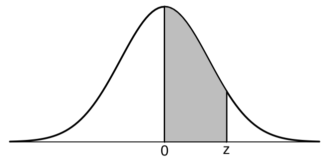
A diameter X of a shaft produced has a normal distribution with parameters $\mu = 1.005, \sigma = 0.01$. The shaft will meet specifications if its diameter is between 0.98 and 1.02 cm. Which percent of shafts will not meet specifications?

Solution.

$$1 - P(0.98 < X < 1.02) = 1 - P\left(\frac{0.98 - 1.005}{0.01} < Z < \frac{1.02 - 1.005}{0.01}\right)$$

$= 1 - (0.4938 + 0.4332) = 0.0730$ □

Table A: standard normal probabilities



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990