

CONTINUOUS RANDOM VARIABLES

DEFINITION : A *continuous random variable* is a function $X(s)$ from an *uncountably infinite* sample space \mathcal{S} to the real numbers \mathbb{R} ,

$$X(\cdot) \quad : \quad \mathcal{S} \quad \rightarrow \quad \mathbb{R} .$$

EXAMPLE :

Rotate a *pointer* about a pivot in a plane (like a hand of a clock).

The *outcome* is the *angle* where it stops : $2\pi\theta$, where $\theta \in (0, 1]$.

A good *sample space* is all values of θ , *i.e.* $\mathcal{S} = (0, 1]$.

A very simple example of a *continuous random variable* is $X(\theta) = \theta$.

Suppose *any outcome*, *i.e.*, any value of θ is "equally likely".

What are the values of

$$P(0 < \theta \leq \frac{1}{2}) \quad , \quad P(\frac{1}{3} < \theta \leq \frac{1}{2}) \quad , \quad P(\theta = \frac{1}{\sqrt{2}}) ?$$

The (*cumulative*) *probability distribution function* is defined as

$$F_X(x) \equiv P(X \leq x) .$$

Thus

$$F_X(b) - F_X(a) \equiv P(a < X \leq b) .$$

We must have

$$F_X(-\infty) = 0 \quad \text{and} \quad F_X(\infty) = 1 ,$$

i.e.,

$$\lim_{x \rightarrow -\infty} F_X(x) = 0 ,$$

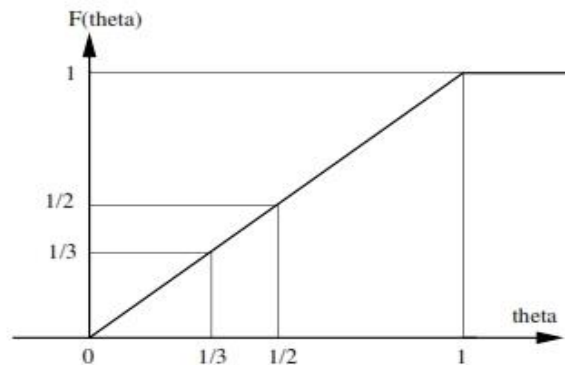
and

$$\lim_{x \rightarrow \infty} F_X(x) = 1 .$$

Also, $F_X(x)$ is a *non-decreasing* function of x . (**Why ?**)

NOTE : All the above is *the same* as for *discrete* random variables !

EXAMPLE : In the "pointer example", where $X(\theta) = \theta$, we have the *probability distribution function*



Note that

$$F\left(\frac{1}{3}\right) \equiv P\left(X \leq \frac{1}{3}\right) = \frac{1}{3} \quad , \quad F\left(\frac{1}{2}\right) \equiv P\left(X \leq \frac{1}{2}\right) = \frac{1}{2} \quad ,$$

$$P\left(\frac{1}{3} < X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} .$$

QUESTION : What is $P\left(\frac{1}{3} \leq X \leq \frac{1}{2}\right)$?

The *probability density function* is the *derivative* of the probability distribution function :

$$f_X(x) \equiv F'_X(x) \equiv \frac{d}{dx} F_X(x) .$$

EXAMPLE : In the "pointer example"

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x \leq 1 \\ 1, & 1 < x \end{cases}$$

Thus

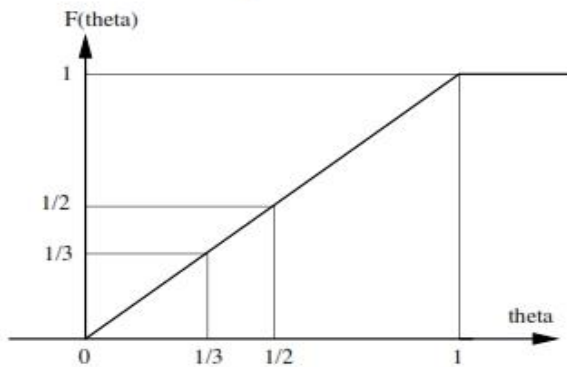
$$f_X(x) = F'_X(x) = \begin{cases} 0, & x \leq 0 \\ 1, & 0 < x \leq 1 \\ 0, & 1 < x \end{cases}$$

NOTATION : When it is clear what X is then we also write

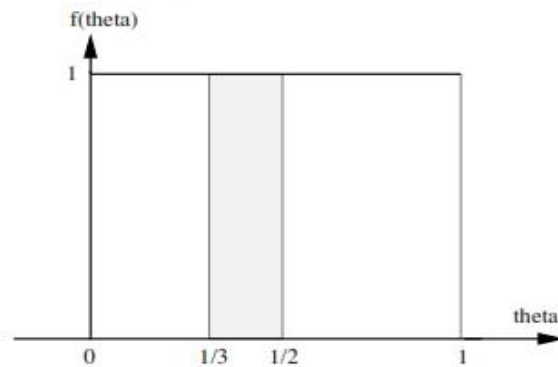
$$f(x) \text{ for } f_X(x) , \quad \text{and} \quad F(x) \text{ for } F_X(x) .$$

EXAMPLE : (continued ...)

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x \leq 1 \\ 1, & 1 < x \end{cases}, \quad f(x) = \begin{cases} 0, & x \leq 0 \\ 1, & 0 < x \leq 1 \\ 0, & 1 < x \end{cases}$$



Distribution function



Density function

NOTE :

$$P\left(\frac{1}{3} < X \leq \frac{1}{2}\right) = \int_{1/3}^{1/2} f(x) dx = \frac{1}{6} = \text{the shaded area .}$$

In general, from

$$f(x) \equiv F'(x),$$

with

$$F(-\infty) = 0 \quad \text{and} \quad F(\infty) = 1,$$

we have from Calculus the following *basic identities* :

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} F'(x) dx = F(\infty) - F(-\infty) = 1,$$

$$\int_{-\infty}^x f(x) dx = F(x) - F(-\infty) = F(x) = P(X \leq x),$$

$$\int_a^b f(x) dx = F(b) - F(a) = P(a < X \leq b),$$

$$\int_a^a f(x) dx = F(a) - F(a) = 0 = P(X = a).$$

EXERCISE : Draw *graphs* of the distribution and density functions

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-x}, & x > 0 \end{cases}, \quad f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-x}, & x > 0 \end{cases},$$

and verify that

- $F(-\infty) = 0, \quad F(\infty) = 1,$
- $f(x) = F'(x),$
- $F(x) = \int_0^x f(x) dx, \quad (\text{Why is zero as lower limit OK?})$
- $\int_0^\infty f(x) dx = 1,$
- $P(0 < X \leq 1) = F(1) - F(0) = F(1) = 1 - e^{-1} \cong 0.63,$
- $P(X > 1) = 1 - F(1) = e^{-1} \cong 0.37,$
- $P(1 < X \leq 2) = F(2) - F(1) = e^{-1} - e^{-2} \cong 0.23.$

EXERCISE : For positive integer n , consider the density functions

$$f_n(x) = \begin{cases} cx^n(1 - x^n), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- Determine the value of c in terms of n .
- Draw the graph of $f_n(x)$ for $n = 1, 2, 4, 8, 16$.
- Determine the distribution function $F_n(x)$.
- Draw the graph of $F_n(x)$ for $n = 1, 2, 3, 4, 8, 16$.
- Determine $P(0 \leq X \leq \frac{1}{2})$ in terms of n .
- What happens to $P(0 \leq X \leq \frac{1}{2})$ when n becomes large?
- Determine $P(\frac{9}{10} \leq X \leq 1)$ in terms of n .
- What happens to $P(\frac{9}{10} \leq X \leq 1)$ when n becomes large?

Joint distributions

A *joint probability density function* $f_{X,Y}(x, y)$ must satisfy

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1 \quad (\text{“Volume”} = 1).$$

The corresponding *joint probability distribution function* is

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^y \int_{-\infty}^x f_{X,Y}(x, y) dx dy.$$

By Calculus we have $\frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} = f_{X,Y}(x, y)$.

Also,

$$P(a < X \leq b, c < Y \leq d) = \int_c^d \int_a^b f_{X,Y}(x, y) dx dy.$$

EXAMPLE :

If

$$f_{X,Y}(x, y) = \begin{cases} 1 & \text{for } x \in (0, 1] \text{ and } y \in (0, 1], \\ 0 & \text{otherwise,} \end{cases}$$

then, for $x \in (0, 1]$ and $y \in (0, 1]$,

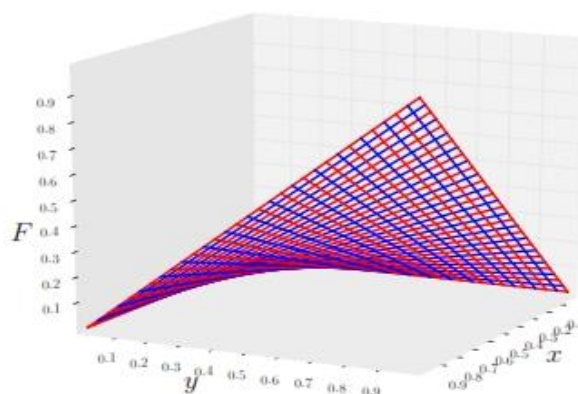
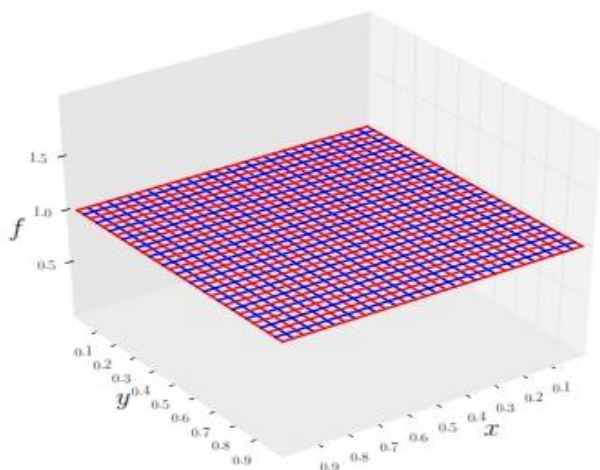
$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = \int_0^y \int_0^x 1 dx dy = xy.$$

Thus

$$F_{X,Y}(x, y) = xy, \quad \text{for } x \in (0, 1] \text{ and } y \in (0, 1].$$

For example

$$P\left(X \leq \frac{1}{3}, Y \leq \frac{1}{2}\right) = F_{X,Y}\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{6}.$$



Also,

$$P\left(\frac{1}{3} \leq X \leq \frac{1}{2}, \frac{1}{4} \leq Y \leq \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{3}}^{\frac{1}{2}} f(x, y) \, dx \, dy = \frac{1}{12} .$$

EXERCISE : Show that we can also compute this as follows :

$$F\left(\frac{1}{2}, \frac{3}{4}\right) - F\left(\frac{1}{3}, \frac{3}{4}\right) - F\left(\frac{1}{2}, \frac{1}{4}\right) + F\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{1}{12} .$$

and explain why !

Marginal density functions

The *marginal density functions* are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \quad , \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx .$$

with corresponding *marginal distribution functions*

$$F_X(x) \equiv P(X \leq x) = \int_{-\infty}^x f_X(x) \, dx = \int_{-\infty}^x \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx ,$$

$$F_Y(y) \equiv P(Y \leq y) = \int_{-\infty}^y f_Y(y) \, dy = \int_{-\infty}^y \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy .$$

By Calculus we have

$$\frac{dF_X(x)}{dx} = f_X(x) \quad , \quad \frac{dF_Y(y)}{dy} = f_Y(y) .$$

EXAMPLE : If $f_{X,Y}(x, y) = \begin{cases} 1 & \text{for } x \in (0, 1] \text{ and } y \in (0, 1] , \\ 0 & \text{otherwise} , \end{cases}$

then, for $x \in (0, 1]$ and $y \in (0, 1]$,

$$f_X(x) = \int_0^1 f_{X,Y}(x, y) dy = \int_0^1 1 dy = 1 ,$$

$$f_Y(y) = \int_0^1 f_{X,Y}(x, y) dx = \int_0^1 1 dx = 1 ,$$

$$F_X(x) = P(X \leq x) = \int_0^x f_X(x) dx = x ,$$

$$F_Y(y) = P(Y \leq y) = \int_0^y f_Y(y) dy = y .$$

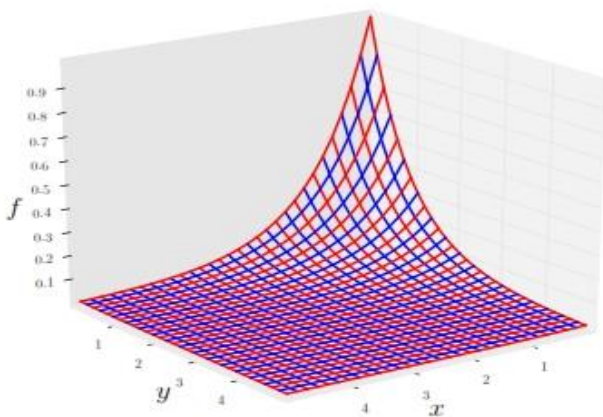
For example

$$P(X \leq \frac{1}{3}) = F_X(\frac{1}{3}) = \frac{1}{3} , \quad P(Y \leq \frac{1}{2}) = F_Y(\frac{1}{2}) = \frac{1}{2} .$$

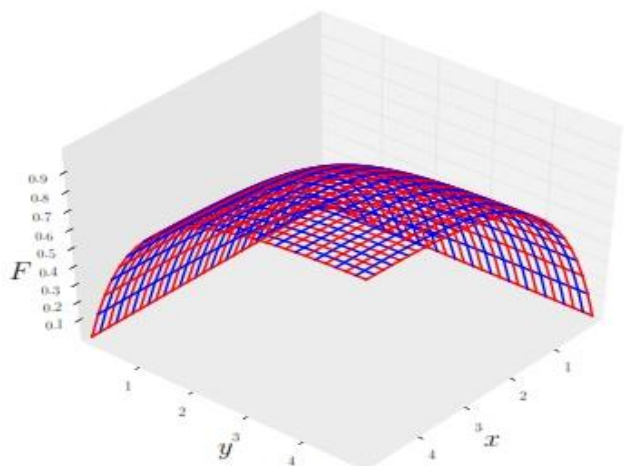
EXERCISE : Let $F_{X,Y}(x, y) = \begin{cases} (1 - e^{-x})(1 - e^{-y}) & \text{for } x \geq 0 \text{ and } y \geq 0 , \\ 0 & \text{otherwise} . \end{cases}$

- Verify that

$$f_{X,Y}(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \begin{cases} e^{-x-y} & \text{for } x \geq 0 \text{ and } y \geq 0 , \\ 0 & \text{otherwise} . \end{cases}$$



Density function $f_{X,Y}(x, y)$



Distribution function $F_{X,Y}(x, y)$

EXERCISE : (continued ...)

$$F_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y}) \quad , \quad f_{X,Y}(x,y) = e^{-x-y} \quad , \quad \text{for } x,y \geq 0 .$$

Also verify the following :

- $F(0,0) = 0 \quad , \quad F(\infty, \infty) = 1 \quad ,$
- $\int_0^\infty \int_0^\infty f_{X,Y}(x,y) dx dy = 1 \quad , \quad (\text{ Why zero lower limits ? })$
- $f_X(x) = \int_0^\infty e^{-x-y} dy = e^{-x} \quad ,$
- $f_Y(y) = \int_0^\infty e^{-x-y} dx = e^{-y} \quad .$
- $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad . \quad (\text{ So ? })$

EXERCISE : (continued ...)

$$F_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y}) \quad , \quad f_{X,Y}(x,y) = e^{-x-y} \quad , \quad \text{for } x,y \geq 0 .$$

Also verify the following :

- $F_X(x) = \int_0^x f_X(x) dx = \int_0^x e^{-x} dx = 1 - e^{-x} \quad ,$
- $F_Y(y) = \int_0^y f_Y(y) dy = \int_0^y e^{-y} dy = 1 - e^{-y} \quad ,$
- $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) \quad . \quad (\text{ So ? })$
- $P(1 < x < \infty) = F_X(\infty) - F_X(1) = 1 - (1 - e^{-1}) = e^{-1} \cong 0.37 \quad ,$
- $P(1 < x \leq 2 \quad , \quad 0 < y \leq 1) = \int_0^1 \int_1^2 e^{-x-y} dx dy$

$$= (e^{-1} - e^{-2})(1 - e^{-1}) \cong 0.15 \quad ,$$

Independent continuous random variables

Recall that two events E and F are *independent* if

$$P(EF) = P(E)P(F).$$

Continuous random variables $X(s)$ and $Y(s)$ are *independent* if

$$P(X \in I_X, Y \in I_Y) = P(X \in I_X) \cdot P(Y \in I_Y),$$

for *all* allowable sets I_X and I_Y (typically *intervals*) of *real numbers*.

Equivalently, $X(s)$ and $Y(s)$ are independent if for all such sets I_X and I_Y the *events*

$$X^{-1}(I_X) \quad \text{and} \quad Y^{-1}(I_Y),$$

are independent *in the sample space* \mathcal{S} .

NOTE : $X^{-1}(I_X) \equiv \{s \in \mathcal{S} : X(s) \in I_X\}$,
 $Y^{-1}(I_Y) \equiv \{s \in \mathcal{S} : Y(s) \in I_Y\}$.

FACT : $X(s)$ and $Y(s)$ are *independent* if for all x and y

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y).$$

EXAMPLE : The random variables with density function

$$f_{X,Y}(x, y) = \begin{cases} e^{-x-y} & \text{for } x \geq 0 \text{ and } y \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

are *independent* because (by the preceding exercise)

$$f_{X,Y}(x, y) = e^{-x-y} = e^{-x} \cdot e^{-y} = f_X(x) \cdot f_Y(y).$$

NOTE :

$$F_{X,Y}(x, y) = \begin{cases} (1 - e^{-x})(1 - e^{-y}) & \text{for } x \geq 0 \text{ and } y \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

also satisfies (by the preceding exercise)

$$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y).$$

PROPERTY :

For *independent* continuous random variables X and Y we have

$$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y) , \quad \text{for all } x, y .$$

PROOF :

$$\begin{aligned} F_{X,Y}(x, y) &= P(X \leq x , Y \leq y) \\ &= \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(x, y) dy dx \\ &= \int_{-\infty}^x \int_{-\infty}^y f_X(x) \cdot f_Y(y) dy dx \quad (\text{by independence}) \\ &= \int_{-\infty}^x [f_X(x) \cdot \int_{-\infty}^y f_Y(y) dy] dx \\ &= [\int_{-\infty}^x f_X(x) dx] \cdot [\int_{-\infty}^y f_Y(y) dy] \\ &= F_X(x) \cdot F_Y(y) . \end{aligned}$$

REMARK : Note how the proof parallels that for the discrete case !

Conditional distributions

Let X and Y be continuous random variables.

For given allowable sets I_X and I_Y (typically *intervals*), let

$$E_x = X^{-1}(I_X) \quad \text{and} \quad E_y = Y^{-1}(I_Y) ,$$

be their corresponding *events* in the sample space \mathcal{S} .

We have
$$P(E_x|E_y) \equiv \frac{P(E_x E_y)}{P(E_y)} .$$

The *conditional probability density function* is defined as

$$f_{X|Y}(x|y) \equiv \frac{f_{X,Y}(x, y)}{f_Y(y)} .$$

When X and Y are *independent* then

$$f_{X|Y}(x|y) \equiv \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x) ,$$

(assuming $f_Y(y) \neq 0$).

EXAMPLE : The random variables with density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x \geq 0 \text{ and } y \geq 0 , \\ 0 & \text{otherwise} , \end{cases}$$

have (by previous exercise) the marginal density functions

$$f_X(x) = e^{-x} , \quad f_Y(y) = e^{-y} ,$$

for $x \geq 0$ and $y \geq 0$, and zero otherwise.

Thus for such x, y we have

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-x-y}}{e^{-y}} = e^{-x} = f_X(x) ,$$

i.e., information about Y does not alter the density function of X .

Indeed, we have already seen that X and Y are *independent* .

Expectation

The *expected value* of a continuous random variable X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx ,$$

which represents the *average value* of X over many trials.

The expected value of a *function of a random variable* is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx .$$

The expected value of a function of *two* random variables is

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dy dx .$$

EXAMPLE :

For the *pointer* experiment

$$f_X(x) = \begin{cases} 0, & x \leq 0 \\ 1, & 0 < x \leq 1 \\ 0, & 1 < x \end{cases}$$

we have

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2},$$

and

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}.$$

EXAMPLE : For the joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

we have (by previous exercise) the marginal density functions

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} e^{-y} & \text{for } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Thus } E[X] = \int_0^{\infty} x e^{-x} dx = -[(x+1)e^{-x}] \Big|_0^{\infty} = 1. \quad (\text{Check!})$$

$$\text{Similarly } E[Y] = \int_0^{\infty} y e^{-y} dy = 1,$$

and

$$E[XY] = \int_0^{\infty} \int_0^{\infty} xy e^{-x-y} dy dx = 1. \quad (\text{Check!})$$

EXERCISE :

Prove the following for *continuous* random variables :

- $E[aX] = a E[X] ,$
- $E[aX + b] = a E[X] + b ,$
- $E[X + Y] = E[X] + E[Y] ,$

and *compare* the proofs to those for *discrete* random variables.

EXERCISE :

A stick of length 1 is split at a randomly selected point X .

(Thus X is uniformly distributed in the interval $[0, 1]$.)

Determine the expected length of the piece containing the point $1/3$.

PROPERTY : If X and Y are *independent* then

$$E[XY] = E[X] \cdot E[Y] .$$

PROOF :

$$\begin{aligned} E[XY] &= \int_{\mathbb{R}} \int_{\mathbb{R}} x y f_{X,Y}(x, y) dy dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} x y f_X(x) f_Y(y) dy dx \quad (\text{by independence}) \\ &= \int_{\mathbb{R}} [x f_X(x) \int_{\mathbb{R}} y f_Y(y) dy] dx \\ &= [\int_{\mathbb{R}} x f_X(x) dx] \cdot [\int_{\mathbb{R}} y f_Y(y) dy] \\ &= E[X] \cdot E[Y] . \end{aligned}$$

REMARK : Note how the proof parallels that for the discrete case !

EXAMPLE : For

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{for } x > 0 \text{ and } y > 0 , \\ 0 & \text{otherwise} , \end{cases}$$

we already found

$$f_X(x) = e^{-x} \quad , \quad f_Y(y) = e^{-y} \quad ,$$

so that

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad ,$$

i.e., X and Y are *independent* .

Indeed, we also already found that

$$E[X] = E[Y] = E[XY] = 1 \quad ,$$

so that

$$E[XY] = E[X] \cdot E[Y] \quad .$$

Variance

Let

$$\mu = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Then the *variance* of the continuous random variable X is

$$\text{Var}(X) \equiv E[(X - \mu)^2] \equiv \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx \quad ,$$

which is the average weighted *square distance* from the mean.

As in the discrete case, we have

$$\begin{aligned} \text{Var}(X) &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - \mu^2 \quad . \end{aligned}$$

The *standard deviation* of X is

$$\sigma(X) \equiv \sqrt{\text{Var}(X)} = \sqrt{E[X^2] - \mu^2} \quad .$$

which is the average weighted *distance* from the mean.

EXAMPLE : For $f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$

we have

$$E[X] = \mu = \int_0^{\infty} x e^{-x} dx = 1 \quad (\text{already done!}),$$

$$E[X^2] = \int_0^{\infty} x^2 e^{-x} dx = -[(x^2 + 2x + 2)e^{-x}] \Big|_0^{\infty} = 2,$$

$$\text{Var}(X) = E[X^2] - \mu^2 = 2 - 1^2 = 1,$$

$$\sigma(X) = \sqrt{\text{Var}(X)} = 1.$$

NOTE : The two integrals can be done by “*integration by parts*”.

EXERCISE :

Also use the *Method of Moments* to compute $E[X]$ and $E[X^2]$.

EXERCISE : For the random variable X with density function

$$f(x) = \begin{cases} 0, & x \leq -1 \\ c, & -1 < x \leq 1 \\ 0, & x > 1 \end{cases}$$

- Determine the value of c
- Draw the graph of $f(x)$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $\text{Var}(X)$ and $\sigma(X)$
- Determine $P(X \leq -\frac{1}{2})$
- Determine $P(|X| \geq \frac{1}{2})$

EXERCISE : For the random variable X with density function

$$f(x) = \begin{cases} x + 1, & -1 < x \leq 0 \\ 1 - x, & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- Draw the graph of $f(x)$
- Verify that $\int_{-\infty}^{\infty} f(x) dx = 1$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $Var(X)$ and $\sigma(X)$
- Determine $P(X \geq \frac{1}{3})$
- Determine $P(|X| \leq \frac{1}{3})$

EXERCISE : For the random variable X with density function

$$f(x) = \begin{cases} \frac{3}{4} (1 - x^2), & -1 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- Draw the graph of $f(x)$
- Verify that $\int_{-\infty}^{\infty} f(x) dx = 1$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $Var(X)$ and $\sigma(X)$
- Determine $P(X \leq 0)$
- Compute $P(X \geq \frac{2}{3})$
- Compute $P(|X| \geq \frac{2}{3})$

EXERCISE : Recall the density function

$$f_n(x) = \begin{cases} cx^n(1 - x^n) , & 0 \leq x \leq 1 \\ 0 , & \text{otherwise} \end{cases}$$

considered earlier, where n is a positive integer, and where

$$c = \frac{(n+1)(2n+1)}{n} .$$

- Determine $E[X]$.
- What happens to $E[X]$ for *large* n ?
- Determine $E[X^2]$
- What happens to $E[X^2]$ for *large* n ?
- What happens to $Var(X)$ for *large* n ?

Covariance

Let X and Y be continuous random variables with *mean*

$$E[X] = \mu_X \quad , \quad E[Y] = \mu_Y .$$

Then the *covariance* of X and Y is

$$\begin{aligned} Cov(X, Y) &\equiv E[(X - \mu_X) (Y - \mu_Y)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X) (y - \mu_Y) f_{X,Y}(x, y) dy dx . \end{aligned}$$

As in the discrete case, we have

$$\begin{aligned} Cov(X, Y) &= E[(X - \mu_X) (Y - \mu_Y)] \\ &= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\ &= E[XY] - E[X] E[Y] . \end{aligned}$$

As in the discrete case, we also have

PROPERTY 1 :

- $Var(X + Y) = Var(X) + Var(Y) + 2 Cov(X, Y) ,$

and

PROPERTY 2 : If X and Y are *independent* then

- $Cov(X, Y) = 0 ,$

- $Var(X + Y) = Var(X) + Var(Y) .$

NOTE :

- The proofs are identical to those for the discrete case !

EXAMPLE : For

$$f_{X,Y}(x, y) = \begin{cases} e^{-x-y} & \text{for } x > 0 \text{ and } y > 0 , \\ 0 & \text{otherwise} , \end{cases}$$

we already found

$$f_X(x) = e^{-x} \quad , \quad f_Y(y) = e^{-y} \quad ,$$

so that

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) ,$$

i.e., X and Y are *independent* .

Indeed, we also already found

$$E[X] = E[Y] = E[XY] = 1 ,$$

so that

$$Cov(X, Y) = E[XY] - E[X] E[Y] = 0 .$$

EXERCISE :

For the random variables X , Y with *joint density function*

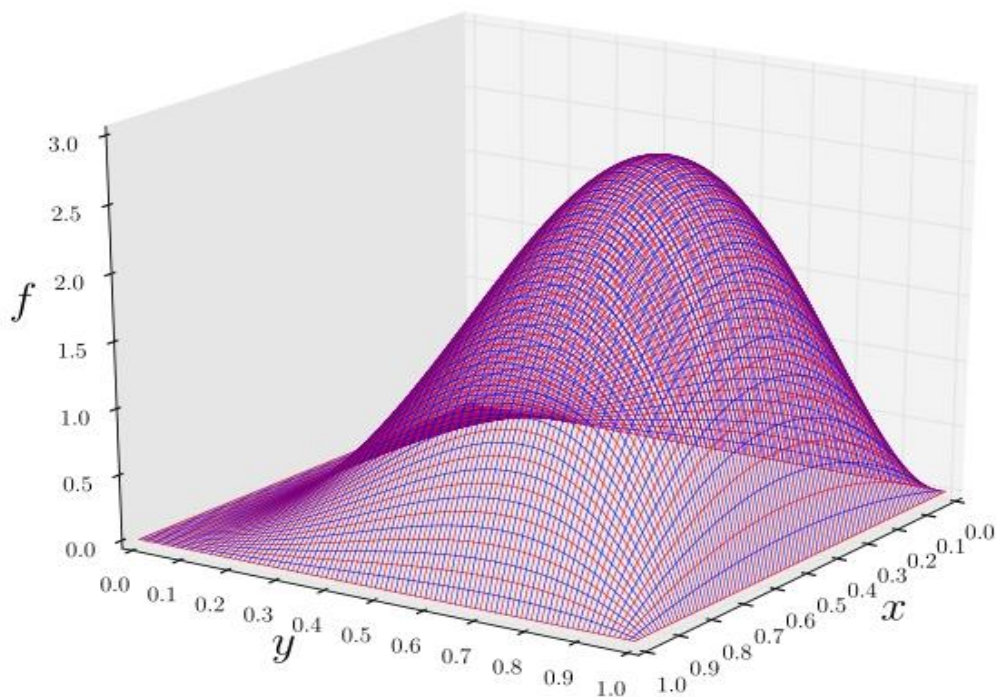
$$f(x, y) = \begin{cases} 45xy^2(1-x)(1-y^2) , & 0 \leq x \leq 1 , 0 \leq y \leq 1 \\ 0 , & \text{otherwise} \end{cases}$$

- Verify that $\int_0^1 \int_0^1 f(x, y) dy dx = 1$.
- Determine the *marginal density functions* $f_X(x)$ and $f_Y(y)$.
- Are X and Y *independent* ?
- What is the value of $Cov(X, Y)$?

EXERCISE :

Verify the following properties :

- $Var(cX + d) = c^2 Var(X)$,
- $Cov(X, Y) = Cov(Y, X)$,
- $Cov(cX, Y) = c Cov(X, Y)$,
- $Cov(X, cY) = c Cov(X, Y)$,
- $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$,
- $Var(X + Y) = Var(X) + Var(Y) + 2 Cov(X, Y)$.



The joint probability density function $f_{XY}(x, y)$.

Markov's inequality.

For a continuous *nonnegative* random variable X , and $c > 0$, we have

$$P(X \geq c) \leq \frac{E[X]}{c} .$$

PROOF :

$$\begin{aligned} E[X] &= \int_0^{\infty} x f(x) dx = \int_0^c x f(x) dx + \int_c^{\infty} x f(x) dx \\ &\geq \int_c^{\infty} x f(x) dx \\ &\geq c \int_c^{\infty} f(x) dx \quad (\text{Why ?}) \\ &= c P(X \geq c) . \end{aligned}$$

EXERCISE :

Show Markov's inequality also holds for *discrete* random variables.

Markov's inequality : For continuous *nonnegative* X , $c > 0$:

$$P(X \geq c) \leq \frac{E[X]}{c} .$$

EXAMPLE : For $f(x) = \begin{cases} e^{-x} & \text{for } x > 0 , \\ 0 & \text{otherwise} , \end{cases}$

we have

$$E[X] = \int_0^{\infty} x e^{-x} dx = 1 \quad (\text{already done !})$$

Markov's inequality gives

$$c = \mathbf{1} : \quad P(X \geq \mathbf{1}) \leq \frac{E[X]}{\mathbf{1}} = \frac{1}{\mathbf{1}} = 1 \quad (!)$$

$$c = \mathbf{10} : \quad P(X \geq \mathbf{10}) \leq \frac{E[X]}{\mathbf{10}} = \frac{1}{\mathbf{10}} = 0.1$$

QUESTION : Are these estimates "*sharp*" ?

Markov's inequality gives

$$c = 1 : \quad P(X \geq 1) \leq \frac{E[X]}{1} = \frac{1}{1} = 1 \quad (!)$$

$$c = 10 : \quad P(X \geq 10) \leq \frac{E[X]}{10} = \frac{1}{10} = 0.1$$

The actual values are

$$P(X \geq 1) = \int_1^{\infty} e^{-x} dx = e^{-1} \cong 0.37$$

$$P(X \geq 10) = \int_{10}^{\infty} e^{-x} dx = e^{-10} \cong 0.000045$$

EXERCISE : Suppose the score of students taking an examination is a random variable with mean 65 .

Give an upper bound on the probability that a student's score is greater than 75 .

Chebyshev's inequality: For (practically) any random variable X :

$$P(|X - \mu| \geq k \sigma) \leq \frac{1}{k^2} ,$$

where $\mu = E[X]$ is the *mean*, $\sigma = \sqrt{\text{Var}(X)}$ the *standard deviation*.

PROOF : Let $Y \equiv (X - \mu)^2$, which is nonnegative.

By Markov's inequality

$$P(Y \geq c) \leq \frac{E[Y]}{c} .$$

Taking $c = k^2 \sigma^2$ we have

$$\begin{aligned} P(|X - \mu| \geq k \sigma) &= P((X - \mu)^2 \geq k^2 \sigma^2) = P(Y \geq k^2 \sigma^2) \\ &\leq \frac{E[Y]}{k^2 \sigma^2} = \frac{\text{Var}(X)}{k^2 \sigma^2} = \frac{\sigma^2}{k^2 \sigma^2} = \frac{1}{k^2} . \quad \text{QED !} \end{aligned}$$

NOTE : This inequality also holds for discrete random variables.

EXAMPLE : Suppose the value of the Canadian dollar in terms of the US dollar over a certain period is a random variable X with

$$\text{mean } \mu = 0.98 \quad \text{and} \quad \text{standard deviation } \sigma = 0.05 .$$

What can be said of the probability that the Canadian dollar is valued

$$\text{between } \$0.88\text{US} \quad \text{and} \quad \$1.08\text{US} ,$$

that is,

$$\text{between } \mu - 2\sigma \quad \text{and} \quad \mu + 2\sigma ?$$

SOLUTION : By Chebyshev's inequality we have

$$P(|X - \mu| \geq 2 \sigma) \leq \frac{1}{2^2} = 0.25 .$$

Thus

$$P(|X - \mu| < 2 \sigma) > 1 - 0.25 = 0.75 ,$$

that is,

$$P(\$0.88\text{US} < \text{Can\$} < \$1.08\text{US}) > 75 \% .$$

EXERCISE :

The score of students taking an examination is a random variable with **mean $\mu = 65$** and **standard deviation $\sigma = 5$** .

- What is the probability a student scores between 55 and 75 ?
- How many students would have to take the examination so that the probability that their average grade is between 60 and 70 is at least 80% ?

HINT : Defining

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \dots + X_n) , \quad (\text{the average grade})$$

we have
$$\mu_{\bar{X}} = E[\bar{X}] = \frac{1}{n} n \mu = \mu = 65 ,$$

and, assuming independence,

$$Var(\bar{X}) = n \frac{\sigma^2}{n^2} = \frac{\sigma^2}{n} = \frac{25}{n} , \quad \text{and} \quad \sigma_{\bar{X}} = \frac{5}{\sqrt{n}} .$$