Review of Probability

Random variable

A random variable is a real valued function whose numerical value is determined by the outcome of a random experiment. In other words the random variable X is a function that associated each element in the sample space Ω from with the real numbers (i.e. $X : \Omega \to \mathcal{R}$)

Notation:

X **(**capital letter**):** denotes the random variable .

x **(**small letter**):** denotes a value of the random variable X.

Discrete random variable

A random variable X is called a discrete random variable if its set of possible values is countable (integer).

Continuous random variable

A random variable X is called a continuous random variable if it can take values on a continuous scales.

Discrete probability distribution

If X is a discrete random variable with distinct values $x_1, x_2, \ldots \ldots . . x_t$, then the function

$$f(x) = \begin{cases} f(X = x_i), & if \quad x = x_1, x_2, \ldots \ldots . . x_t \\ 0, & Otherwise \end{cases}$$

Is defined to be the probability mass function pmf of X .

This means that a discrete random variable is a listing of all possible distinct (elementary) events and their probabilities of occurring for a random variable.

x_1	x_2	....	x_t
$f(x_1)$	$f(x_2)$	....	$f(x_t)$

The pmf $f(x)$ is a real valued function and satisfies the following properties:

3. $0 \le f(x) \le 1$ 2. $\sum_{\forall x} f(x) = 1$ 3. $P(X = A) = \sum_{x \epsilon A} f(x)$ where $A \subset x$ is

Continuous probability distribution

The probability density function pdf of a continuous random variable X is a mathematical function $f(x)$ which is defined as follows:

$$f(x) = \begin{cases} f(X = x_i), & if -\infty \le x \le \infty \\ 0, & Otherwise \end{cases}$$

The pdf $f(x)$ is satisfies the following properties:

1. $f(x) \ge 0 \; \forall \; x \in \mathcal{R}$ 2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$ 3. $P(X \epsilon A) = \int_A f(x)\, dx$, $A \subset x$

Cumulative distribution function (CDF)

For any random variable we define the cumulative distribution function cdf , $F(x)$ by:

$$f(x) = P(X \leq x)$$

Where, x is any real value.

$$F(x) = \begin{cases} \sum_{u=-\infty}^{x} f(u)\,, & if\ X\ is\ a\ discrete \\ \int_{-\infty}^{x} f(u)du\,, & if\ X\ is\ a\ continuous \end{cases}$$

$F(x)$ is monotonic increasing i.e.

$F(a) \leq F(b) \quad whenever\ a \leq b$

And the limit of $F(x)$ to the left is 0 and to the right is 1:

$$\lim_{x\to-\infty} F(x) = 0 \qquad and \qquad \lim_{x\to\infty} F(x) = 1$$

For a continuous case :

1. $P(a < X < b) = P(X < b) - P(X < a) = F(b) - F(a)$
2. $f(x) = \frac{dF(x)}{dx}$

Mathematical Expectation

Let X be a random variable with a probability distribution $f(x)$ the expected value (mean) of X is denoted by E(X) or μ_x and is defined by:

$$E(X) = \mu_X = \begin{cases} \sum_{all\ x} x\ f(x)\,; & if\ \ X\ \ is\ discrete \\ \int_{-\infty}^{\infty} x\ f(x)dx\ ; & if\ \ X\ \ is\ continuous \end{cases}$$

Linear property:

Let X be a random variable with the pdf $f(x)$, and let a and b are a constants, then:

$$E(a + bX) = a + bE(X)$$

The variance

Let X be a random variable with a probability distribution $f(x)$ the variance of X is denoted by $Var(X)$ or σ_X^2 and is defined by:

$$Var(X) = \sigma_X^2 = E[(X-\mu)^2] = \begin{cases} \sum_{all\ x} (x-\mu)^2\ f(x)\,; & if\ \ X\ \ is\ discrete \\ \int_{-\infty}^{\infty} (x-\mu)^2\ f(x)dx\ ; & if\ \ X\ \ is\ continuous \end{cases}$$

and it's also can be written as:

$$Var(X) = E(X^2) - \mu_X^2$$

Linear property:

Let X be a random variable with the pdf *f(x)*, and let *a* and *b* are a constants, then:

$$Var(a+bX)=b^2Var(X)$$

The moments:

Let X be a random variable with the pdf *f(x)*, the *r*th moment about the origin of *X*, is given by:

$$\acute{\mu}_r = E(X^r) = \begin{cases} \sum_{all\ x} x^r f(x); & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} x^r\ f(x)\,dx; & if\ X\ is\ continuous \end{cases}$$

if the expectation exists

As special case:

$$\acute{\mu}_1 = EX \quad = \text{mean of } X = \mu$$

Let X be a random variable with the pdf *f(x)*, the *r*th central moment of *X* about μ , is defined as:

$$\mu_r = \mathrm{E}(X-\mu)^r = \begin{cases} \sum_{all\ x} (x-\mu)^r f(x); & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} (x-\mu)^r\ f(x)\,dx; & if\ X\ is\ continuous \end{cases}$$

As special case:

$$\mu_2 = \mathrm{E}(X-\mu)^2 = \sigma^2 \text{ the variance of X.}$$

Moment- Generating Function MGF:

Let X be a random variable with the pdf *f(x)*, the moment - generating function of *X*, is given by $E(e^{tx})$ and is denoted by $M_X(t)$. Hence :

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum_{all\ x} e^{tx} f(x); & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} e^{tx}\ f(x)\,dx; & if\ X\ is\ continuous \end{cases}$$

Moment-generating functions will exist only if the sum or integral of the above definition converges. If a moment-generating function of a random variable X does exist, it can be used to generate all the moments of that variable.

Definition:

Let X be a random variable with moment - generating function of *X*, is given by $M_X(t)$. then :

$\frac{d^r M_X(t)}{dt^2}\Big|_{t=0} = \acute{\mu}_r$ Therefore, $\frac{d\ \ M_X(t)}{dt}\Big|_{t=0} = \acute{\mu}_1 = \mu$

$\frac{d^2 M_X(t)}{dt^2}\Big|_{t=0} = \acute{\mu}_2$ $\sigma^2 = \acute{\mu}_2 - \acute{\mu}_1^2$

Example. Find the moment-generating function of the binomial random variable X and then use it to verify that $\mu = np$ and $\sigma^2 = npq.$

$$f(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}; & x = 0,1, \ldots\ldots, n \\ 0; & otherwise \end{cases}$$

$M_X(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$

Recognizing this last sum as the binomial expansion of $(pe^t + q)^n$, we obtain:

$$M_X(t) = (pe^t + q)^n$$

now: $\frac{dM_X(t)}{dt} = npe^t(pe^t + q)^{n-1}$

and: $\frac{d^2M_X(t)}{dt^2} = npe^t[pe^t(n-1)(pe^t + q)^{n-2} + (pe^t + q)^{n-1}]$

Setting t = 0 , we get: $\acute{\mu}_1 = np$ and : $\acute{\mu}_2 = np[p(n-1) + 1]$

Therefore, $\mu = \acute{\mu}_1 = np$

$$\sigma^2 = \acute{\mu}_2 - \acute{\mu}_1^2 = np[p(n-1) + 1] - (np)^2 = npq$$

Probability Generating Function PGF:

Let X be a random variable defined over the non-negative intergers. The probability generating function PGF is given by the polynomial

$$G_X(s) = E(s^X) = p_0 + p_1 s + p2s^2 + \cdots = \sum_{x=0}^{\infty} s^x P(X = x)$$

Example. Let X have a binomial distribution function such that $X \sim B(n, p)$. The PGF is given by

$$G_X(s) = E(s^X) = \sum_{x=0}^{n} \binom{n}{x} (sp)^x q^{n-x} = (q + sp)^n$$

An important property of a PGF is that it converges for |s| ≤ 1 since

$G_X(1) = \sum_x P(X = x) = 1.$

The PGF can be used to directly derive the probability function of the random variable, as well as its moments. Single probabilities can be calculated as

$$P(X = j) = p_j = (j!)^{-1} \frac{d^j G_X(s)}{ds^j}\Big|_{s=0}$$

Example: A binomial distributed random variable has PGF $G_X(s) = (q + sp)^n$. Thus,

$P(X = 0) = G_X(0) = q^n$

$P(X = 1) = G_X{}'(0) = nq^{n-1}p^1$

$P(X = 2) = (2!)^{-1} G_X{}''(0) = (2!)^{-1} n(n-1) q^{n-2} p^2$

$\vdots \qquad \qquad \vdots$

The expectation $E(X)$ satisfies the relation

$$E(X) = \sum_{x=0}^{\infty} x\, G_X(s) = G_X{}'(1)$$

Example: A binomial distributed random variable has mean

$$G_X{}'(1) = np(p+q)^{n-1} = np$$

Calculating first

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\, G_X(s) = G_X{}''(1)$$

the variance is obtained as

$$Var(X) = E[X(X-1)] + E(X) - [E(X)]^2 = G_X''^{(1)} + G_X'^{(1)} - \left[G_X'^{(1)} \right]^2$$

Example: A binomial distributed random variable has variance

$$Var(X) = n(n-1)p^2 + np + [np]^2 = np(1-p)$$

Joint and marginal probability Distributions

Joint probability distribution (Discrete case)

If X and Y are two discrete random variables, then $f(x,y) = P(X = x, Y = y)$ is called joint probability mass function jpmf of X and Y, and $f(x,y)$ has the following properties:

1. $0 \le f(x,y) \le 1$ for all x and y. 2. $\sum_x \sum_y f(x,y) = 1$
3. $P[(X,Y) \in A] = \sum\sum_A f(x,y)$ for any region A in the X,Y plane.

Marginal probability distribution (Discrete case)

If X and Y are jointly discrete random variables with the jpmf $f(x,y)$, then $\mathrm{g}(x)$ and $\mathrm{h}(y)$ are called marginal probability mass functions of X and Y respectively which can be calculated as

- $\mathrm{g}(x) = \sum_{\forall y} f(x,y)$ - $\mathrm{h}(y) = \sum_{\forall x} f(x,y)$

Joint probability distribution (Continuous case)

If X and Y are two continuous random variables, then $f(x,y) = P(X = x, Y = y)$ is called joint probability density function jpdf of X and Y, and $f(x,y)$ has the following properties:

1. $f(x,y) \ge 0$ for all x and y.
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$
3. $P[(X,Y) \in A] = \int_A f(x,y)\, dx\, dy$ for any region A in the X,Y plane.

Marginal probability distribution (Continuous case)

If X and Y are jointly continuous random variables with the j.p.d.f $f(x,y)$, then $\mathrm{g}(x)$ and $\mathrm{h}(y)$ are called marginal probability density function of X and Y respectively which can be calculated as

- $\mathrm{g}(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$
- $\mathrm{h}(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

Conditional Distributions and conditional Expectation

Conditional distribution

If X and Y are jointly random variables discrete or continuous with the jpf $f(x,y)$, $\mathrm{g}(x)$ and $\mathrm{h}(y)$ are marginal probability distributions of X and Y respectively, then the conditional distribution of the random variable Y given that $X = x$ is

$$f(y \mid x) = \frac{f(x,y)}{\mathrm{g}(x)}, \quad \mathrm{g}(x) > 0$$

Similarly the conditional distribution of the random variable X given that $Y = y$ is

$$f(x \mid y) = \frac{f(x,y)}{\mathrm{h}(y)}, \quad \mathrm{h}(y) > 0$$

Statistical independence

If X and Y be two random variables discrete or continuous with the jpf $f(x,y)$, $\mathrm{g}(x)$ and $\mathrm{h}(y)$ are marginal probability distributions of X and Y respectively. The random variables X and Y are said to be statistically independent if and only if:

$$f(x,y) = \mathrm{g}(x)\mathrm{h}(y)$$

for all (x,y) within their ranges.

Conditional Expectation

If X and Y are jointly random variables discrete or continuous with the jpf $f(x,y)$, $\mathrm{g}(x)$ and $\mathrm{h}(y)$ are marginal probability distributions of X and Y respectively, then the conditional expectation of the random variable X given that $Y = y$ for all values of Y such that $\mathrm{h}(y) > 0$ is

$$E(X \mid Y = y) = \begin{cases} \sum_{all\ x} x f(x \mid y); & \text{for discrete} \\ \int_{\mathrm{R}} x f(x \mid y)\, dx; & \text{for continuous} \end{cases}$$

Note that $E(X \mid Y = y)$ is a function of Y.

Covariance

Let X and Y be a random variables with joint probability distribution $f(x,y)$ the covariance of X and Y which denoted by $\mathrm{Cov}(X,Y)$) or σ_{XY} is :

$$E(X-\mu_X)(Y-\mu_Y) = \begin{cases} \sum_{all\ x}\sum_{all\ y}(X-\mu_X)(Y-\mu_Y)\, f(x,y)\,; & \text{for discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X-\mu_X)(Y-\mu_Y)\, f(x,y)\, dx\, dy; & \text{for continuous} \end{cases}$$

The alternative and preferred formula for σ_{XY} is:

$$E(XY) - \mu_X\mu_Y$$

Linear combination

Let X and Y be a random variables with joint probability distribution $f(x,y)$, a and b are constants, then

$$\mathrm{Var}(aX \pm \mathrm{b}Y) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) \pm 2ab\mathrm{Cov}(X,Y)$$

If X and Y are independent random variables, then

$$\mathrm{Var}(aX \pm \mathrm{b}Y) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

Correlation coefficient

Let X and Y be two random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y respectively. The correlation coefficient of X and Y is

$$\rho_{\mathrm{XY}} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$$

2: Definition of Stochastic Process

Definition

A stochastic process *(random process)* is a family of random variables, $\{X(t), t \in T\}$ or $\{X_t, t \in T\}$ That is, for each t in the index set T, $X(t)$ is a random variable.

Random process also defined as a random variable which a function of time t, that means , $X(t)$ is a random variable for every time instant t or it's a random variable indexed by time.

We know that a random variable is a function defined on the sample space Ω. Thus a random process $\{X(t), t \in T\}$ is a real function of two arguments $\{X(t, \omega), t \in T,\ \omega \in \Omega\}$. For fixed $t(= t_k)$, $X(t_k, \omega) = X_k(\omega)$ is a random variable denoted by $X(t_k)$, as ω varies over the sample space Ω. On the other hand for fixed sample space $\omega_h \in \Omega$, $X(t, \omega_h) = X_h(t)$ is a single function of time t, called a sample function or a *realization* of the process. The totality of all sample functions is called an *ensemble*.

If both ω and t are fixed, $X(t_k, \omega_h)$ is a real number. We used the notation $X(t)$ to represent$X(t, \omega)$.

Description of a Random Process

In a random process $\{X(t), t \in T\}$ the index t called the *time-parameter* (or simply the time) *and* $T \in \mathbb{R}$ called the parameter set of the random process. Each $X(t)$ takes values in some set $S \in \mathbb{R}$ called the *state space*; then $X(t)$ is the state of the process at time *t*, and if $X(t) = i$ we said the process in state i at time t .

Definition:-

$\{X(t), t \in T\}$ is a discrete - time (discrete parameter) process if the index set T of the random process is discrete . A discrete-parameter process is also called a random sequence and is denoted by $\{X(n), n = 1,2, \ldots\}$ or $\{X_n, n = 1,2, \ldots\}$.

In practical this generally means T={1,2,3,…….}.

Thus a discrete-time process is $\{X(0), X(1),\ X(2), \ldots\}$: a new random number recorded at every time 0, 1, 2, 3, . . .

Definition:-

$\{X(t), t \in T\}$ is continuous - time (continuous parameter) process if the index set T is continuous .

In practical this generally means T = [0, ∞), or T = [0,K] for some K.

Thus a continuous-time process $\{X(t), t \in T\}$ has a random number $X(t)$ recorded at every instant in time.

(Note that $X(t)$ needs not change at every instant in time, but it is allowed to change at any time; i.e. not just at t = 0, 1, 2, . . . , like a discrete-time process.)

Definition:-

The state space, S: is the set of real values that $X(t)$ can take.

Every $X(t)$ takes a value in $\mathbb{R}$, but S will often be a smaller set: $S \subset \mathbb{R}$. For example, if $X(t)$ is the outcome of a coin tossed at time *t*, then the state space is S = {0, 1}.

Definition:-

The state space S is called a discrete-state process if it is discrete , often referred to as a *chain*. In this case, the state space S is often assumed to be $\{0,1,2,\ldots\}$If the state space S is continuous then we have a continuous-state process.

Examples:

Discrete-time, discrete-state processes

Example 1: Tossing a balanced die more than once, if we interest on the number on the uppermost face at toss n , say $X(1)$ the number appears on the first toss , $X(2)$ number appears in the second one ,…………….. ect, then $\{X(n), n \in T\}$ is the random process, and the random variable $X(n)$ denotes the number appears at toss n . where n is the parameter. $T = \{1,2,3,\ldots\}$ and $S = \{1,2,3,4,5,6\}$.

*Example*2:The number of emails in your inbox at time *t* .$T = \{1,2,3,\ldots\}$ and $S = \{0,1,2,\ldots\}$.

Example 3: your bank balance on day *t* .

Example 4: the number of occupied channels in a telephone link at the arrival time of the n^{th} customer, n = 1,2,...

Continuous-time, discrete-state processes

Example 6: The number of occupied channels in a telephone link at time $t > 0$

Example 7: The number of packets in the buffer of a statistical multiplexer at time $t > 0$

3: Characterization of Stochastic Process

Distribution function CDF and Probability distribution PDF for (t) :

Consider the stochastic process $\{X(t), t \in T\}$, for any $t_0 \in T$, $X(t_0) = X$ is a random variable, and it's a CDF $F_{X(t_0)}(x)$ or $F_X(x; t_0)$ is defined as:

$$F_X(x; t_0) \quad = P(X(t_0) \leq x)$$

$F_X(x; t_0)$is known as a *first - order distribution function* of the random process $X(t)$.

Similarly, Given t_1 and t_2 , $X(t_1) = X_1$ and $X(t_2) = X_2$ represent two random variables their joint CDF $F_{X(t_1)X(t_2)}(x_1, x_2)$ or $F_X(x_1, x_2; t_1, t_2)$ is given by

$$F_X(x_1, x_2; t_1, t_2) \quad = P(X(t_1) \leq x_1, X(t_2) \leq x_2)$$

$F_X(x_1, x_2; t_1, t_2)$ is known as the *second - order distribution* of $X(t)$.

In general we define the *nth-order distribution function* of $X(t)$ by

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) \quad = P(X(t_1) \leq x_1, X(t_2) \leq x_2, \dots, X(t_n) \leq x_n)$$

Similarly, we can write joint PDFs or PMFs depending on whether $X(t)$ is continuous-valued (the $X(t_i)$'s are continuous random variables) or discrete-valued (the $X(t_i)$'s are discrete random variables). For example the *second – order* PDF and PMF given respectively by

$$f_X(x_1, x_2; t_1, t_2) \ = P(X(t_1) = x_1, X(t_2) = x_2) = \frac{\partial^2 F_{X(t_1)X(t_2)}(x_1,x_2)}{\partial x_1 \partial x_2}$$

$$P_X(x_1, x_2; t_1, t_2) \ = P(X(t_1) = x_1, X(t_2) = x_2)$$

Mean and Variance functions of random process:

As in the case of r.v.'s, random processes are often described by using statistical averages.

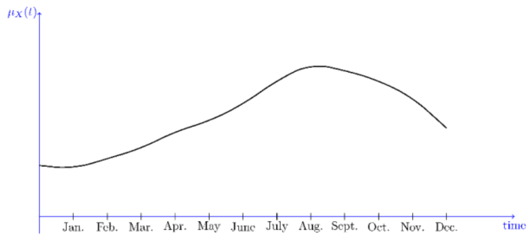
For the random process $\{X(t), t \in T\}$, the *mean function* $\mu_X(t): T \to \mathbb{R}$ is defined as

$$\mu_X(t) = E[X(t)] = \int x f_X(t)\, dx$$

The above definition is valid for both continuous-time and discrete-time random processes. In particular, if $\{X(n), n \in T\}$ is a discrete-time random process, then

$$\mu_X(n) = E[X(n)] \quad \forall \ n \in \mathbb{R}$$

The mean function gives us an idea about how the random process behaves on average as time evolves (a function of time). For example, if $X(t)$ is the temperature in a certain city, the mean function $\mu_X(t)$ might look like the function shown in Figure below. As we see, the expected value of $X(t)$ is lowest in the winter and highest in summer.



The variance of a random process $X(t)$, also a function of time, given by:

$$\sigma_X^2(t) = Var[X(t)] = E[X(t) - \mu_X(t)]^2 = E[X_t]^2 - [\mu_X(t)]^2$$

Autocorrelation, and Covariance Functions:

The mean function $\mu_X(t)$ gives us the expected value of $X(t)$ at time t, but it does not give us any information about how $X(t_1)$ and $X(t_2)$ are related. To get some insight on the relation between $X(t_1)$ and $X(t_2)$, we define correlation and covariance functions.

Given two random variables $X(t_1)$, $X(t_2)$ the *autocorrelation function* or simply *correlation function* $R_{XX}(t_1, t_2)$, defined by:

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\, x_2\, f_X(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

Where $f_X(x_1, x_2; t_1, t_2)$ is a joint probability function for t_1 and t_2.

For a random process, t_1 and t_2 go through all possible values, and therefore, $E[X(t_1)X(t_2)]$ can change and is a function of t_1 and t_2.

Note that:

$$R_{XX}(t_1, t_2) = R_{XX}(t_2, t_1)$$

The *autocovariance function* of $X(t)$ is defined by:

$$C_{XX}(t_1, t_2) = \mathrm{Cov}[X(t_1), X(t_2)] = E\big[\big(X(t_1) - \mu_X(t_1)\big)\big(X(t_2) - \mu_X(t_2)\big)\big]$$

$$= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \quad \text{for} \quad \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{T}$$

It is clear that if the mean of $X(t)$ is zero, then $C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2)$.

If $t_1 = t_2 = t$ we obtain

$$R_{XX}(t_1, t_2) = R_{XX}(t, t) = E[X(t)X(t)] = E[X(t)]^2$$

$$C_{XX}(t_1, t_2) = C_{XX}(t, t) = \mathrm{Cov}[X(t), X(t)]$$

$$= Var\big(X(t)\big) \quad \text{for t} \in \mathrm{T}$$

The normalized autocovariance function is defined by:

$$\rho(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)C_{XX}(t_2, t_2)}}$$

Example: wireless signal model

Consider RP $X(t) = \alpha\cos(2\pi F_c(t) + \theta)$

Where α : amplitude (capacity) $F_c(t)$: carrier frequency θ: phase

$\theta \sim U(-\pi, \pi)$ that is $f_\theta(\theta) = \frac{1}{2\pi}$ if $-\pi \le \theta \le \pi$

α, F_c: are constant $F_c(t)$: function of a time

Find

i. Mean function of $X(t)$
ii. Autocorrelation function of $X(t)$

Solution

i. Mean function of $\boldsymbol{X(t)}$

$$\mu_X(t) = E[X(t)] = E\{\alpha\cos(2\pi F_c(t) + \theta)\} = \int \alpha\cos(2\pi F_c(t) + \theta)\, f_\theta(\theta) d\theta$$

$$= \int_{-\pi}^{\pi} \alpha\cos(2\pi F_c(t) + \theta)\ 1/2\pi\ d\theta = \frac{a}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi F_c(t) + \theta)\ d\theta$$

$$= \frac{a}{2\pi}\ \sin(2\pi F_c(t) + \theta)\Big|_{-\pi}^{\pi} = \frac{a}{2\pi}\ \{\sin(2\pi F_c(t) + \pi) - \sin(2\pi F_c(t) - \pi)\}$$

$$= \frac{a}{2\pi} \times \{0\} = 0$$

$$\Longrightarrow \mu_X(t) = 0$$

ii. Autocorrelation function of $X(t)$

$$R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

Let $t_1 = t$ $t_2 = t + \tau$ and $\tau \in \mathbb{R}$ $\tau = t_2 - t_1$ time shift

$$R_{XX}(t, t+\tau) = E\{[\alpha\cos(2\pi F_c(t) + \theta)][\alpha\cos(2\pi F_c(t+\tau) + \theta)]\}$$

$$= \alpha^2 E\{[\cos(2\pi F_c(t) + \theta)][\cos(2\pi F_c(t+\tau) + \theta)]\}$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

Let $\alpha = 2\pi F_c(t) + \theta$ $\beta = 2\pi F_c(t + \tau) + \theta$ then

$\alpha + \beta = 2\pi F_c(2t + \tau) + 2\theta$ $\alpha - \beta = 2\pi F_c(\tau)$

$$R_{XX}(t, t+\tau) = \frac{a^2}{2} E\{\cos(2\pi F_c(2t+\tau) + 2\theta) + \cos\big(2\pi F_c(\tau)\big)\}$$

$$= \frac{a^2}{2}\Big(E\{\cos(2\pi F_c(2t+\tau) + 2\theta)\} + E\{\cos\big(2\pi F_c(\tau)\big)\}\Big)$$

The first term is 0 , and $E\{\cos\big(2\pi F_c(\tau)\big)\} = \cos\big(2\pi F_c(\tau)\big)$ is the constant (no θ)

$$R_{XX}(t, t+\tau) = \frac{a^2}{2}\cos\big(2\pi F_c(\tau)\big)$$

Example:

A random process $\{X(t), t \in T\}$ with $\mu_X(t) = 5$ and $R_{XX}(t_1, t_2) = 25 + 3e^{-0.6|t_1 - t_2|}$ Determine the mean, the variance and the covariance of the random variables $U = X(6)$ and $V = X(9)$.

Solution:

$E(U) = E[X(6)] = \mu_X(6) = 5,\; E(V) = E[X(9)] = \mu_X(9) = 5$

$Var(U) = E\{[X(6)]^2\} - \{E[X(6)]\}^2$

since $R_{XX}(t_1, t_1) = E\{[X(t_1)]^2\}$

$$Var(U) = R_{XX}(t_1, t_1) - \{\mu_X(6)\}^2 = R_{XX}(6,6) - 25$$

$$= 25 + 3e^{-0.6|6-6|} - 25 = 28 - 25 = 3$$

Similarly,

$$Var(V) = R_{XX}(t_1, t_1) - \{\mu_X(9)\}^2 = R_{XX}(9,9) - 25$$

$$= 25 + 3e^{-0.6|9-9|} - 25 = 28 - 25 = 3$$

$$\mathrm{Cov}[X(t_1), X(t_2)] = C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

$\mathrm{Cov}(U, V) = C_{XX}(6,9) = R_{XX}(6,9) - \mu_X(6)\mu_X(9)$

Since, $R_{XX}(6,9) = 25 + 3e^{-0.6|6-9|} = 25 + 3e^{-1.8} = 25.496$

$$\mathrm{Cov}(U, V) = 25.496 - 25 = 0.496$$

4: Classification of Stochastic Processes

We can classify random processes based on many different criteria.

Stationary and Wide-Sense Stationary Random Processes

A. *Stationary Processes:*

A random process $\{X(t), t \in T\}$ is *stationary* or *strict-sense stationary* SSS if its statistical properties do not change by time. For example, for stationary process, $X(t)$ and $X(t+\Delta)$ have the same probability distributions. In particular, we have

$$F_X(x,t) = F_X(x; t+\Delta) \qquad \forall \quad t, t+\Delta \in T$$

More generally, for stationary process a random $\{X(t), t \in T\}$, the joint distributions of the two random variables $X(t_1)$, $X(t_2)$ is the same as the joint distribution of $X(t_1+\Delta)$, $X(t_2+\Delta)$, for example, if you have stationary process $X(t)$, then

$$P\big[\big(X(t_1), X(t_2)\big) \in A\big] = P\big[\big(X(t_1+\Delta), X(t_2+\Delta)\big) \in A\big]$$

For any set of $A \in \mathbb{R}^2$.

In short, a random process is stationary if a time shift does not change its statistical properties.

<u>*Definition*</u>. A discrete-time random process $\{X(n), n \in \mathbb{Z}\}$ is strict-sense stationary or simply stationary if, for all $n_1, n_2, \dots, n_r \in \mathbb{N}$ and all $D \in \mathbb{Z}$, the joint CDF of

$$X(n_1), X(n_2), \dots\dots, X(n_r)$$

Is the same CDF as

$$X(n_1+D), X(n_2+D), \dots, X(n_r+D)$$

That is, for real numbers $x_1, x_2, \dots, x_r$ we have

$$F_X(x_1, x_2, \dots, x_r; t_1, t_2, \dots, t_r) = F_X(x_1, x_2, \dots, x_r; t_1+D, t_2+D, \dots, t_r+D)$$

This can be written as

$$F_{X(t_1),X(t_2),\dots,X(t_r)}(x_1, x_2, \dots, x_r) = F_{X(t_1+D),X(t_2+D),\dots,X(t_r+D)}(x_1, x_2, \dots, x_r)$$

<u>*Definition.*</u> A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ is strict-sense stationary or simply stationary if, for all $t_1, t_2, \dots, t_r \in \mathbb{R}$ and all $\Delta \in \mathbb{R}$, the joint CDF of

$$X(t_1), X(t_2), \dots\dots, X(t_r)$$

Is the same CDF as

$$X(t_1+\Delta), X(t_2+\Delta), \dots, X(t_r+\Delta)$$

That is, for real numbers $x_1, x_2, \dots, x_r$ we have

$$F_X(x_1, x_2, \dots, x_r; t_1, t_2, \dots, t_r) = F_X(x_1, x_2, \dots, x_r; t_1 + \Delta, t_2 + \Delta, \dots, t_r + \Delta)$$

This can be written as

$$F_{X(t_1),X(t_2),\dots,X(t_r)}(x_1, x_2, \dots, x_r) = F_{X(t_1+\Delta),X(t_2+\Delta),\dots,X(t_r+\Delta)}(x_1, x_2, \dots, x_r)$$

B. *Wide-Sense Stationary Processes :*

A random process is called *weak-sense stationary* or *wide-sense stationary* (WSS) if its mean function and its autocorrelation function do not change by shifts in time. More precisely, $X(t)$ is WSS if, for all $t_1, t_2 \in \mathbb{R}$,

1. $E[X(t_1)] = E[X(t_2)] = \mu_X$ constant (stationary mean in time)

For $t_1 = t$ $t_2 = t + \tau$ and $\tau \in \mathbb{R}$ $\tau = t_2 - t_1$ time shift

2. $R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$

Note that the first condition states that the mean function $\mu_X(t)$ is not a function of time t, thus we can write $\mu_X(t) = \mu_X$. The second condition states that the correlation function $R_{XX}(t, t + \tau)$ is only a function of time shift τ and not on specific times t_1, t_2.

Definition

A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ *weak-sense stationary* or *wide-sense stationary* (WSS) if

1. $\mu_X(t) = \mu_X \quad \forall\ t \in \mathbb{R}$
2. $R_{XX}(t, t + \tau) = R_{XX}(t_1 - t_2) = R_{XX}(\tau) \quad \forall\ t_1, t_2 \in \mathbb{R}$

Definition

A discrete-time random process $\{X(n), n \in \mathbb{Z}\}$ *weak-sense stationary* or *wide-sense stationary* (WSS) if

1. $\mu_X(n) = \mu_X \quad \forall\ n \in \mathbb{Z}$
2. $R_{XX}(n_1, n_2) = R_X(n_1 - n_2) \quad \forall\ n_1, n_2 \in \mathbb{Z}$

Note that a strict-sense stationary process is also a *WSS* process, but in general, the converse is not true.

Example: wireless signal model

Consider RP $X(t) = \alpha \cos(2\pi F_c(t) + \theta)$

Where α : amplitude (capacity) $F_c(t)$: carrier frequency θ: phase

$\theta \sim U(-\pi, \pi)$ that is $f_\theta(\theta) = \frac{1}{2\pi}$ if $-\pi \le \theta \le \pi$

α, F_c: are constant $F_c(t)$: function of a time

Show that $X(t)$ is WSS.

Solution

The Mean function of $\boldsymbol{X(t)}$ is $\mu_X(t) = 0$ constant

The autocorrelation function $R_{XX}(t, t+\tau) = \frac{a^2}{2}\cos\big(2\pi F_c(\tau)\big)$ function of τ

Since $\mu_X(t)$ is a constant doesn't depend on time and the $R_{XX}(t, t+\tau)$ depends only on time shift τ, therefore, $X(t)$ is WSS.

Example:

Consider RP $X(t) = A\sin(\omega_c(t) + \theta)$ A is a r.v. with mean μ_A and variance σ_A^2 , $\theta \sim U(-\pi, \pi)$. A and θ are independent. Find

i. Mean function of $X(t)$
ii. Autocorrelation function of $X(t)$ and
iii. Show that $X(t)$ is WSS

Solution:

i. Mean function:

$$\mu_X(t) = E[X(t)] = E\{A\sin(\omega_c(t) + \theta)\} = E\{A\}\, E\{\sin(\omega_c(t) + \theta)\} \quad A, \theta \text{ Independent}$$

$$= \mu_A \int \sin(\omega_c(t) + \theta)\, f(\theta) d\theta = \mu_A \int_{-\pi}^{\pi} \sin(\omega_c(t) + \theta)\ 1/2\pi\ d\theta$$

$$= -\frac{\mu_A}{2\pi}\cos(\omega_c(t) + \theta)\Big|_{-\pi}^{\pi}$$

$$= -\frac{\mu_A}{2\pi}\left[\cos(\omega_c(t) + \pi) - \cos(\omega_c(t) - \pi)\right]$$

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Let $u = \omega_c(t) + \pi$ $\quad v = u = \omega_c(t) - \pi$ $\quad u + v = 2\omega_c(t)$ $\quad u - v = 2\pi$

$$\cos(\omega_c(t) + \pi) - \cos(\omega_c(t) - \pi) = -2\sin\big(\omega_c(t)\big)\sin(\pi)$$

$$-\frac{\mu_A}{2\pi}\left[\cos(\omega_c(t) + \pi) - \cos(\omega_c(t) - \pi)\right] = \frac{2\mu_A}{2\pi}\sin\big(\omega_c(t)\big)\sin(\pi)$$

Since $\sin(\pi) = 0$ therefore

$$\frac{\mu_A}{\pi}\sin\big(\omega_c(t)\big)\sin(\pi) = 0$$

$$\mu_X(t) = 0$$

ii. Correlation function:

$$R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{A^2\sin(\omega_c(t_1) + \theta)\sin(\omega_c(t_2) + \theta)\}$$

Because A, θ Independent

$$E\{A^2\sin(\omega_c(t_1) + \theta)\sin(\omega_c(t_2) + \theta)\} = E\{A^2\}E\{\sin(\omega_c(t_1) + \theta)\sin(\omega_c(t_2) + \theta)\}$$

$$sin(\alpha)\,sin(\beta) = \frac{1}{2}[cos(\alpha-\beta) - cos(\alpha+\beta)]$$

Let $\alpha = \omega_c(t_1) + \theta$ $\beta = \omega_c(t_2) + \theta$ then

$$\alpha - \beta = \omega_c(t_1 - t_2) \qquad \alpha + \beta = \omega_c(t_1 + t_2) + 2\theta$$

Therefore

$$E\{A^2\}E\{\sin(\omega_c(t_1)+\theta)\sin(\omega_c(t_2)+\theta)\}$$

$$= E\{A^2\}\,E\left[\frac{1}{2}\cos\big(\omega_c(t_1-t_2)\big) - \frac{1}{2}\cos(\omega_c(t_1+t_2)+2\theta)\right]$$

$$= E\{A^2\}\,\left[\frac{1}{2}E\big\{\cos\big(\omega_c(t_1-t_2)\big)\big\} - \frac{1}{2}E\{\cos(\omega_c(t_1+t_2)+2\theta)\}\right]$$

The second term is zero.

$$R_{XX}(t_1,t_2) = \frac{1}{2}\,E\{A^2\}\cos\big(\omega_c(t_1-t_2)\big) = \frac{\mu_A}{2}\,\cos\big(\omega_c(\tau)\big)$$

$X(t)$ is WSS random process because the mean function is a constant (=0) and the autocorrelation function is only a function of a time difference $t_1 - t_2$.

Independent and independent identically distributed iid Random Processes

A. *Independent Processes:*

In a random process $X(t)$, if $X(t_i)$ for $i = 1,2,\dots,n$ are independent r.v.'s, so that for $n = 1,2,\dots$,

$$f_X(x_1,x_2,\dots,x_n;t_1,t_2,\dots,t_n) = \prod_{i=1}^{n} f_X(x_i;t_i)$$

and $$F_X(x_1,x_2,\dots,x_n;t_1,t_2,\dots,t_n) = \prod_{i=1}^{n} F_X(x_i;t_i)$$

Or $$P(X(t_1)\le x_1, X(t_2)\le x_2,\dots,X(t_n)\le x_n)$$

$$= P(X(t_1)\le x_1).P(X(t_2)\le x_2),\dots P(X(t_n)\le x_n)$$

then we call $X(t)$ an independent random process. Thus, a first-order distribution is sufficient to characterize an independent random process $X(t)$.

B. *Independent and identically distributed iid random process*

A Random process $\{X(t), t \in T\}$ is said to be independent and identically distributed (iid) if any finite number, say *k*, of random variables $X(t_1)$, $X(t_2)$, ……, $X(t_k)$ are mutually independent and have a common cumulative distribution function $F_X(.)$. The joint cdf and pdf for $X(t_1)$, $X(t_2)$, ……, $X(t_k)$ are given respectively by:

$$F_X(x_1, x_2, \dots, x_k; t_1, t_2, \dots, t_k) = \prod_{i=1}^{k} F_X(x_i; t_i)$$

$$f_X(x_1, x_2, \dots, x_k; t_1, t_2, \dots, t_k) = \prod_{i=1}^{k} f_X(x_i; t_i)$$

Example.

Consider the random process $\{X_n,\ n = 0,1,2, \dots\}$ in which $X_i's$ are iid standard normal random variables.

(a) Write down $f_{x_n}(x)$ for $n = 0,1,2, \dots$

(b) Write down $f_{x_n,x_m}(x_1, x_2)$ for $m \neq n$

Solution.

(a) Since $X_n \sim N(0,1)$, we have

$$f_{x_n}(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2} \qquad \forall\ x \in \mathbb{R}$$

(b) If $m \neq n$, then x_n and x_m are independent (because of the i.i.d. assumption) so,

$$\begin{aligned} f_{x_n,x_m}(x_1, x_2) &= f_{x_n}(x_1) f_{x_m}(x_2) \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x_1^2} \ \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x_2^2} \\ &= \frac{1}{2\pi} e^{\frac{1}{2}(x_1^2 + x_2^2)} \qquad \forall\ x_1, x_2 \in \mathbb{R} \end{aligned}$$

Solved Problems (1)

Problem 1

Let $Y_1, Y_2, ...$ be a sequence of iid random variables with mean $E[Y_i] = 0$ and $\text{var}[Y_i] = 4$. Define the discrete time random process $\{X_n,\ n \in N\}$ as

$$X_n = Y_1 + Y_2, ... + Y_n \qquad\qquad \forall\, n \in N$$

(a) Find the mean function X_n

(b) Find the autocorrelation function and covariance function for X_n

Problem 2

Consider the random process $X_n = 1000(1 + R)^n$, for $n = 0,1,2, ...$ $R \sim U(0.04, 0.05)$.

(a) Find the mean function X_n

(b) Find the autocorrelation function and covariance function for X_n

Problem 3

Consider the random process $\{X(t), t \in \mathbb{R}\}$ defined as

$$X(t) = Cos(t + U)\}$$

where $U \sim Uniform(0,2\pi)$. Show that $X(t)$ is a WSS process.

Problem 4

Given a random process $\{X(t), t \in T\}$ with $\mu_X(t) = 4$ and $R_X(t_1, t_2) = 20 + 2e^{-0.4|t_1 - t_2|}$. Suppose $Y_1 = 2X(3)$ and $Y_2 = X(6)$. Find:

(a) $\mathrm{E}(Y_1)$.

(b) $\mathrm{Var}(Y_2)$.

(c) $\mathrm{Cov}(Y_1, Y_2)$

Solutions (1)

Problem1 (Solution)

(a)
$$\mu_n = E[X_n]$$
$$= E[Y_1 + Y_2, \dots . + Y_n]$$
$$= E[Y_1] + E[Y_2] + \cdots + E[Y_n]$$
$$= 0$$

(b) Let $m \leq n$
$$R_{XX}(m,n) = E[X_m X_n]$$
$$= E[(Y_1 + Y_2, \dots . + Y_m)(Y_1 + Y_2, \dots . + Y_n)]$$
$$= E[Y_1^2] + E[Y_2^2] + \cdots + E[Y_m^2]$$

since $E[Y_i Y_j] = E[Y_i]E[Y_j] = 0$ then $E[Y_1^2] = \mathrm{var}[Y]$

$R_{XX}(m,n) = \mathrm{var}[Y_1] + \mathrm{var}[Y_2] + \cdots + \mathrm{var}[Y_m]$

$= 4 + 4 + \cdots + 4 = 4m$

Similarly for $m \geq n$
$$R_{XX}(m,n) = E[X_m X_n]$$
$$= E[(Y_1 + Y_2, \dots . + Y_m)(Y_1 + Y_2, \dots . + Y_n)]$$
$$= E[Y_1^2] + E[Y_2^2] + \cdots + E[Y_n^2]$$
$$= \mathrm{var}[Y_1] + \mathrm{var}[Y_2] + \cdots + \mathrm{var}[Y_n]$$
$$= 4\mathrm{n}$$

Problem2 (Solution)

(a) Let $Y = 1 + R$ so, $Y \sim U(1.04, 1.05)$.
$$\mu_n = E[X_n]$$
$$= 1000\, E[Y^n]$$
$$= 1000 \int_{1.04}^{1.05} y^n \frac{1}{0.01}\, dy$$
$$= 100000 \int_{1.04}^{1.05} y^n\, dy$$
$$= \frac{10^5}{n+1} [y^{n+1}]_{1.04}^{1.05}$$
$$= \frac{10^5}{n+1} [(1.05)^{n+1} - (1.04)^{n+1}] \qquad \forall\, n \in \{0,1,2, \dots\}$$

(b) $R_{XX}(m,n) = E[X_m X_n]$

$$= 10^6\, E[Y^m Y^n]$$

$$= 10^6\, E[Y^{m+n}]$$

$$= 10^8 \int_{1.04}^{1.05} y^{m+n}\, dy$$

$$= \frac{10^8}{n+m+1} [y^{m+n+1}]_{1.04}^{1.05}$$

$$= \frac{10^8}{n+m+1} [(1.05)^{m+n+1} - (1.04)^{m+n+1}] \qquad \forall \;\; m,n \in \{0,1,2,\dots\}$$

To find covariance function

$$C_{XX}(m,n) = R_{XX}(m,n) - E[X_m]E[X_n]$$

$$= \frac{10^8}{n+m+1} [(1.05)^{m+n+1} - (1.04)^{m+n+1}]$$

$$- \frac{10^{10}}{(m+1)(n+1)} [(1.05)^{m+1} - (1.04)^{m+1}][(1.05)^{n+1} - (1.04)^{n+1}]$$

Problem 3 (Solution)

We need to check two conditions

1. $\mu_X(t) = \mu_X \qquad \forall \;\; t \in \mathbb{R}$ and
2. $R_{XX}(t_1,t_2) = R_{XX}(t_1 - t_2) \quad \forall \;\; t_1, t_2 \in \mathbb{R}$

We have

$$\mu_n = E[X_n]$$

$$= E[\cos(t+U)]$$

$$= \int_0^{2\pi} \cos(t+u) \frac{1}{2\pi}\, du$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(t+u)\, du$$

$$= 0 \qquad \forall\, t \in \mathbb{R}$$

we can also find $R_{XX}(t_1,t_2)$

$$R_{XX}(t_1,t_2) = E[X_1 X_2]$$

$$= E[\cos(t_1+U)\cos(t_2+U)]$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

Let $\quad \alpha = t_1 + U \qquad \beta = t_2 + U \quad$ then

$$\alpha + \beta = t_1 + t_2 + 2U \qquad \alpha - \beta = t_1 - t_2$$