Lesson1: Definition of Stochastic Process

Definition

A stochastic process *(random process)* is a family of random variables, $\{X(t), t \in T\}$ or $\{X_t, t \in T\}$ That is, for each *t* in the index set *T*, $X(t)$ is a random variable.

Random process also defined as a random variable which a function of time t, that means , $X(t)$ is a random variable for every time instant t or it's a random variable indexed by time.

We know that a random variable is a function defined on the sample space Ω. Thus a random process $\{X(t), t \in T\}$ is a real function of two arguments $\{X(t, \omega), t \in T,\ \omega \in \Omega\}$. For fixed $t(= t_k)$, $X(t_k, \omega) = X_k(\omega)$ is a random variable denoted by $X(t_k)$, as ω varies over the sample space Ω. On the other hand for fixed sample space $\omega_h \in \Omega$, $X(t, \omega_h) = X_h(t)$ is a single function of time t, called a sample function or a *realization* of the process. The totality of all sample functions is called an *ensemble*.

If both ω and t are fixed, $X(t_k, \omega_h)$ is a real number. We used the notation $X(t)$ to represent$X(t, \omega)$.

Description of a Random Process

In a random process $\{X(t), t \in T\}$ the index t called the *time-parameter* (or simply the time) *and* $T \in \mathbb{R}$ called the parameter set of the random process. Each $X(t)$ takes values in some set $S \in \mathbb{R}$ called the *state space*; then $X(t)$ is the state of the process at time *t*, and if $X(t) = i$ we said the process in state i at time t .

Definition:-

$\{X(t), t \in T\}$ is a discrete - time (discrete parameter) process if the index set T of the random process is discrete . A discrete-parameter process is also called a random sequence and is denoted by $\{X(n), n = 1,2, \ldots\}$ or $\{X_n, n = 1,2, \ldots\}$.

In practical this generally means T={1,2,3,…….}.

Thus a discrete-time process is $\{X(0), X(1),\ X(2), \ldots\}$: a new random number recorded at every time 0, 1, 2, 3, . . .

Definition:-

$\{X(t), t \in T\}$ is continuous - time (continuous parameter) process if the index set T is continuous .

In practical this generally means T = [0, ∞), or T = [0,K] for some K.

Thus a continuous-time process $\{X(t), t \in T\}$ has a random number $X(t)$ recorded at every instant in time.

(Note that $X(t)$ needs not change at every instant in time, but it is allowed to change at any time; i.e. not just at t = 0, 1, 2, . . . , like a discrete-time process.)

Definition:-

The state space, S: is the set of real values that $X(t)$ can take.

Every $X(t)$ takes a value in $\mathbb{R}$, but S will often be a smaller set: $S \subset \mathbb{R}$. For example, if $X(t)$ is the outcome of a coin tossed at time *t*, then the state space is S = {0, 1}.

Definition:-

The state space S is called a discrete-state process if it is discrete , often referred to as a *chain*. In this case, the state space S is often assumed to be $\{0,1,2, ...\}$If the state space S is continuous then we have a continuous-state process.

Examples:

Discrete-time, discrete-state processes

Example 1: Tossing a balanced die more than once, if we interest on the number on the uppermost face at toss n , say $X(1)$ the number appears on the first toss , $X(2)$ number appears in the second one ,…………….. ect, then $\{X(n), n \in T\}$ is the random process, and the random variable $X(n)$ denotes the number appears at toss n . where n is the parameter. $T = \{1,2,3, ...\}$ and $S = \{1,2,3,4,5,6\}$.

*Example*2:The number of emails in your inbox at time t .$T = \{1,2,3, ...\}$ and $S = \{0,1,2, ...\}$.

Example 3: your bank balance on day *t* .

Example 4: the number of occupied channels in a telephone link at the arrival time of the n^{th} customer, n = 1,2,...

Continuous-time, discrete-state processes

Example 6: The number of occupied channels in a telephone link at time $t > 0$

Example 7: The number of packets in the buffer of a statistical multiplexer at time $t > 0$

Lesson 2: Characterization of Stochastic Process

Distribution function CDF and Probability distribution PDF for (t) :

Consider the stochastic process $\{X(t), t \in T\}$, for any $t_0 \in T$, $X(t_0) = X$ is a random variable, and it's a CDF $F_{X(t_0)}(x)$ or $F_X(x; t_0)$ is defined as:

$$F_X(x; t_0) \quad = P(X(t_0) \leq x)$$

$F_X(x; t_0)$is known as a *first - order distribution function* of the random process $X(t)$.

Similarly, Given t_1 and t_2 , $X(t_1) = X_1$ and $X(t_2) = X_2$ represent two random variables their joint CDF $F_{X(t_1)X(t_2)}(x_1, x_2)$ or $F_X(x_1, x_2; t_1, t_2)$ is given by

$$F_X(x_1, x_2; t_1, t_2) \quad = P(X(t_1) \leq x_1, X(t_2) \leq x_2)$$

$F_X(x_1, x_2; t_1, t_2)$ is known as the *second - order distribution* of $X(t)$.

In general we define the *nth-order distribution function* of $X(t)$ by

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) \;\; = P(X(t_1) \leq x_1, X(t_2) \leq x_2, \dots, X(t_n) \leq x_n)$$

Similarly, we can write joint PDFs or PMFs depending on whether $X(t)$ is continuous-valued (the $X(t_i)$'s are continuous random variables) or discrete-valued (the $X(t_i)$'s are discrete random variables). For example the *second – order* PDF and PMF given respectively by

$$f_X(x_1, x_2; t_1, t_2) \;\; = P(X(t_1) = x_1, X(t_2) = x_2) = \frac{\partial^2 F_{X(t_1)X(t_2)}(x_1, x_2)}{\partial x_1 \partial x_2}$$

$$P_X(x_1, x_2; t_1, t_2) \;\; = P(X(t_1) = x_1, X(t_2) = x_2)$$

Mean and Variance functions of random process:

As in the case of r.v.'s, random processes are often described by using statistical averages.

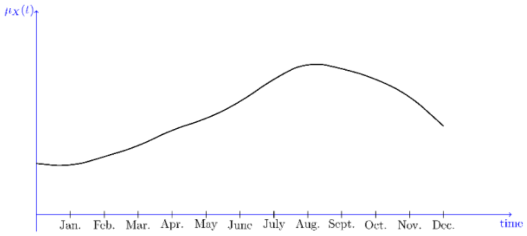
For the random process $\{X(t), t \in T\}$, the *mean function* $\mu_X(t): T \to \mathbb{R}$ is defined as

$$\mu_X(t) = E[X(t)] = \int x f_X(t)\, dx$$

The above definition is valid for both continuous-time and discrete-time random processes. In particular, if $\{X(n), n \in T\}$ is a discrete-time random process, then

$$\mu_X(n) = E[X(n)] \quad \forall \;\, n \in \mathbb{R}$$

The mean function gives us an idea about how the random process behaves on average as time evolves (a function of time). For example, if $X(t)$ is the temperature in a certain city, the mean function $\mu_X(t)$ might look like the function shown in Figure below. As we see, the expected value of $X(t)$ is lowest in the winter and highest in summer.



The variance of a random process $X(t)$, also a function of time, given by:

$$\sigma_X^2(t) = Var[X(t)] = E[X(t) - \mu_X(t)]^2 = E[X_t]^2 - [\mu_X(t)]^2$$

Autocorrelation, and Covariance Functions:

The mean function $\mu_X(t)$ gives us the expected value of $X(t)$ at time t, but it does not give us any information about how $X(t_1)$ and $X(t_2)$ are related. To get some insight on the relation between $X(t_1)$ and $X(t_2)$, we define correlation and covariance functions.

Given two random variables $X(t_1)$, $X(t_2)$ the *autocorrelation function* or simply *correlation function* $R_{XX}(t_1, t_2)$, defined by:

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\, x_2\, f_X(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

Where $f_X(x_1, x_2; t_1, t_2)$ is a joint probability function for t_1 and t_2.

For a random process, t_1 and t_2 go through all possible values, and therefore, $E[X(t_1)X(t_2)]$ can change and is a function of t_1 and t_2.

Note that:

$$R_{XX}(t_1, t_2) = R_{XX}(t_2, t_1)$$

The *autocovariance function* of $X(t)$ is defined by:

$$C_{XX}(t_1, t_2) = \mathrm{Cov}[X(t_1), X(t_2)] = E\big[\big(X(t_1) - \mu_X(t_1)\big)\big(X(t_2) - \mu_X(t_2)\big)\big]$$

$$= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \quad \text{for} \quad \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{T}$$

It is clear that if the mean of $X(t)$ is zero, then $C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2)$.

If $t_1 = t_2 = t$ we obtain

$$R_{XX}(t_1, t_2) = R_{XX}(t, t) = E[X(t)X(t)] = E[X(t)]^2$$

$$C_{XX}(t_1, t_2) = C_{XX}(t, t) = \mathrm{Cov}[X(t), X(t)]$$

$$= Var\big(X(t)\big) \quad \text{for t} \in \mathrm{T}$$

The normalized autocovariance function is defined by:

$$\rho(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)C_{XX}(t_2, t_2)}}$$

Example: wireless signal model

Consider RP $X(t) = \alpha\cos(2\pi F_c(t) + \theta)$

Where α : amplitude (capacity) $F_c(t)$: carrier frequency θ: phase

$\theta \sim U(-\pi, \pi)$ that is $f_\theta(\theta) = \frac{1}{2\pi}$ if $-\pi \le \theta \le \pi$

α, F_c: are constant $F_c(t)$: function of a time

Find

i. Mean function of $X(t)$

ii. Autocorrelation function of $X(t)$

Solution

i. Mean function of $\boldsymbol{X(t)}$

$$\mu_X(t) = E[X(t)] = E\{\alpha\cos(2\pi F_c(t) + \theta)\} = \int \alpha\cos(2\pi F_c(t) + \theta)\, f_\theta(\theta) d\theta$$

$$= \int_{-\pi}^{\pi} \alpha\cos(2\pi F_c(t) + \theta)\ 1/2\pi\, d\theta = \frac{a}{2\pi}\int_{-\pi}^{\pi}\cos(2\pi F_c(t) + \theta)\ d\theta$$

$$= \frac{a}{2\pi}\ \sin(2\pi F_c(t) + \theta)\Big|_{-\pi}^{\pi} = \frac{a}{2\pi}\ \{\sin(2\pi F_c(t) + \pi) - \sin(2\pi F_c(t) - \pi)\}$$

$$= \frac{a}{2\pi} \times \{0\} = 0$$

$$\Longrightarrow \mu_X(t) = 0$$

ii. Autocorrelation function of $X(t)$

$$R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

Let $t_1 = t$ $t_2 = t + \tau$ and $\tau \in \mathbb{R}$ $\tau = t_2 - t_1$ time shift

$$R_{XX}(t, t+\tau) = E\{[\alpha\cos(2\pi F_c(t) + \theta)][\alpha\cos(2\pi F_c(t+\tau) + \theta)]\}$$

$$= \alpha^2 E\{[\cos(2\pi F_c(t) + \theta)][\cos(2\pi F_c(t+\tau) + \theta)]\}$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

Let $\alpha = 2\pi F_c(t) + \theta$ $\beta = 2\pi F_c(t+\tau) + \theta$ then

$\alpha + \beta = 2\pi F_c(2t + \tau) + 2\theta$ $\alpha - \beta = 2\pi F_c(\tau)$

$$R_{XX}(t, t+\tau) = \frac{a^2}{2} E\{\cos(2\pi F_c(2t+\tau) + 2\theta) + \cos(2\pi F_c(\tau))\}$$

$$= \frac{a^2}{2}\left(E\{\cos(2\pi F_c(2t+\tau) + 2\theta)\} + E\{\cos(2\pi F_c(\tau))\}\right)$$

The first term is 0 , and $E\{\cos(2\pi F_c(\tau))\} = \cos(2\pi F_c(\tau))$ is the constant (no θ)

$$R_{XX}(t, t+\tau) = \frac{a^2}{2}\cos(2\pi F_c(\tau))$$

Example:

A random process $\{X(t), t \in T\}$ with $\mu_X(t) = 5$ and $R_{XX}(t_1, t_2) = 25 + 3e^{-0.6|t_1 - t_2|}$ Determine the mean, the variance and the covariance of the random variables $U = X(6)$ and $V = X(9)$.

Solution:

$E(U) = E[X(6)] = \mu_X(6) = 5,\ E(V) = E[X(9)] = \mu_X(9) = 5$

$Var(U) = E\{[X(6)]^2\} - \{E[X(6)]\}^2$

since $R_{XX}(t_1, t_1) = E\{[X(t_1)]^2\}$

$$Var(U) = R_{XX}(t_1, t_1) - \{\mu_X(6)\}^2 = R_{XX}(6,6) - 25$$

$$= 25 + 3e^{-0.6|6-6|} - 25 = 28 - 25 = 3$$

Similarly,

$$Var(V) = R_{XX}(t_1, t_1) - \{\mu_X(9)\}^2 = R_{XX}(9,9) - 25$$

$$= 25 + 3e^{-0.6|9-9|} - 25 = 28 - 25 = 3$$

$$\text{Cov}[X(t_1), X(t_2)] = C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

$\text{Cov}(U, V) = C_{XX}(6,9) = R_{XX}(6,9) - \mu_X(6)\mu_X(9)$

Since, $R_{XX}(6,9) = 25 + 3e^{-0.6|6-9|} = 25 + 3e^{-1.8} = 25.496$

$$\text{Cov}(U, V) = 25.496 - 25 = 0.496$$

Lesson 3: Classification of Stochastic Processes

We can classify random processes based on many different criteria.

Stationary and Wide-Sense Stationary Random Processes

A. *Stationary Processes:*

A random process $\{X(t), t \in T\}$ is *stationary* or *strict-sense stationary* SSS if its statistical properties do not change by time. For example, for stationary process, $X(t)$ and $X(t+\Delta)$ have the same probability distributions. In particular, we have

$$F_X(x, t) = F_X(x; t+\Delta) \qquad \forall \quad t, t+\Delta \in T$$

More generally, for stationary process a random $\{X(t), t \in T\}$, the joint distributions of the two random variables $X(t_1)$, $X(t_2)$ is the same as the joint distribution of $X(t_1+\Delta)$, $X(t_2+\Delta)$, for example, if you have stationary process $X(t)$, then

$$P\big[\big(X(t_1), X(t_2)\big) \in A\big] = P\big[\big(X(t_1+\Delta), X(t_2+\Delta)\big) \in A\big]$$

For any set of $A \in \mathbb{R}^2$.

In short, a random process is stationary if a time shift does not change its statistical properties.

<u>Definition</u>. A discrete-time random process $\{X(n), n \in \mathbb{Z}\}$ is strict-sense stationary or simply stationary if, for all $n_1, n_2, \dots, n_r \in \mathbb{N}$ and all $D \in \mathbb{Z}$, the joint CDF of

$$X(n_1), X(n_2), \dots\dots, X(n_r)$$

Is the same CDF as

$$X(n_1+D), X(n_2+D), \dots, X(n_r+D)$$

That is, for real numbers $x_1, x_2, \dots, x_r$ we have

$$F_X(x_1, x_2, \dots, x_r; t_1, t_2, \dots, t_r) = F_X(x_1, x_2, \dots, x_r; t_1+D, t_2+D, \dots, t_r+D)$$

This can be written as

$$F_{X(t_1), X(t_2), \dots, X(t_r)}(x_1, x_2, \dots, x_r) = F_{X(t_1+D), X(t_2+D), \dots, X(t_r+D)}(x_1, x_2, \dots, x_r)$$

<u>Definition.</u> A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ is strict-sense stationary or simply stationary if, for all $t_1, t_2, \dots, t_r \in \mathbb{R}$ and all $\Delta \in \mathbb{R}$, the joint CDF of

$$X(t_1), X(t_2), \dots\dots, X(t_r)$$

Is the same CDF as

$$X(t_1+\Delta), X(t_2+\Delta), \dots, X(t_r+\Delta)$$

That is, for real numbers $x_1, x_2, \dots, x_r$ we have

$$F_X(x_1, x_2, \dots, x_r; t_1, t_2, \dots, t_r) = F_X(x_1, x_2, \dots, x_r; t_1 + \Delta, t_2 + \Delta, \dots, t_r + \Delta)$$

This can be written as

$$F_{X(t_1),X(t_2),\dots,X(t_r)}(x_1, x_2, \dots, x_r) = F_{X(t_1+\Delta),X(t_2+\Delta),\dots,X(t_r+\Delta)}(x_1, x_2, \dots, x_r)$$

B. *Wide-Sense Stationary Processes :*

A random process is called *weak-sense stationary* or *wide-sense stationary* (WSS) if its mean function and its autocorrelation function do not change by shifts in time. More precisely, $X(t)$ is WSS if, for all $t_1, t_2 \in \mathbb{R}$,

1. $E[X(t_1)] = E[X(t_2)] = \mu_X$ constant (stationary mean in time)

For $t_1 = t$ $t_2 = t + \tau$ and $\tau \in \mathbb{R}$ $\tau = t_2 - t_1$ time shift

2. $R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$

Note that the first condition states that the mean function $\mu_X(t)$ is not a function of time t, thus we can write $\mu_X(t) = \mu_X$. The second condition states that the correlation function $R_{XX}(t, t + \tau)$ is only a function of time shift τ and not on specific times t_1, t_2.

Definition

A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ *weak-sense stationary* or *wide-sense stationary* (WSS) if

1. $\mu_X(t) = \mu_X \quad \forall\ t \in \mathbb{R}$
2. $R_{XX}(t, t + \tau) = R_{XX}(t_1 - t_2) = R_{XX}(\tau) \quad \forall\ t_1, t_2 \in \mathbb{R}$

Definition

A discrete-time random process $\{X(n), n \in \mathbb{Z}\}$ *weak-sense stationary* or *wide-sense stationary* (WSS) if

1. $\mu_X(n) = \mu_X \quad \forall\ n \in \mathbb{Z}$
2. $R_{XX}(n_1, n_2) = R_X(n_1 - n_2) \quad \forall\ n_1, n_2 \in \mathbb{Z}$

Note that a strict-sense stationary process is also a *WSS* process, but in general, the converse is not true.

Example: wireless signal model

Consider RP $X(t) = \alpha \cos(2\pi F_c(t) + \theta)$

Where α : amplitude (capacity) $F_c(t)$: carrier frequency θ: phase

$\theta \sim U(-\pi, \pi)$ that is $f_\theta(\theta) = \frac{1}{2\pi}$ if $-\pi \le \theta \le \pi$

α, F_c: are constant $F_c(t)$: function of a time

Show that $X(t)$ is WSS.

Solution

The Mean function of $\boldsymbol{X(t)}$ is $\mu_X(t) = 0$ constant

The autocorrelation function $R_{XX}(t, t+\tau) = \frac{a^2}{2}\cos\big(2\pi F_c(\tau)\big)$ function of τ

Since $\mu_X(t)$ is a constant doesn't depend on time and the $R_{XX}(t, t+\tau)$ depends only on time shift τ, therefore, $X(t)$ is WSS.

Example:

Consider RP $X(t) = A\sin(\omega_c(t) + \theta)$ A is a r.v. with mean μ_A and variance σ_A^2 , $\theta \sim U(-\pi, \pi)$. A and θ are independent. Find

i. Mean function of $X(t)$
ii. Autocorrelation function of $X(t)$ and
iii. Show that $X(t)$ is WSS

Solution:

i. Mean function:

$$\mu_X(t) = E[X(t)] = E\{A\sin(\omega_c(t) + \theta)\} = E\{A\}\, E\{\sin(\omega_c(t) + \theta)\} \quad A, \theta \text{ Independent}$$

$$= \mu_A \int \sin(\omega_c(t) + \theta)\, f(\theta) d\theta = \mu_A \int_{-\pi}^{\pi} \sin(\omega_c(t) + \theta)\ 1/2\pi\ d\theta$$

$$= -\frac{\mu_A}{2\pi} \cos(\omega_c(t) + \theta) \Big|_{-\pi}^{\pi}$$

$$= -\frac{\mu_A}{2\pi} [\cos(\omega_c(t) + \pi) - \cos(\omega_c(t) - \pi)]$$

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Let $u = \omega_c(t) + \pi$ $\quad v = u = \omega_c(t) - \pi$ $\qquad u + v = 2\omega_c(t)$ $\qquad u - v = 2\pi$

$$\cos(\omega_c(t) + \pi) - \cos(\omega_c(t) - \pi) = -2\sin\big(\omega_c(t)\big)\sin(\pi)$$

$$-\frac{\mu_A}{2\pi} [\cos(\omega_c(t) + \pi) - \cos(\omega_c(t) - \pi)] = \frac{2\mu_A}{2\pi} \sin\big(\omega_c(t)\big)\sin(\pi)$$

Since $\sin(\pi) = 0$ therefore

$$\frac{\mu_A}{\pi} \sin\big(\omega_c(t)\big)\sin(\pi) = 0$$

$$\mu_X(t) = 0$$

ii. Correlation function:

$$R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{A^2 \sin(\omega_c(t_1) + \theta)\sin(\omega_c(t_2) + \theta)\}$$

Because A, θ Independent

$$E\{A^2 \sin(\omega_c(t_1) + \theta)\sin(\omega_c(t_2) + \theta)\} = E\{A^2\}E\{\sin(\omega_c(t_1) + \theta)\sin(\omega_c(t_2) + \theta)\}$$

$$sin(\alpha)\,sin(\beta) = \frac{1}{2}[cos(\alpha-\beta) - cos(\alpha+\beta)]$$

Let $\alpha = \omega_c(t_1) + \theta$ $\beta = \omega_c(t_2) + \theta$ then

$$\alpha - \beta = \omega_c(t_1 - t_2) \qquad \alpha + \beta = \omega_c(t_1 + t_2) + 2\theta$$

Therefore

$$E\{A^2\}E\{\sin(\omega_c(t_1)+\theta)\sin(\omega_c(t_2)+\theta)\}$$

$$= E\{A^2\}\,E\left[\frac{1}{2}\cos\big(\omega_c(t_1-t_2)\big) - \frac{1}{2}\cos(\omega_c(t_1+t_2)+2\theta)\right]$$

$$= E\{A^2\}\,\left[\frac{1}{2}E\big\{\cos\big(\omega_c(t_1-t_2)\big)\big\} - \frac{1}{2}E\{\cos(\omega_c(t_1+t_2)+2\theta)\}\right]$$

The second term is zero.

$$R_{XX}(t_1,t_2) = \frac{1}{2}\,E\{A^2\}\cos\big(\omega_c(t_1-t_2)\big) = \frac{\mu_A}{2}\,\cos\big(\omega_c(\tau)\big)$$

$X(t)$ is WSS random process because the mean function is a constant (=0) and the autocorrelation function is only a function of a time difference $t_1 - t_2$.

Independent and independent identically distributed iid Random Processes

A. *Independent Processes:*

In a random process $X(t)$, if $X(t_i)$ for $i = 1,2,\ldots.,n$ are independent r.v.'s, so that for $n = 1,2,\ldots.,$

$$f_X(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n) = \prod_{i=1}^{n} f_X(x_i;t_i)$$

and $$F_X(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n) = \prod_{i=1}^{n} F_X(x_i;t_i)$$

Or $$P(X(t_1)\le x_1, X(t_2)\le x_2,\ldots,X(t_n)\le x_n)$$

$$= P(X(t_1)\le x_1).\,P(X(t_2)\le x_2),\ldots P(X(t_n)\le x_n)$$

then we call $X(t)$ an independent random process. Thus, a first-order distribution is sufficient to characterize an independent random process $X(t)$.

B. *Independent and identically distributed iid random process*

A Random process $\{X(t), t\in T\}$ is said to be independent and identically distributed (iid) if any finite number, say *k*, of random variables $X(t_1)$, $X(t_2)$, ……, $X(t_k)$ are mutually independent and have a common cumulative distribution function $F_X(.)$. The joint cdf and pdf for $X(t_1)$, $X(t_2)$, ……, $X(t_k)$ are given respectively by:

$$F_X(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots, t_k) = \prod_{i=1}^{k} F_X(x_i; t_i)$$

$$f_X(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots, t_k) = \prod_{i=1}^{k} f_X(x_i; t_i)$$

Example.

Consider the random process $\{X_n,\ n = 0,1,2,\ldots\}$ in which $X_i's$ are iid standard normal random variables.

(a) Write down $f_{x_n}(x)$ for $n = 0,1,2,\ldots$

(b) Write down $f_{x_n,x_m}(x_1, x_2)$ for $m \neq n$

Solution.

(a) Since $X_n \sim N(0,1)$, we have

$$f_{x_n}(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2} \qquad \forall\ x \in \mathbb{R}$$

(b) If $m \neq n$, then x_n and x_m are independent (because of the i.i.d. assumption) so,

$$\begin{aligned} f_{x_n,x_m}(x_1, x_2) &= f_{x_n}(x_1) f_{x_m}(x_2) \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x_1^2} \ \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x_2^2} \\ &= \frac{1}{2\pi} e^{\frac{1}{2}(x_1^2 + x_2^2)} \qquad \forall\ x_1, x_2 \in \mathbb{R} \end{aligned}$$