Problems

BINOMIAL DISTRIBUTION

6.1. Compute $P(k)$ for the binomial distribution $B(n, p)$ where:

(a) $n = 5,\ p = \dfrac{1}{4},\ k = 2$ (b) $n = 10,\ p = \dfrac{1}{2},\ k = 7$ (c) $n = 8,\ p = \dfrac{2}{3},\ k = 5$

Use Theorem 6.1, that $P(k) = \binom{n}{k} p^k q^{n-k}$ where $q = 1 - p$.

(a) Here $q = \dfrac{3}{4}$, so $P(2) = \binom{5}{2}\left(\dfrac{1}{4}\right)^2\left(\dfrac{3}{4}\right)^3 = 10\left(\dfrac{1}{16}\right)\left(\dfrac{27}{64}\right) \approx 0.264.$

(b) Here $q = \dfrac{1}{2}$, so $P(7) = \binom{10}{7}\left(\dfrac{1}{2}\right)^7\left(\dfrac{1}{2}\right)^3 = 120\left(\dfrac{1}{128}\right)\left(\dfrac{1}{8}\right) \approx 0.117.$

(c) Here $q = \dfrac{1}{3}$, so $P(5) = \binom{8}{5}\left(\dfrac{2}{3}\right)^5\left(\dfrac{1}{3}\right)^3 = 56\left(\dfrac{32}{243}\right)\left(\dfrac{1}{27}\right) \approx 0.273.$

6.2. The probability that John hits a target is $p = \frac{1}{4}$. He fires $n = 6$ times. Find the probability that he hits the target: (a) exactly 2 times, (b) more than 4 times, (c) at least once.

This is a binomial experiment with $n = 6$, $p = \frac{1}{4}$, and $q = 1 - p = \frac{3}{4}$; hence use Theorem 6.1.

(a) $P(2) = \binom{6}{2}(1/4)^2(3/4)^4 = 15(3^4)/(4^6) = 1215/4096 \approx 0.297$

(b) John hits the target more than 4 times if he hits it 5 or 6 times. Hence

$$P(X > 4) = P(5) + P(6) = \binom{6}{5}(1/4)^5(3/4)^1 + (1/4)^6$$
$$= 18/4^6 + 1/4^6 = 19/4^6 = 19/4096 \approx 0.0046$$

(c) Here $q^6 = (3/4)^6 = 729/4096$ is the probability that John misses all six times; hence

$$P(\text{one or more}) = 1 - 729/4096 = 3367/4096 \approx 0.82$$

6.3. Suppose 20 percent of the items produced by a factory are defective. Suppose 4 items are chosen at random. Find the probability that: (a) 2 are defective, (b) 3 are defective, (c) none are defective.

This is a binomial experiment with $n = 4$, $p = 0.2$ and $q = 1 - p = 0.8$; that is, $B(4, 0.2)$. Hence use Theorem 6.1.

(*a*) Here $k=2$ and $P(2)=\binom{4}{2}(0.2)^2(0.8)^2 \approx 0.1536$.

(*b*) Here $k=3$ and $P(3)=\binom{4}{3}(0.2)^3(0.8) \approx 0.0256$.

(*c*) Here $P(0)=q^4=(0.8)^4=0.4095$. Hence $P(X>0)=1-P(0)=1-0.4095=0.5904$

6.4. A family has six children. Find the probability P that there are: (*a*) three boys and three girls, (*b*) fewer boys than girls. Assume that the probability of any particular child being a boy is $\frac{1}{2}$.

Here $n=6$ and $p=q=\frac{1}{2}$.

(*a*) $P=P(3\text{ boys})=\binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3=\frac{20}{64}=\frac{5}{16}$

(*b*) There are fewer boys than girls if there are zero, one or two boys. Hence:

$$P=P(0\text{ boys})+P(1\text{ boy})+P(2\text{ boys})=\left(\frac{1}{2}\right)^6+\binom{6}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^5+\binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4=\frac{11}{32}$$

6.5. A certain type of missile hits its target with probability $p=0.3$. Find the number of missiles that should be fired so that there is at least a 90 percent probability of hitting the target.

The probability of missing the target is $q=1-p=0.7$. Hence the probability that n missiles miss the target is $(0.7)^n$. Thus we seek the smallest n for which

$$1-(0.7)^n>0.9 \qquad \text{or equivalently} \qquad (0.7)^n<0.1$$

Compute:

$$(0.7)^1=0.7, \qquad (0.7)^2=0.49, \qquad (0.7)^3=0.343, \qquad (0.7)^4=0.240,$$
$$(0.7)^5=0.168, \qquad (0.7)^8=0.118, \qquad (0.7)^9=0.0823$$

Thus at least nine missiles should be fired.

6.6. The mathematics department has eight graduate assistants who are assigned the same office. Each assistant is just as likely to study at home as in the office. Find the minimum number m of desks that should be put in the office so that each assistant has a desk at least 90 percent of the time.

This problem can be modeled as a binomial experiment where:

$n=8=$ number of assistants assigned to the office
$p=\frac{1}{2}=$ probability that an assistant will study in the office
$X=$ number of assistants studying in the office

Suppose there are k desks in the office, where $k \le 8$. Then a graduate assistant will not have a desk if $X>k$. Note that

$$P(X>k)=P(k+1)+P(k+2)+\cdots+P(8)$$

We want the smallest value of k for which $P(X>k)<0.10$.

Compute $P(8)$, $P(7)$, $P(6)$, ... until the sum exceeds 10 percent. Using Theorem 6.1, with $n=8$ and $p=q=\frac{1}{2}$, we obtain:

$$P(8)=(1/2)^8=1/256$$
$$P(7)=8(1/2)^7(1/2)=8/256$$
$$P(6)=28(1/2)^6(1/2)^2=28/256$$

Now $P(8)+P(7)+P(6)=37/256>10\%$ but $P(7)+P(8)<10\%$. Thus $m=6$ desks are needed.

6.7. A man fires at a target $n = 6$ times and hits it $k = 2$ times. (*a*) List the different ways that this can happen. (*b*) How many ways are there?

(*a*) List all sequences with two Ss (successes) and four Fs (failures):

SSFFFF, SFSFFF, SFFSFF, SFFFSF, SFFFFS, FSSFFF, FSFSFF, FSFFSF,
FSFFFS, FFSSFF, FFSFSF, FFSFFS, FFFSSF, FFFSFS, FFFFSS

(*b*) There are 15 different ways as indicated by the list. Observe that this is equal to $\binom{6}{2}$, since we are distributing $k = 2$ letters S among the $n = 6$ positions in the sequence.

6.8. Prove Theorem 6.1. The probability of exactly k successes in a binomial experiment $B(n, p)$ is given by $P(k) = P(k \text{ successes}) = \binom{n}{k} p^k q^{n-k}$. The probability of one or more successes is $1 - q^n$.

The sample space of the n repeated trials consists of all n-tuples (i.e. n-element sequences) whose components are either S (success) or F (failure). Let A be the event of exactly k successes. Then A consists of all n-tuples of which k components are S and $n - k$ components are F. The number of such n-tuples in the event A is equal to the number of ways that k letters S can be distributed among the n components of an n-tuple; hence A consists of $C(n, k) = \binom{n}{k}$ sample points. The probability of each point in A is $p^k q^{n-k}$; hence

$$P(A) = \binom{n}{k} p^k q^{n-k}$$

In particular, the probability of no successes is

$$P(0) = \binom{n}{0} p^0 q^n = q^n$$

Thus the probability of one or more successes is $1 - q^n$.

EXPECTED VALUE AND STANDARD DEVIATION

6.9. Four fair coins are tossed. Let X denote the number of heads occurring. Calculate the expected value of X directly, and compare with Theorem 6.2.

X is binomially distributed with $n = 4$ and $p = q = \frac{1}{2}$. We have:

$$P(0) = \frac{1}{16}, \qquad P(1) = \frac{4}{16}, \qquad P(2) = \frac{6}{16}, \qquad P(3) = \frac{4}{16}, \qquad P(4) = \frac{1}{16}$$

Thus the expected value is:

$$E(X) = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right) = \frac{32}{16} = 2$$

This agrees with Theorem 6.2, which states that $E(X) = np = 4\left(\frac{1}{2}\right) = 2$.

6.10. A family has eight children. (*a*) Determine the expected number of girls if male and female children are equally probable. (*b*) Find the probability P that the expected number of girls does occur.

(*a*) The number of girls is binomially distributed with $n = 8$ and $p = q = 0.5$. By Theorem 6.2,

$$\mu = np = 8(0.5) = 4$$

(*b*) We seek the probability of 4 girls. By Theorem 6.1, with $k = 4$,

$$P = P(4 \text{ girls}) = \binom{8}{4}(0.5)^4(0.5)^4 \approx 0.27 = 27\%$$

6.11. The probability that a man hits a target is $p = 0.1$. He fires $n = 100$ times. Find the expected number E of times he will hit the target, and the standard deviation σ.

This is a binomial experiment $B(n, p)$ where $n = 100$, $p = 0.1$, and $q = 1 - p = 0.9$. Thus apply Theorem 6.2 to obtain

$$E = np = 100(0.1) = 10 \qquad \text{and} \qquad \sigma = \sqrt{npq} = \sqrt{100(0.1)(0.9)} = 3$$

6.12. A fair die is tossed 300 times. Find the expected number E and the standard deviation σ of the number of 2's.

The number of 2's is binomially distributed with $n = 300$ and $p = \frac{1}{6}$. Hence $q = 1 - p = \frac{5}{6}$. By Theorem 6.2,

$$E = np = 300\left(\frac{1}{6}\right) = 50 \qquad \text{and} \qquad \sigma = \sqrt{npq} = \sqrt{300\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} = \sqrt{41.67} \approx 6.45$$

6.13. A student takes an 18 question multiple-choice exam, with four choices per question. Suppose one of the choices is obviously incorrect, and the student makes an "educated" guess of the remaining choices. Find the expected number E of correct answers, and the standard deviation σ.

This is a binomial experiment $B(n, p)$ where $n = 18$, $p = \frac{1}{3}$, and $q = 1 - p = \frac{2}{3}$. Hence

$$E = np = 18\left(\frac{1}{3}\right) = 6 \qquad \text{and} \qquad \sigma = \sqrt{npq} = \sqrt{18\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)} = 2$$

6.14. Prove Theorem 6.2: Let X be the binomial random variable $B(n, p)$. Then: (i) $\mu = E(X) = np$, (ii) $\mathrm{Var}(X) = npq$.

On the sample space of n Bernoulli trials, let X_i (for $i = 1, 2, \ldots, n$) be the random variable which has the value 1 or 0 according as the ith trial is a success or a failure. Then each X_i has the distribution

x	0	1
$P(x)$	q	p

and the total number of successes is $X = X_1 + X_2 + \cdots + X_n$.

(i) For each i, we have

$$E(X_i) = 0(q) + 1(p) = p$$

Using the linearity property of E (Theorem 5.4 and Corollary 5.5), we have

$$\begin{aligned} E(X) &= E(X_1 + X_2 + \cdots + X_n) \\ &= E(X_1) + E(X_2) + \cdots + E(X_n) \\ &= p + p + \cdots + p = np \end{aligned}$$

(ii) For each i, we have

$$E(X_i^2) = 0^2(q) + 1^2(p) = p$$

and

$$\text{Var}(X_i) = E(X_i^2) - [E(X)_i]^2 = p - p^2 = p(1-p) = pq$$

The n random variables X_i are independent. Therefore, by Theorem 5.9,

$$\begin{aligned} \text{Var}(X) &= \text{Var}(X_1 + X_2 + \cdots + X_n) \\ &= \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n) \\ &= pq + pq + \cdots + pq = npq \end{aligned}$$

6.15. Give a direct proof of Theorem 6.2: Let X be the binomial random variable $B(n, p)$. Then: (i) $\mu = E(X) = np$, (ii) $\text{Var}(X) = npq$.

(i) Using the notation $b(k; n, p) = P(k) = \binom{n}{k} p^k q^{n-k}$, we obtain:

$$\begin{aligned} E(X) = \sum_{k=0}^{n} k \cdot b(k; n, p) &= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k q^{n-k} \\ &= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k} \end{aligned}$$

(we drop the term $k = 0$ since its value is zero, and we factor out np from each term). We let $s = k - 1$ in the above sum. As k runs through the values 1 to n, s runs through the values 0 to $n - 1$. Thus

$$E(X) = np \sum_{s=0}^{n-1} \frac{(n-1)!}{s!(n-1-s)!} p^s q^{n-1-s} = np \sum_{s=0}^{n-1} b(s; n-1, p) = np$$

since, by the binomial theorem,

$$\sum_{s=0}^{n-1} b(s; n-1, p) = (p+q)^{n-1} = 1^{n-1} = 1$$

(ii) We first compute $E(X^2)$ as follows:

$$\begin{aligned} E(X^2) = \sum_{k=0}^{n} k^2 b(k; n, p) &= \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} p^k q^{n-k} \\ &= np \sum_{k=1}^{n} k \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k} \end{aligned}$$

Again we let $s = k - 1$ and obtain

$$E(X^2) = np \sum_{s=0}^{n-1} (s+1) \frac{(n-1)!}{s!(n-1-s)!} p^s q^{n-1-s} = np \sum_{s=0}^{n-1} (s+1) b(s; n-1, p)$$

But

$$\begin{aligned} \sum_{s=0}^{n-1} (s+1) b(s; n-1, p) &= \sum_{s=0}^{n-1} s \cdot b(s; n-1, p) + \sum_{s=0}^{n-1} b(s; n-1, p) \\ &= (n-1)p + 1 = np + 1 - p = np + q \end{aligned}$$

where we use (i) to obtain $(n-1)p$. Accordingly,

$$E(X^2) = np(np+q) = (np)^2 + npq$$

and

$$\operatorname{Var}(X) = E(X^2) - \mu_x^2 = (np)^2 + npq - (np)^2 = npq$$

Thus the theorem is proved.

NORMAL DISTRIBUTION

6.16. The mean and standard deviation on an examination are $\mu = 74$ and $\sigma = 12$, respectively. Find the scores in standard units of students receiving: (*a*) 65, (*b*) 74, (*c*) 86, (*d*) 92.

(*a*) $z = \dfrac{x-\mu}{\sigma} = \dfrac{65-74}{12} = -0.75$ (*c*) $z = \dfrac{x-\mu}{\sigma} = \dfrac{86-74}{12} = 1.0$

(*b*) $z = \dfrac{x-\mu}{\sigma} = \dfrac{74-74}{12} = 0$ (*d*) $z = \dfrac{x-\mu}{\sigma} = \dfrac{92-74}{12} = 1.5$

6.17. The mean and standard deviation on an examination are $\mu = 74$ and $\sigma = 12$, respectively. Find the grades corresponding to standard scores: (*a*) -1, (*b*) 0.5, (*c*) 1.25, (*d*) 1.75.

Solving $z = \dfrac{x-\mu}{\sigma}$ for x yields $x = \sigma z + \mu$. Thus:

(*a*) $x = \sigma z + \mu = (12)(-1) + 74 = 62$ (*c*) $x = \sigma z + \mu = (12)(1.25) + 74 = 89$

(*b*) $x = \sigma z + \mu = (12)(0.5) + 74 = 80$ (*d*) $x = \sigma z + \mu = (12)(1.75) + 74 = 95$

6.18. Table A-1 (see Appendix) uses $\Phi(z)$ to denote the area under the standard normal curve ϕ between 0 and z. Find: (*a*) $\Phi(1.47)$, (*b*) $\Phi(0.52)$, (*c*) $\Phi(1.1)$, (*d*) $\Phi(4.1)$.

Use Table A-1 as follows:

(*a*) To find $\Phi(1.47)$, look on the left for the row labeled 1.4, and then look on the top for the column labeled 7. The entry in the table corresponding to row 1.4 and column 7 is 0.4292. Hence $\Phi(1.47) = 0.4292$.

(*b*) To find $\Phi(0.52)$, look on the left for the row labeled 0.5, and then look on the top for the column labeled 2. The entry in the table corresponding to row 0.5 and column 2 is 0.1985. Hence $\Phi(0.52) = 0.1985$.

(*c*) To find $\Phi(1.1)$, look on the left for the row labeled 1.1. The first entry in this row is 0.3643 which corresponds to $1.1 = 1.10$. Hence $\Phi(1.1) = 0.3643$.

(*d*) The value of $\Phi(z)$ for any $z \geq 3.9$ is 0.5000. Thus $\Phi(4.1) = 0.5000$ even though 4.1 is not in the table.

6.19. Let Z be the random variable with standard normal distribution ϕ. Determine the value of z if: (*a*) $P(0 \leq Z \leq z) = 0.4236$, (*b*) $P(Z \leq z) = 0.7967$, (*c*) $P(z \leq Z \leq 2) = 0.1000$.

(*a*) Here $z > 0$. Thus draw a picture of z and $P(0 \leq Z \leq z)$ as in Fig. 6-8(*a*). Here Table 6-1 can be used directly. The entry 0.4236 appears to the right of row 1.4 and under column 3. Thus $z = 1.43$.

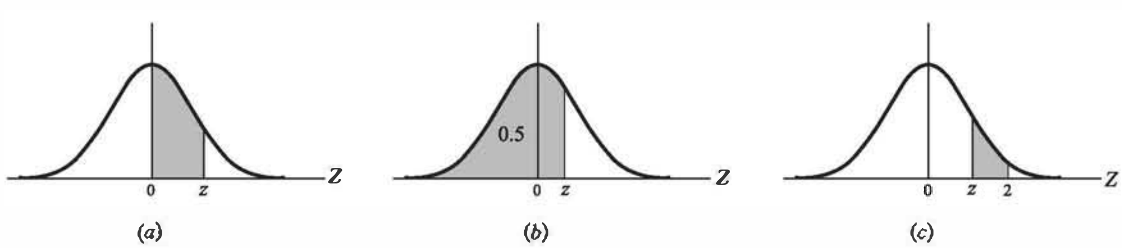


Fig. 6-8

(*b*) Note z must be positive since the probability is greater than 0.5. Thus draw z and $P(Z \le z)$ as in Fig. 6-8(*b*). We have

$$\Phi(z) = P(0 \le Z \le z) = P(Z \le z) - 0.5 = 0.7967 - 0.5000 = 0.2967$$

Since 0.2967 appears in row 0.8 and column 3 in Table 6-1, we have $z = 0.83$.

(*c*) Since $\Phi(2) = 0.4772 > 0.1000$, z must lie between 0 and 2. Thus draw z and $P(z \le Z \le 2)$ as in Fig. 6-8(*c*). We have

$$\Phi(z) = \Phi(2) - P(z \le Z \le 2) = 0.4772 - 0.1000 = 0.3772$$

From Table 6-1, we get $z = 1.16$.

6.20. Let Z be the random variable with standard normal distribution ϕ. Find: (*a*) $P(0 \le Z \le 1.28)$, (*b*) $P(-0.73 \le Z \le 0)$, (*c*) $P(Z = 1.1)$.

(*a*) By definition $\Phi(z)$ is the area under the curve ϕ between 0 and z. Therefore, using Table A-1,

$$P(0 \le Z \le 1.28) = \Phi(1.28) = 0.3997$$

(*b*) By symmetry and Table A-1,

$$P(-0.73 \le Z \le 0) = P(0 \le Z \le 0.73) = \Phi(0.73) = 0.2673$$

(*c*) The area under a single point $a = 1.1$ is 0; hence $P(Z = 1.1) = 0$.

6.21. Let Z be the random variable with standard normal distribution ϕ. Find: (*a*) $P(-1.37 \le Z \le 0.82)$, (*b*) $P(0.65 \le Z \le 1.26)$, (*c*) $P(-1.04 \le Z \le -0.12)$.

Use the following formula (pictured in Fig. 6-3):

$$P(z_1 \le Z \le z_2) = \begin{cases} \Phi(z_2) + \Phi(|z_1|) & \text{if } z_1 \le 0 \le z_2 \\ \Phi(z_2) - \Phi(z_1) & \text{if } 0 \le z_1 \le z_2 \\ \Phi(|z_1|) - \Phi(|z_2|) & \text{if } z_1 \le z_2 \le 0 \end{cases}$$

(*a*) Since $-1.37 < 0 < 0.82$,

$$\begin{aligned} P(-1.37 \le Z \le 0.82) &= \Phi(0.82) + \Phi(1.37) \\ &= 0.2939 + 0.4147 = 0.7086 \end{aligned}$$

(*b*) Since $0 < 0.65 < 1.26$,

$$\begin{aligned} P(0.65 \le Z \le 1.26) &= \Phi(1.26) - \Phi(0.65) \\ &= 0.3962 - 0.2422 = 0.1540 \end{aligned}$$

(*c*) Since $-1.04 < -0.12 < 0$,

$$\begin{aligned} P(-1.04 \le Z \le -0.12) &= \Phi(1.04) - \Phi(0.12) \\ &= 0.3508 - 0.0478 = 0.3030 \end{aligned}$$

6.22. Let Z be the random variable with standard normal distribution ϕ. Find the following one-sided probabilities: (*a*) $P(Z \le -0.7)$, (*b*) $P(Z \le 1.03)$, (*c*) $P(Z \ge 0.36)$, (*d*) $P(Z \ge -1.1)$.

Figure 6-4 shows how to compute the one-sided probabilities.

(*a*) $P(Z \le -0.7) = 0.5 - \Phi(0.7) = 0.5 - 0.2580 = 0.2420$

(*b*) $P(Z \le 1.03) = 0.5 + \Phi(1.03) = 0.5 + 0.3485 = 0.8485$

(c) $P(Z \geq 0.36) = 0.5 - \Phi(0.36) = 0.5 - 0.1406 = 0.3594$

(d) $P(Z \geq -1.1) = 0.5 + \Phi(-1.1) = 0.5 + 0.3643 = 0.8643$

6.23. Suppose that the student IQ scores form a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 20$. Find the percentage of students whose scores fall between: (a) 80 and 120, (b) 60 and 140, (c) 40 and 160, (d) 100 and 120, (e) over 160.

All the scores are units of the standard deviation $\sigma = 20$ from the mean $\mu = 100$; hence we can use the 68–95–99.7 rule or Fig. 6.2 to obtain:

(a) 68 percent, (b) 95 percent, (c) 97.7 percent

(d) $\frac{1}{2}$(68 percent) = 34 percent, (e) $\frac{1}{2}$(0.3 percent) = 0.15 percent

6.24. Suppose the temperature T during May is normally distributed with mean $\mu = 68°$ and standard deviation $\sigma = 6°$. Find the probability p that the temperature during May is: (a) between 70° and 80°, (b) less than 60°.

First convert the T values into Z values in standard units, then use Table A-1 (see Appendix).

(a) We have:

$$70° \text{ in standard units} = (70 - 68)/6 = 0.33$$
$$80° \text{ in standard units} = (80 - 68)/6 = 2.00$$

Here $0 < 0.33 < 2.00$. Therefore (Fig. 6-9(a)),

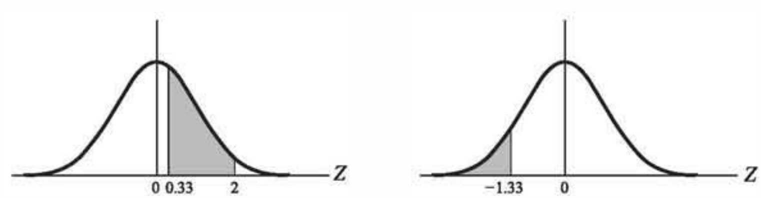
$$p = P(70 \leq T \leq 80) = P(0.33 \leq Z \leq 2.00)$$
$$= \Phi(2.00) - \Phi(0.33) = 0.4772 - 0.1293 = 0.3479$$

(b) We have:

$$60° \text{ in standard units} = (60 - 68)/6 = -1.33$$

This is a one-sided probability with $-1.33 < 0$. Using Fig. 6-9(b), symmetry, and that half the area under the curve is 0.5000, we obtain

$$p = P(T \leq 60) = P(Z \leq -1.33) = P(Z \geq 1.33)$$
$$= 0.5 - \Phi(1.33) = 0.5000 - 0.4082 = 0.0918$$



(a) $P(0.33 \leq Z \leq 2.00)$ (b) $P(Z \leq -1.33)$

Fig. 6-9

6.25. Suppose the weights W of 800 male students are normally distributed with mean $\mu = 140$ pounds and standard deviation $\sigma = 10$ pounds. Find the number N of students with weights: (a) between 138 and 148 pounds, (b) more than 152 pounds.

First convert the W values into Z values in standard units, then use Table A-1 (see Appendix).

(a) We have:

$$138 \text{ in standard units} = (138 - 140)/10 = -0.2$$
$$148 \text{ in standard units} = (148 - 140)/10 = 0.8$$

Here $-0.2 < 0 < 0.8$. Therefore (Fig. 6-10(a)),

$$\begin{aligned} P(138 \leq W \leq 148) &= P(-0.2 \leq Z \leq 0.8) \\ &= \Phi(0.8) + \Phi(-0.2) = 0.2881 + 0.0793 = 0.3674 \end{aligned}$$

Thus $N = 800(0.3674) \approx 294$.

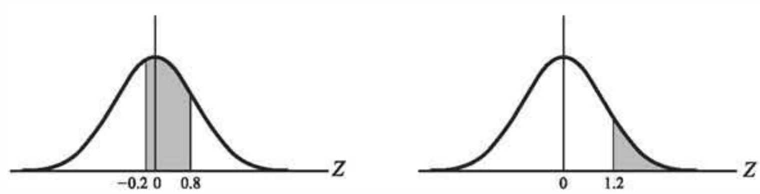
(b) We have:

$$152 \text{ in standard units} = (152 - 140)/10 = 1.20$$

This is a one-sided probability with $0 < 1.20$. Using Fig. 6-10(b) and that half the area under the curve is 0.5000, we get

$$P(W \geq 152) = P(Z \geq 1.2) = 0.5 - \Phi(1.2) = 0.5000 - 0.3849 = 0.1151$$

Thus $N = 800(0.1151) \approx 92$.



(a) $P(-0.2 \leq Z \leq 0.8)$ (b) $P(Z \leq 1.2)$

Fig. 6-10

NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

This section of problems uses BP to denote the binomial probability and NP to denote the normal probability.

6.26. A fair coin is tossed 12 times. Determine the probability P that the number of heads occurring is between 4 and 7 inclusive by using: (a) the binomial distribution, (b) the normal approximation to the binomial distribution.

(a) Let heads denote a success. By Theorem 6.1, with $n = 12$ and $p = q = \frac{1}{2}$:

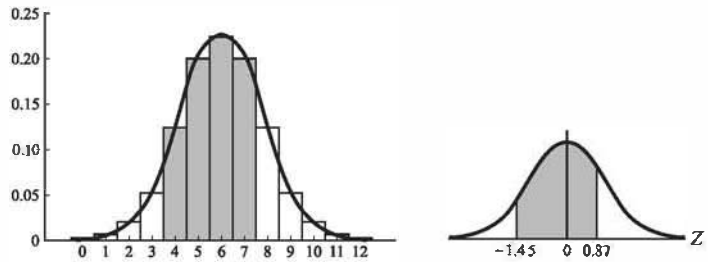
$$BP(4) = \binom{12}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^8 = \frac{495}{4096}, \qquad BP(6) = \binom{12}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^6 = \frac{924}{4096}$$

$$BP(5) = \binom{12}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^7 = \frac{792}{4096}, \qquad BP(7) = \binom{12}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^5 = \frac{792}{4096}$$

Hence $P = \dfrac{495}{4096} + \dfrac{792}{4096} + \dfrac{924}{4096} + \dfrac{792}{4096} = \dfrac{3003}{4096} = 0.7332$.

(b) Here $\mu = np = 12\left(\frac{1}{2}\right) = 6$ and $\sigma = \sqrt{npq} = \sqrt{12\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = 1.73$. Let X denote the number of heads occurring. We seek $BP(4 \leq X \leq 7)$, which corresponds to the shaded area in Fig. 6-11(a). On the other hand, if we assume that the data is continuous, in order to apply the binomial approximation, we must find $NP(3.5 \leq X \leq 7.5)$, as indicated in Fig. 6-11(a). We have:

$$\begin{aligned} 3.5 \text{ in standard units} &= (3.5 - 6)/1.73 = -1.45 \\ 7.5 \text{ in standard units} &= (7.5 - 6)/1.73 = 0.87 \end{aligned}$$



$BP(4 \leq X \leq 7)$ $NP(-1.45 \leq Z \leq 0.87)$

(*a*) (*b*)

Fig. 6-11

Then, as indicated by Fig. 6-11(*b*),

$$P = NP(3.5 \leq X \leq 7.5) = NP(-1.45 \leq Z \leq 0.87)$$
$$= \phi(0.87) + \Phi(1.45) = 0.3087 + 0.4265 = 0.7343$$

(Note that the relative error $e = |(0.7332 - 0.7343)/0.7332| = 0.0015$ is less than 0.2 percent.)

6.27. A fair die is tossed 180 times. Determine the probability P that the face 6 will appear: (*a*) between 29 and 32 times inclusive, (*b*) between 31 and 35 times inclusive, (*c*) less than 22 times.

This is a binomial experiment $B(n,p)$ with $n = 180$, $p = \frac{1}{6}$ and $q = 1 - p = \frac{5}{6}$. Then

$$\mu = np = 180\left(\frac{1}{6}\right) = 30 \qquad \text{and} \qquad \sigma = \sqrt{npq} = \sqrt{180\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} = 5$$

Let X denote the number of times the face 6 appears.

(*a*) We seek $BP(29 \leq X \leq 32)$ or, assuming the data is continuous, $NP(28.5 \leq X \leq 32.5)$. We have:

$$28.5 \text{ in standard units} = (28.5 - 30)/5 = -0.3$$
$$32.5 \text{ in standard units} = (32.5 - 30)/5 = 0.5$$

(This is the case $z_1 \leq 0 \leq z_2$.) Therefore (Fig. 6-3(*a*)),

$$P = NP(28.5 \leq X \leq 32.5) = NP(-0.3 \leq Z \leq 0.5)$$
$$= \Phi(0.5) + \Phi(0.3) = 0.1915 + 0.1179 = 0.3094$$

(*b*) We seek $BP(31 \leq X \leq 35)$ or, assuming the data is continuous, $NP(30.5 \leq X \leq 35.5)$. We have:

$$30.5 \text{ in standard units} = (30.5 - 30)/5 = 0.1$$
$$35.5 \text{ in standard units} = (35.5 - 30)/5 = 1.1$$

(This is the case $0 \leq z_1 \leq z_2$.) Therefore (Fig. 6-3(*b*)),

$$P = NP(30.5 \leq X \leq 35.5) = NP(0.1 \leq Z \leq 1.1)$$
$$= \Phi(1.1) - \Phi(0.1) = 0.3643 - 0.0398 = 0.3245$$

(*c*) We seek $BP(X < 22)$ or, approximately, $NP(X \leq 21.5)$. (See remark in Section 6.5 on the one-sided normal approximation.) We have:

$$21.5 \text{ in standard units} = (21.5 - 30)/5 = -1.7$$

Therefore, using symmetry and that half the area under the normal curve is 0.5000, we get

$$P = NP(X \leq 21.5) = NP(Z \leq -1.7)$$
$$= 0.5000 - \Phi(1.7) = 0.5000 - 0.4554 = 0.0446$$

6.28. Assume that 4 percent of the population over 65 years old has Alzheimer's disease. Suppose a random sample of 9600 people over 65 is taken. Find the probability P that fewer than 400 of them have the disease.

This is a binomial experiment $B(n,p)$ with $n = 9600$, $p = 0.04$, and $q = 1 - p = 0.96$. Then

$$\mu = np = (9600)(0.04) = 384 \qquad \text{and} \qquad \sigma = \sqrt{npq} = \sqrt{(9600)(0.04)(0.96)} = 19.2$$

Let X denote the number of people with Alzheimer's disease.

We seek $BP(X < 400)$ or, approximately, $NP(X \leq 399.5)$. (See remark in Section 6.5 on the one-sided normal approximation.) We have:

$$399.5 \text{ in standard units} = (399.5 - 384)/19.2 = 0.81$$

Therefore,

$$P = NP(X \leq 399.5) = NP(Z \leq 0.81)$$
$$= 0.5000 + \Phi(0.81) = 0.5000 + 0.2897 = 0.7897$$

POISSON DISTRIBUTION

6.29. Find: (*a*) $e^{-1.3}$, (*b*) $e^{-2.5}$.

Use Table 6-1 and the law of exponents.

(*a*) $e^{-1.3} = (e^{-1})(e^{-0.3}) = (0.368)(0.741) = 0.273.$

(*b*) $e^{-2.5} = (e^{-2})(e^{-0.5}) = (0.135)(0.607) = 0.0819.$

6.30. For the Poisson distribution $f(k; \lambda) = \dfrac{\lambda^k e^{-\lambda}}{k!}$, find: (*a*) $f(2; 1)$, (*b*) $f(3; \frac{1}{2})$, (*c*) $f(2; 0.7)$.

Use Table 6-1 to obtain $e^{-\lambda}$.

(*a*) $f(2; 1) = \dfrac{1^2 e^{-1}}{2!} = \dfrac{e^{-1}}{2} = \dfrac{0.368}{2} = 0.184.$

(*b*) $f(3; \frac{1}{2}) = \dfrac{(\frac{1}{2})^3 e^{-0.5}}{3!} = \dfrac{e^{-0.5}}{48} = \dfrac{0.607}{48} = 0.013.$

(*c*) $f(2; 0.7) = \dfrac{(0.7)^2 e^{-0.7}}{2!} = \dfrac{(0.49)(0.497)}{2} = 0.12.$

6.31. Suppose 300 misprints are distributed randomly throughout a book of 500 pages. Find the probability P that a given page contains (*a*) exactly 2 misprints, (*b*) 2 or more misprints.

We view the number of misprints on one page as the number of successes in a sequence of Bernoulli trials. Here $n = 300$ since there are 300 misprints, and $P = 1/500$, the probability that a misprint appears on the given page. Since p is small, we use the Poisson approximation to the binomial distribution with $\lambda = np = 0.6$.

(*a*) $P = f(2; 0.6) = \dfrac{(0.6)^2 e^{-0.6}}{2!} = (0.36)(0.549)/2 = 0.0988 \approx 0.1.$

(*b*) We have:

$$P(0 \text{ misprints}) = \frac{(0.6)^0 e^{-0.6}}{0!} = e^{-0.6} = 0.549$$

$$P(1 \text{ misprint}) = \frac{(0.6)e^{-0.6}}{1!} = (0.6)(0.549) = 0.329$$

Then $P = 1 - P(0 \text{ or } 1 \text{ misprint}) = 1 - (0.549 + 0.329) = 0.122$.

6.32. Show that the Poisson distribution $f(k; \lambda)$ is a probability distribution, that is,

$$\sum_{k=0}^{\infty} f(k; \lambda) = 1$$

By known results of analysis, $e^{\lambda} = \sum_{k=0}^{\infty} \lambda^k / k!$. Hence

$$\sum_{k=0}^{\infty} f(k; \lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \lambda^k / k! = e^{-\lambda} e^{\lambda} = 1$$

6.33. Prove Theorem 6.5: Let X be a random variable with the Poisson distribution $f(k; \lambda)$. Then: (i) $E(X) = \lambda$, (ii) $\mathrm{Var}(X) = \lambda$. Hence $\sigma_X = \sqrt{\lambda}$.

(i) Using $f(k; \lambda) = \lambda^k e^{-\lambda}/k!$, we obtain

$$E(X) = \sum_{k=0}^{\infty} k \cdot f(k; \lambda) = \sum_{k=0}^{\infty} k \, \frac{\lambda^k e^{-\lambda}}{k!} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!}$$

(we drop the term $k = 0$ since its value is zero, and we factor out λ from each term). Let $s = k - 1$ in the above sum. As k runs through the values 1 to ∞, s runs through the values 0 to ∞. Thus

$$E(X) = \lambda \sum_{k=0}^{\infty} \frac{\lambda^s e^{-\lambda}}{s!} = \lambda \sum_{k=0}^{\infty} f(s; \lambda) = \lambda$$

since $\sum_{k=0}^{\infty} f(s; \lambda) = 1$, by Problem 6.36.

(ii) We first compute $E(X^2)$. We have

$$E(X^2) = \sum_{k=0}^{\infty} k^2 f(k; \lambda) = \sum_{k=0}^{\infty} k^2 \, \frac{\lambda^k e^{-\lambda}}{k!} = \lambda \sum_{k=1}^{\infty} k \, \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!}$$

Again we let $s = k - 1$ and obtain

$$E(X^2) = \lambda \sum_{s=0}^{\infty} (s+1) \, \frac{\lambda^s e^{-\lambda}}{s!} = \lambda \sum_{s=0}^{\infty} (s+1) f(s; \lambda)$$

But

$$\sum_{s=0}^{\infty} (s+1) f(s; \lambda) = \sum_{s=0}^{\infty} s f(s; \lambda) = \sum_{s=0}^{\infty} f(s; \lambda) = \lambda + 1$$

where we use (i) to obtain λ and Problem 6.36 to obtain 1. Accordingly,

$$E(X^2) = \lambda(\lambda + 1) = \lambda^2 + \lambda$$

and

$$\mathrm{Var}(X) = E(X^2) - \mu_X^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Thus the theorem is proved.

6.34. Show that if p is small and n is large, then the binomial distribution $B(n,p)$ is approximated by the Poisson distribution $POI(\lambda)$ where $\lambda = np$, that is, using

$$BP(k) = \binom{n}{k} p^k q^{n-k} \qquad \text{and} \qquad f(k;\lambda) = \lambda^k e^{-\lambda}/k!$$

then $BP(k) \approx f(k;\lambda)$ where $np = \lambda$.

We have $BP(0) = (1-p)^n = (1-\lambda/n)^n$. Taking the natural logarithm of both sides,

$$\ln BP(0) = n \ln(1-\lambda/n)$$

The Taylor expansion of the natural logarithm is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

so

$$\ln\left(1-\frac{\lambda}{n}\right) = -\frac{\lambda}{n} - \frac{\lambda^2}{2n^2} - \frac{\lambda^3}{3n^3} - \cdots$$

Therefore, if n is large

$$\ln BP(0) = n \ln\left(1-\frac{\lambda}{n}\right) = -\lambda - \frac{\lambda^2}{2n} - \frac{\lambda^3}{3n^2} \approx -\lambda$$

and hence $BP(0) \approx e^{-\lambda}$.

Furthermore, if p is very small and hence $q \approx 1$, we have

$$\frac{BP(k)}{BP(k-1)} = \frac{(n-k+1)p}{kq} = \frac{\lambda-(k-1)p}{kq} \approx \frac{\lambda}{k}$$

That is, $BP(k) \approx \frac{\lambda}{k} BP(k-1)$. Thus, using $BP(0) \approx e^{-\lambda}$, we obtain $BP(1) \approx \lambda e^{-\lambda}$, $BP(2) \approx \lambda^2 e^{-\lambda}/2$ and, by induction, $BP(k) \approx \lambda^k e^{-\lambda}/k! = f(k;\lambda)$.

MISCELLANEOUS PROBLEMS

6.35. The painted light bulbs produced by a company are 50 percent red, 30 percent green, and 20 percent blue. In a sample of 5 bulbs, find the probability P that 2 are red, 1 is green, and 2 are blue.

By Theorem 6.6 on the multinomial distribution,

$$P = \frac{5!}{2!1!2!}(0.5)^2(0.3)(0.2)^2 = 0.09$$

6.36. Show that the normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2(x-\mu)^2/\sigma^2}$$

is a continuous probability distribution, i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Substituting $t = (x-\mu)/\sigma$ in $\int_{-\infty}^{\infty} f(x)\,dx$, we obtain the integral

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2}\,dt$$

It suffices to show that $I^2 = 1$. We have

$$I^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt \int_{-\infty}^{\infty} e^{-s^2/2}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(s^2-t^2)/2}\,ds\,dt$$

We introduce polar coordinates in the above double integral. Let $s = r\cos\theta$ and $t = r\sin\theta$. Then $ds\,dt = r\,dr\,d\theta$, $0 \le \theta \le 2\pi$, and $0 \le r \le \infty$. That is,

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} re^{-r^2/2}\,dr\,d\theta$$

But

$$\int_0^{\infty} re^{-r^2/2}\,dr = \left[-e^{-r^2/2}\right]_0^{\infty} = 1$$

Hence $I^2 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} d\theta = 1$ and the theorem is proved.

6.37. Prove Theorem 6.3: Let X be a random variable with the normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-1/2(x-\mu)^2/\sigma^2}$$

Then (i) $E(X) = \mu$ and (ii) $\text{Var}(X) = \sigma^2$. Hence $\sigma_X = \sigma$.

(i) By definition, $E(X) = \dfrac{1}{\sigma\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} x\,e^{-1/2(x-\mu)^2/\sigma^2}\,dx$. Setting $t = (x-\mu)/\sigma$, we obtain

$$E(X) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma t + \mu)\,e^{-t^2/2}\,dt = \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t\,e^{-t^2/2}\,dt + \mu\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt$$

But $g(t) = t\,e^{-t^2/2}$ is an odd function, i.e. $g(-t) = -g(t)$; hence $\displaystyle\int_{-\infty}^{\infty} t\,e^{-t^2/2}\,dt = 0$. Furthermore, $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = 1$, by the preceding problem. Accordingly, $E(X) = \dfrac{\sigma}{\sqrt{2\pi}}\cdot 0 + \mu\cdot 1 = \mu$ as claimed.

(ii) By definition, $E(X^2) = \dfrac{1}{\sigma\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} x^2\,e^{-1/2(x-\mu)^2/\sigma^2}\,dx$. Again setting $t = (x-\mu)/\sigma$, we obtain

$$\begin{aligned} E(X^2) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma t + \mu)^2\,e^{-t^2/2}\,dt \\ &= \sigma^2\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2\,e^{-t^2/2}\,dt + 2\mu\sigma\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t\,e^{-t^2/2}\,dt + \mu^2\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt \end{aligned}$$

which reduces as above to $E(X^2) = \sigma^2\,\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} t^2\,e^{-t^2/2}\,dt + \mu^2$.

We integrate the above integral by parts. Let $u = t$ and $dv = te^{-t^2/2}\,dt$. Then $v = -e^{-t^2/2}$ and $du = dt$. Thus

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2\,e^{-t^2/2}\,dt = \frac{1}{\sqrt{2\pi}}\left[-t\,e^{-t^2/2}\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = 0 + 1 = 1$$

Consequently, $E(X^2) = \sigma^2\cdot 1 + \mu^2 = \sigma^2 + \mu^2$ and

$$\text{Var}(X) = E(X^2) - \mu_X^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

Thus the theorem is proved.