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Ordinary Differential Equations

LECTURE NOTES

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Ordinary Differential Equations

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Module Describtion

Module Aims

The aim of this module is to introduce the students to the basic theory of ordinary differential equations and give a competence in solving ordinary differential equations by using different methods of solution of differential equations.

General Description of the module

The subject of differential equations is a very important branch of applied mathematics. Many phenomena from physics, biology and engineering may be described using ordinary differential equations. They are also used to model the behaviour of systems in the natural world, and predict how these systems will behave in the further. For instance, exponential growth (the rate of change of a population is proportional to the size of the population) is expressed by the differential equation $dP/dt = kP$. Newton's Law of Gravitation (acceleration is inversely proportional to the square of distance) translates to the equation $y'' = -ky^2$. Many examples are found in the fields of physics, engineering, biology, chemistry and economics.

The traditional course in differential equations focused on the small number of differential equations for which exact solutions exist. However, the methods used by scientists today have changed dramatically due to computer (using different type of computational package like Maple, Mathematica, reduce, Singular, etc). Here we will cover almost all methods for solving every kind of ordinary differential equations.

Homework

Homework will be given at every lecture. You should start working on the homework problems for a section as soon as that section is covered in class. Although you are encouraged to consult with other students and seek help from tutor and me, homework should ultimately represent your own work. Answers unsupported by work will not receive credit. Not all problems may be graded. Homework should be neatly handwritten or typed, on one side of the page only. Copy the problem in its original form from the lecture (book) and provide the solution to the problem.

prerequisite

One must be familiar with the basic differential and integral calculus, which are the main contents of college level introductory Calculus course. Although the course does not require more more details in linear algebra, it will be very helpful if one has a little bit of knowledge on Linear Algebra such as the determinant of a square matrix, linear (in)dependence of vectors, and Cramers rule of solving a determined system of simultaneous linear algebraic equations.

Learning Objectives

- The student will learn to formulate ordinary differential equations (ODEs) and seek understanding of their solutions.

- The student will recognise basic types of differential equations which are solvable, and will understand the features of linear equations in particular.

- Students will be familiar to derive methods to solve ordinary differential equations.

Grades

Grades will be assigned on the basis of 100 points distributed as follows:

30 points midterm test.

10 points discussion.

60 points final examination.

Attendence

Class attendance is mandatory. Although I do not have a rigid policy, anyone who has missed lots of class and is doing poorly in the course should not except much sympathy from me. If you do miss a class, it is your responsibility to make up the material and make sure your homework is turned in on time.

Hours per week	Notice	Initial Warning	Last Warning
3	3	6	9

Contents

References

1) Differential Equations: a modelling approach. By *Frank R. Giordano and Maurice D. Weir*.

2) Elementary Differential Equations. By *Earl D. Rainvlle and Philip E. Bedient*.

3) Elementary Differential Equations with Linear Algebra. By *Ross L Finney and Donald R. Ostbery*.

4) Ordinary Differential Equations. By *Tyn Myint-V*.

5) Differential Equations and Boundary Value Problems. By *C. Henry Edward and David E. Penney*.

6) Applied Differential Equations. By *Murray R. Spoegel*.

7) Differential Equations. By *C. Ray Wylie*.

8) Schaum's Outline Series, Theory and problems of Differential Equations. By *Frank Ayres, JR. including 560 solved problems.*.

9) Schaum's: 2500 solved problem in Differential Equations. By *Richard Bronson*.

10) A first course in Differential Equations with Application.

11) Introduction to Differential Equations, Lecture notes. By Jeffrey R. Chasnov.

Chapter 1

Basic definitions and elimination of essential constants

1.1 Introduction: How to read a differential equation

Welcome to the world of differential equations! We hope you will enjoy it. Differential equations describe many processes in the world around you, but of course we shall have to convince you of that. Today we are going to give an example, and find out what it means to read a differential equation.

Definition 1. *A differential equation is an equation (not identity) that involves unknown function and any of its derivatives or differentials.*

If only one independent variable is assumed, the differential equation is called ordinary differential equation. The following are examples of ordinary differential equations:

$$\begin{aligned} \frac{dy}{dx} + 2xy &= e^x, \\ y\,dy - x\,x\,e^y dx &= 0, \\ y'' - y' - 2y &= cos(x), \\ (\frac{dy}{dx})^2 - x^2 e^y &= 1. \end{aligned}$$

If two or more independent variables appears, the equation is known as a partial differential equation. For instance:

$$\begin{aligned} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} &= 0, \\ \frac{\partial^2 v}{\partial x^2} &= \frac{\partial v}{\partial t}, \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \ln u. \end{aligned}$$

Example 1 (The Banker's equation)**.** *Consider the differential equa-*

tion

$$\frac{dy}{dt} = 0.028y.$$

It does not look too exciting does it? Really it is, though. It might for example represent your bank account, where the balance is y *at a time* t *years after you open the account, and the account is earning 2.8% interest. Regardless of the specific interpretation, let's see what the equation says. Since we see the term* $\frac{dy}{dt}$ *we can tell that* y *is a function of* t*, and that the rate of change is a multiple, namely* 0.028*, of the value of* y *itself. We definitely should always write* $y(t)$ *instead of just* y*, and we will sometimes, but it is traditional to be sloppy. For example, if* y *happens to be 2000 at a particular time* t*, the rate of change of* y *is then 0.028(2000) = 56, and the units of this rate in the bank account case are dollars/year. Thus* y *is increasing, whenever* y *is positive.*

Question: How to interpret the differential equation

$$\frac{dy}{dt} = 0.028y - 10.$$

Remark 1. *This is not supposed to be a hard question. By the way, when I ask a question, do not cheat yourself by ignoring it. Think about it, and future things will be easier. I promise.*

1.2 Notation, order and degree

The most general form of an ordinary differential equation is

$$f(x, y, y', \ldots, y^{(n)}) = 0, \tag{1.1}$$

where x is independent variable while y is dependent variable.

Definition 2. *The order of a differential equation is the order n of the highest derivative appearing in the equation. Thus*

$$\frac{d^2y}{dx^2} + y = 0$$

is a second order differential equation, whereas

$$\frac{dy}{dx} - xy = \sin(x)$$

is an example of a first order differential equation.

Definition 3. *If a differential equation can be rationalised and cleared of fractions with regard to all derivatives present, then the exponent of the highest order derivatives is called the degree of the differential equation.*

Remark 2. *Not every differential equation has a degree. If the degree* *exists, it should be a positive integer.*

Example 2. *The differential equation* $(y'')^{\frac{2}{3}} = 1 + y'$ *can be rationalised by cubing both sides to obtain* $(y'')^2 = (1 + y')^3$. *The exponent of the highest order derivatives present (namely* y''*) is 2. hence, the differential equation is of degree two. however, note that* $y''' = \sqrt{x + y}$ *is of degree one.*

Homework 1. *Give an example of a differential equation for which a degree is not defined.*

Question: Is it possible for a differential equation to have more than one dependent variable?

1.3 Solutions of differential equations

Definition 4. *Any function which is free of derivatives and which satisfy identically a differential equation on an interval* I, *is said to be a solution of the differential equation. Thus, we say that the function* $u = u(x)$ *is a solution of the differential equation* (1.1) *on the interval* I, *provided that the derivatives* $u', u'', \ldots, u^{(n)}$ *exists on* I *and* $f(x, u, u', u'', \ldots, u^{(n)}) = 0$ *for all* x *in* I. *The graph of* u *is then called a solution curve of the equation.*

Example 3. *For any constant k the function $y = ke^{x/2}$ is a solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}y$ over the interval $-\infty < x < \infty$. Since $y = ke^{x/2} \implies \frac{dy}{dx} = \frac{d}{dx}(ke^{x/2}) = \frac{k}{2}e^{x/2}$ and substitute in the differential equation gives $\frac{k}{2}e^{x/2} = \frac{1}{2}(ke^{x/2})$ which is true for all real number x.*

Remark 3. *A solution to a differential equation MUST be continuous since the derivative appears in the equation.*

Homework 2. *Show that every function of the form $y = \frac{1}{x}e^{cx}$, where c is a constant is a solution of the differential equation $xy' + y - y\ln(xy) = 0$ for all $x \neq 0$.*

Definition 5. *The general solution to an n-th order ordinary differential equation is a solution that contains all possible solutions over an interval I. This general solution contains n arbitrary essential constants.*

Definition 6. *If a solution to an n-th order ordinary differential equations is free of arbitrary constants, then it is called a particular solution to the differential equation.*

Question: Does a given differential equation have always a solution over an interval?

Example 4. *The equation* $y' = 2x$ *is defined for all* x *and has*

$$y(x) = x^2 + c \tag{1.2}$$

as its general solution. To find particular solution that satisfies the initial condition $y(2) = 3$, *and substitute in* (1.2) *and solve for* c, *we get* $c = -1$ *and we conclude that* $y = x^2 - 1$.

Definition 7. *A singular solution is a solution of differential equation which can not obtain from the general solution by giving values to the arbitrary constants. Also its called envelope of solutions.*

Example 5. *For the differential equation* $y^2 + x^2 \frac{dy}{dx} = 0$, *we found that the solution* $y(x) \equiv 0$ *was a singular point and this solution can not be obtain from the general solution* $y(x) = \frac{x}{(cx-1)}$ *by any choice of the constant* c.

Definition 8. *If a relation involving a certain set of constants which is a general solution of a differential equation, the constants are called arbitrary constants, and if these constants can not be replaced by a smaller number of constants, so such constants are called essential arbitrary constants.*

1.4 The elimination of essential arbitrary constants

We now find the differential equation if its general solution is known. We start with a relation involving essential arbitrary constants, and, by elimination of these constants, come to a differential equation. Since each differentiation yields a new relation, the number of derivatives that need be used is the same as the number of essential constants to be eliminated.

We shall in each case determine the differential equation that is
1) Of order equal to the number of essential constants in the given relation.
2) Free from essential constants.

Example 6. *Find the differential equation from the relation*

$$y = Ae^{3x} + Be^{-2x} + Ce^{2x},$$

where A, B and C are arbitrary constants.

Solution: We have

$$y = Ae^{3x} + Be^{-2x} + Ce^{2x}, \tag{1.3}$$

then

$$y' = 3Ae^{3x} - 2Be^{-2x} + 2Ce^{2x}. \tag{1.4}$$

We may eliminate one of arbitrary essential constants, say B, by multiplying equation (1.3) by 2 and adding the result to equation (1.4), thus obtaining

$$2y + y' = 5Ae^{3x} + 4Ce^{2x}. \tag{1.5}$$

Now, differentiating (1.5), we get:

$$2y' + y'' = 15Ae^{3x} + 8Ce^{2x}. \tag{1.6}$$

Multiplying equation (1.5) by 3 and subtracting equation (1.6), we see

$$6y + y' - y'' = 4Ce^{2x}. \tag{1.7}$$

having one essential constant. Differentiating equation (1.7), yields

$$6y' + y'' - y''' = 8Ce^{2x}. \tag{1.8}$$

Finally, multiplying equation (1.7) by 2 and subtracting equation (1.8), we eliminate C and find

$$y''' - 3y'' - 4y' + 12y = 0,$$

which is a third order differential equation having solution (1.3).

Example 7. *Find the differential equation from the relation*

$$y = A\cos\alpha x + B\sin\alpha x,$$

where A and B are arbitrary constants and α being a fixed number.

Solution: We take the derivative

$$y = A\cos\alpha t + B\sin\alpha t, \tag{1.9}$$

with respect to t, we have

$$y' = -\alpha A\sin\alpha t + \alpha B\cos\alpha t.$$

Again take the derivative of this equation, we get

$$y'' = -\alpha^2 A\cos\alpha t - \alpha^2 B\sin\alpha t. \tag{1.10}$$

Adding equations (1.9) and (1.10), yielding

$$y'' + \alpha^2 y = 0.$$

which is the desired differential equation.

Homework 3. *1) Find a differential equation for the family of all circles having radius* 1 *and centre anywhere in the xy-plane.*
2) Eliminate the constant a from the equation $(x-a)^2 + y^2 = a^2$.
3) Eliminate α and β from the relation $x = \beta\cos(\omega t + \alpha)$, in which ω is a parameter (not to be eliminate).

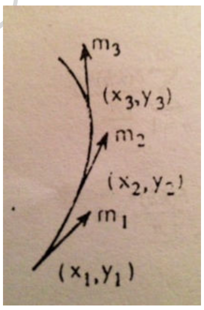
1.5 Geometrical interpretation of differential equations

Consider a differential equation

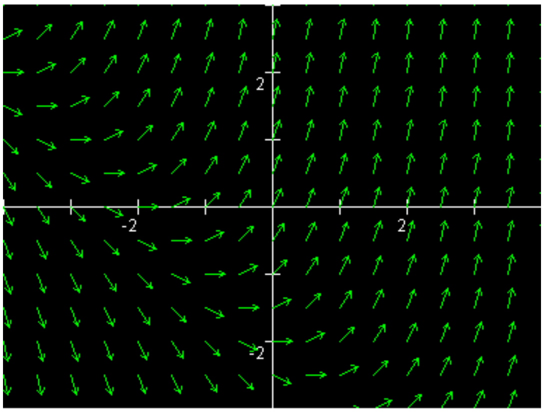
$$\frac{dy}{dx} = f(x, y), \tag{1.11}$$

which is a first order and first degree. Since, from calculus, the derivative is the slope of the tangent line, we interpret this equation geometrically to mean that at any point (x, y) in the plane, the tangent line must have slope $f(x, y)$.

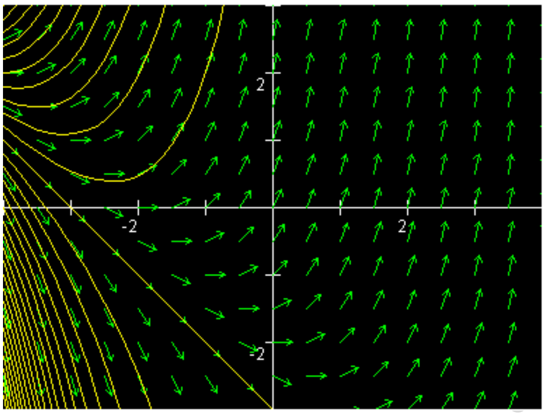
Take any point (x_1, y_1) in xy-plane, equation (1.11) will determine corresponding value of $\frac{dy}{dx}$, say m_1. A point that moves, subject to the restriction imposed by (1.11), on passing through (x_1, y_1) must go in the direction m_1. Let it moves infinitesimal distance to a point (x_2, y_2) and m_2 be the value of $\frac{dy}{dx}$ corresponding to (x_2, y_2) as determined by (1.11). Thence under the same condition to $(x_3.y_3)$ and so on through successive points. In the proceeding thus the point will describe the coordinate of every point of which and the direction of the tangent thereat will satisfy the differential equation (1.11).



For instance, the slope field for $\frac{dy}{dx} = x + y + 2$ is illustrated as follows:



The solution to a differential equation is a curve that is tangent to the arrows of the slope field. Since differential equations are solved by integrating, we call such a curve an integral curve. This picture illustrates some of the integral curves for $\frac{dy}{dx} = x + y + 2$. You can see there are a lot of possible integral curves, infinitely many in fact. This corresponds to the fact that there are infinitely many solutions to a typical differential equation. To specify a particular integral curve, you must specify a point on the curve. Once you specify one specific point, the rest of the curve is determined by following the arrows. This corresponds to finding a particular solution by specifying an initial value.



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Chapter 2

Equations of first order and first degree

2.1 Equations of first order and first degree

We shall study several elementary methods for solving first order ordinary differential equations.

$$y' = \frac{dy}{dx} = f(x, y), \quad \text{(standard, normal or explicit form)} \tag{2.1}$$

or

$$M(x,y)dx + N(x,y)dy = 0, \quad \text{(differential form)} \tag{2.2}$$

or

$$F(x,y,y') = 0, \quad \text{(implicit form)} \tag{2.3}$$

From equation (2.1), we can get equation (2.2) as follows:
If $f(x,y) = -\frac{M(x,y)}{N(x,y)}$, $N(x,y) \neq 0$, equation (2.1) can be written equivalently in differential form as:

$$\begin{aligned}\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} &\Longrightarrow N(x,y)dy = -M(x,y)dx\\ &\Longrightarrow M(x,y)dx + N(x,y)dy = 0.\end{aligned}$$

Remark 4. *We expect that the general solution of* (2.1), (2.2) *or* (2.3) *to have one arbitrary essential constants.*

2.2 Separation of variables (Separable differential equations)

Consider the differential equation

$$y' = \frac{dy}{dx} = f(x,y). \tag{2.4}$$

If we may be able to factor $f(x, y)$ into factors containing only x or y, but not both, then we say $f(x, y)$ is separable. Thus;

$$f(x, y) = p(x)q(y) = \frac{p(x)}{Q(y)}$$

where $q(y) = \frac{1}{Q(y)}$. When the variables are separable, differential equation (2.4) becomes

$$\frac{dy}{dx} = p(x)q(y) \quad \Longrightarrow \quad \frac{dy}{q(y(x))} = p(x)dx. \tag{2.5}$$

Since

$$\frac{dy}{q(y(x))} = \frac{y'(x)}{q(y(x))}dx,$$

and substitute in (2.5), gives

$$\frac{y'(x)}{q(y(x))}dx = p(x)dx.$$

Let $u = y(x)$ and $du = y'(x)dx$, then integration of both sides, we have

$$\int \frac{du}{q(u)} = \int p(x)dx + C$$

where C is a constant.

Remark 5. *It is possibility that either p or q may be constant function.*

Example 8. *Solve the differential equation* $y' = 3x^2e^{-y}$.

Solution: We have

$$\begin{aligned} y' = 3x^2e^{-y} \quad &\Longrightarrow \quad \frac{dy}{dx} = 3x^2e^{-y} \quad \Longrightarrow \quad e^y dy = 3x^2 dx \\ &\Longrightarrow \quad \int e^y dy = \int 3x^2 dx \quad \Longrightarrow \quad e^y = x^3 + C. \end{aligned}$$

By taking the natural logarithm of both sides, we get

$$y = \ln{(x^3 + C)},$$

which is the general solution where C is an arbitrary essential constant.

Example 9. *Solve the differential equation* $y' = 2(x + y^2x)$.

Solution:

$$\begin{aligned} y' = 2(x + y^2x) \quad &\Longrightarrow \quad \frac{dy}{dx} = 2(x + y^2x) \quad \Longrightarrow \quad dy = 2x(1 + y^2)dx \\ &\Longrightarrow \quad \int \frac{dy}{1 + y^2} = \int 2x dx \quad \Longrightarrow \quad \tan^{-1} y = x^2 + C \\ &\Longrightarrow \quad y = \tan{(x^2 + C)}. \end{aligned}$$

is the general solution where C is an arbitrary essential constant.

Remark 6. *If the differential equation* (2.4) *is of differential for*

$$M(x, y)dx + N(x, y)dy = 0. \tag{2.6}$$

If $M(x,y)$ and $N(x,y)$ are separable with variables x and y, so (2.6) *takes the form*

$$M_1(x)M_2(y)dx + N_1(x)N_2(y)dy = 0. \tag{2.7}$$

Now, equation (2.7) *can be converted into an equation that can be integrated by multiplying its coefficients by $\frac{1}{M_2(y)N_1(x)}$. This yields*

$$\frac{M_1(x)}{N_1(x)}dx + \frac{N_2(y)}{M_2(y)}dy = 0.$$

Thus, the general solution of (2.6), *is determined by the expression*

$$\int \frac{M_1(x)}{N_1(x)}dx + \int \frac{N_2(y)}{M_2(y)}dy = C,$$

where C is arbitrary essential constant.

Example 10. *Solve $s\theta ds + (s^3\theta^3 - 3s^3\theta)d\theta = 0$.*

Solution: $s\theta ds + (s^3\theta^3 - 3s^3\theta)d\theta = 0 \Longrightarrow s\theta ds + s^3\theta(\theta^2 - 3)d\theta = 0$. multiplying both sides by $s^{-3}\theta^{-1}$ and integrating both sides, we have

$$\int s^{-2}ds + \int (\theta^2 - 3)d\theta = C \Longrightarrow -s^{-1} + \frac{1}{3}\theta^3 - 3\theta = C$$

or

$$s = (\frac{1}{3}\theta^3 - 3\theta - C)^{-1},$$

where C is arbitrary essential constant.

Homework 4. *Solve the following differential equations:*

1) $ydx - xdy = xydx$.

2) $(x+y)(dx - dy) = dx + dy$.

3) $x^2(1-y)dx + y^2(1+x)dy = 0$.

4) $3e^x \tan y dx + (1 - e^x)\sec^2 y dy = 0$.

2.3 Homogeneous differential equations

Definition 9. *A function $f(x, y)$ is said to be homogeneous function of degree k in x and y, if, and only if,*

$$f(\lambda x, \lambda y) = \lambda^k f(x, y), \quad \lambda \in \mathbb{R}.$$

Note that $f(x, y) = x^2 + y^2$ and $g(x, y) = xy$ are homogeneous functions of degree 2, since

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2 = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2(x^2 + y^2) = \lambda^2 f(x, y)$$

and

$$g(\lambda x, \lambda y) = (\lambda x)(\lambda y) = \lambda^2 xy = \lambda^2 g(x, y).$$

Definition 10. *A function $f(x, y)$ is said to be homogeneous function of degree 0, if it can be written in the form $F(\frac{y}{x})$ or $F(\frac{x}{y})$.*

The functions $e^{\frac{y}{x}}$ and $(2x+y)/y$ are homogeneous function of degree 0 and $\frac{1}{\sqrt{x+y}}$ is homogeneous function of degree $-\frac{1}{2}$.

Definition 11. *A differential equation of the form*

$$M(x, y)dx + N(x, y)dy = 0,$$

is called homogeneous differential equation, if both M and N are homogeneous functions of the same degree.

Remark 7. *If the right hand side of the equation*

$$\frac{dy}{dx} = f(x, y),$$

can be expressed as a function of the ratio $\frac{y}{x}$ or $\frac{x}{y}$ only, then the equation is said to be homogeneous.

Homework 5. *Determine whether the following functions are homogeneous:*

1) $f(x,y) = 2y^3 \exp\left(\frac{y}{x}\right) - \frac{x^4}{x+3y}$.

2) $g(x,y) = xe^{y/x} - y$.

3) $h(x,y) = \frac{x}{\sqrt{(x^2+y^2)}}$.

4) $k(x,y) = \frac{y}{x+\sqrt{xy}}$.

5) $\ell(x,y) = \frac{2xe^{y/x}}{(x^2+y^2\sin(x/y))}$.

Theorem 1. *If the coefficients M and N are homogeneous of the same degree in x and y, then, the differential equation*

$$M(x,y)dx + N(x,y)dy = 0, \tag{2.8}$$

can be reduced to separable equation by the transformation $v = \frac{y}{x}$.

Proof: Since M and N are homogeneous functions of the same degree, say k, we have:

$$M(x,y) = M(x,xv) = x^k M(1,v),$$
$$N(x,y) = N(x,xv) = x^k N(1,v),$$

and hence

$$\frac{M(x,y)}{N(x,y)} = \frac{M(1,v)}{N(1,v)} = f(v). \tag{2.9}$$

From $v = \frac{y}{x}$, we have

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Therefore, equation (2.8) transform into

$$v + x\frac{dv}{dx} = -f(v).$$

Consequently,

$$\frac{dv}{f(v) + v} + \frac{dx}{x} = 0$$

This is a separable equation.

Remark 8. *The normal form of equation* (2.8) *is*

$$y' = \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)},$$

where the function $-\frac{M(x,y)}{N(x,y)} = f(x, y)$ *is homogeneous of degree* 0*. Thus, the substitution* $y = vx$ *will convert* $y' = f(x, y)$ *into an equation whose variables are separable whenever* f *is homogeneous of degree* 0*.*

Homework 6. *Suppose that* $\frac{dy}{dx} = g(\frac{y}{x})$*, derive a formula for solving this type of differential equation.*

Example 11. *Solve the differential equation*

$$y^2 dx - x(x+y)dy = 0.$$

Solution: Clearly, this differential equation is not separable. Since the coefficients are homogeneous functions of degree 2, so let

$$
\begin{aligned}
v = \frac{y}{x} &\implies y = vx \implies dy = vdx + xdv \\
&\implies v^2x^2dx - x(x+vx)(vdx + xdv) = 0 \\
&\implies v^2x^2dx - (x^2 + vx^2)(vdx + xdv) = 0 \\
&\implies v^2x^2dx - vx^2dx - x^3dv - x^2v^2dx - vx^3dv = 0 \\
&\implies -vx^2dx - x^3(1+v)dv = 0 \\
&\implies \frac{dx}{x} = -\frac{1+v}{v}dv \\
&\implies \frac{dx}{x} = -(\frac{1}{v} + 1)dv \\
&\implies \ln|x| = -\ln|v| - v + c_1 \\
&\implies \ln|x| + \ln|v| = -v + c_1 \\
&\implies \ln|xv| = -v + c_1 \\
&\implies xv = c_2e^{-v}, \quad \text{where } c_2 = e^{c_1},\ c_2 > 0
\end{aligned}
$$

Since $v = \frac{y}{x}$, then $y = c_2e^{-\frac{y}{x}}$ is a general solution where c_2 is an arbitrary essential constant.

Example 12. *Solve the differential equation*

$$2xydy - (x^2 + 3y^2)dx = 0, \quad x > 0.$$

Solution: It is obvious this differential equation is not separable. Now divide its both sides by x^2, then we have

$$2(\frac{y}{x})dx - (1 + \frac{3y^2}{x^2})dx = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1 + 3(\frac{y}{x})^2}{2(\frac{y}{x})}.$$

Since the right hand side of the equation above is a function of $\frac{y}{x}$, then the differential equation is a homogeneous of degree 0.

Let $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, then

$$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v} = \frac{1}{2v} + \frac{3v}{2}$$

Thus,

$$x\frac{dv}{dx} = \frac{1}{2v} + \frac{v}{2} = \frac{1 + v^2}{2v} \quad \Longrightarrow \quad \frac{2vdv}{1 + v^2} = \frac{dx}{x}$$

Therefore,

$$\ln(1 + v^2) = \ln x + C$$

Since $x > 0$, combining logarithms, we get

$$\ln(\frac{1+v^2}{x}) = C \quad \Longrightarrow \quad 1 + (\frac{y}{x})^2 = C_1 x$$

or

$$y^2 = x^2(C_1 x - 1),$$

is the general solution where $C_1 = e^C$ is an arbitrary constant.

Homework 7. *Solve the following differential equations:*

1) $xydx + (x^2 + y^2)dy = 0.$

2) $(x^2 + xy + y^2)dx - xydy = 0.$

3) $y' = \frac{x+y}{x-y}.$

4) $\frac{dy}{dx} = \frac{xe^{y/x}+y}{x}.$

5) $(2x\sinh(\frac{y}{x}) + 3y\cosh(\frac{y}{x}))dx - 3x\cosh(\frac{y}{x})dy = 0.$

2.4 Coefficients linear in the two variables

Consider the differential equation

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0, \tag{2.10}$$

in which the a_i's, b_i's and c_i's are constants. If $c_1 = c_2 = 0$, the (2.10) is a homogeneous differential equation, can be solved by $v = \frac{y}{x}$. We now consider the lines

$$\begin{aligned} a_1x + b_1y + c_1 = 0, \\ a_2x + b_2y + c_2 = 0. \end{aligned} \tag{2.11}$$

There are two cases. First they may intersect or, second they may parallel.

1) If the lines (2.11) are intersect (has a solution) ($slope_1 = -\frac{a_1}{b_1} \neq -\frac{a_2}{b_2} = slope_2$), let the point of intersection be (h, k) and let

$$\begin{aligned} x = u + h \quad &\Longrightarrow \quad dx = du, \\ y = v + k \quad &\Longrightarrow \quad dy = dv. \end{aligned}$$

Substitute theses into equation (2.10), we have

$$[a_1(u+h) + b_1(v+k) + c_1]du + [a_2(u+h) + b_2(v+k) + c_2]dv = 0 \Longrightarrow$$

$$[a_1u+b_1v+(a_1h+b_1k+c_1)]du+[a_2u+b_2v+(a_2h+b_2k+c_2)]dv = 0. \tag{2.12}$$

Since the point (h, k) is the intersection point of lines in (2.11), so,

$$a_1h + b_1k + c_1 = 0, \quad a_2h + b_2k + c_2 = 0.$$

Then (2.12) reduces to

$$(a_1u + b_1v)du + (a_2u + b_2v)dv = 0.$$

which is a homogeneous differential equation and can be solved by $w = \frac{u}{v}$.

Example 13. *Solve the differential equation*

$$(x + 2y - 4)dx - (2x + y - 5)dy = 0. \tag{2.13}$$

Solution: This differential equation, clearly, is not separable and not homogeneous differential equation. It is a differential equation with coefficients linear.

Let

$$\begin{aligned} x + 2y - 4 = 0 \quad &\Longrightarrow \quad -2x - 4y + 8 &= 0 \\ 2x + y - 5 = 0 \quad &\Longrightarrow \quad 2x + y - 5 &= 0 \end{aligned}$$

adding the last two equations, we have: $-3y + 3 = 0 \quad \Longrightarrow \quad y = 1.$ Substitute the value of $y = 1$ in $x + 2y - 4 = 0$, we see $x = 2$. Thus,

$(2,1)$ is the only point of intersection. Put

$$x = u + 2 \quad \Longrightarrow \quad du = dx$$

$$y = v + 1 \quad \Longrightarrow \quad dy = dv$$

in (2.13), then we obtain

$$[(u+2) + 2(v+1) - 4]du - [2(u+2) + (v+1) - 5]dv = 0 \quad \Longrightarrow$$

$$(u + 2v)du - (2u + v)dv = 0, \tag{2.14}$$

which is a homogeneous differential equation of degree one in u and v.

Let $z = \frac{u}{v} \quad \Longrightarrow \quad u = zv \quad \Longrightarrow \quad du = vdz + zdv.$

Substitutes in (2.14), yields

$$(vz + 2v)(vdz + zdv) - (2vz + v)dv = 0 \quad \Longrightarrow$$

$$(z + 2)(vdz + zdv) - (2z + 1)dv = 0 \quad \Longrightarrow$$

$$2vdz + z^2dv + 2vdz + 2zdv - 2zdv - dv = 0 \quad \Longrightarrow$$

$$v(z + 2)dz + (z^2 - 1)dv = 0 \quad \Longrightarrow \quad (\frac{z+2}{z^2-1})dz + \frac{dv}{v} = 0$$

By using partial fraction, we see

$$\frac{z+2}{z^2-1} = \frac{A}{z+1} + \frac{B}{z-1} = \frac{A(z-1)+B(z+1)}{z^2-1} \quad \Longrightarrow$$

$A+B=1$ and $B-A=2 \implies B=2+A \implies A+A+2=1 \implies 2a=-1 \implies A=-\frac{1}{2} \implies B=\frac{3}{2}$.

Therefore,

$$-\frac{\frac{1}{2}}{z+1}dz + \frac{\frac{3}{2}}{z-1}dz + \frac{dv}{v} = 0 \implies \frac{3}{z-1}dz - \frac{1}{z+1}dz + 2\frac{dv}{v} = 0$$

$$\implies 3\ln|z-1| - \ln|z+1| + 2\ln|v| = c \implies \ln[\frac{(z-1)^3v^2}{z+1}] = c$$

$$\implies \frac{(z-1)^3v^2}{z+1} = c_1 \quad \text{where } c_1 = e^c$$

Thus,

$$(z-1)^3v^2 = c_1(z+1) \implies (vz-v)^3 = c_1(vz+v)$$

Since $u=vz$, then $(u-v)^3 = c_1(u+v)$. But $u=x-2$ and $v=y-1$.

So

$$(x-y-1)^3 = c_1(x+y-3)$$

is a general solution where c_1 is an arbitrary constant.

2) If the lines (2.11) do not intersect, i.e. they are parallel (they have no solution), ($slope_1 = -\frac{a_1}{b_1} = -\frac{a_2}{b_2} = slope_2$) then there exists a constant k such that

$$a_2x + b_2y = k(a_1x + b_1y) \quad \text{(why?)}$$

which implies that

$$[a_1x + b_1y + c_1]dx + [k(a_1x + b_1y) + c_2]dy = 0,$$

Let $w = a_1x + b_1y$. Then the new equation, in w and x or in w and y, is one with variables separable, since its coefficients contains only w and constants.

Example 14. *Solve the differential equation*

$$(2x + 3y - 1)dx + (2x + 3y + 2)dy = 0,$$

with the condition that $y(1) = 3$*, i.e.* $y = 3$ *when* $x = 1$

Solution: The lines

$$2x + 3y - 1 = 0,$$

$$2x + 3y + 2 = 0,$$

are parallel as they have not got any solution. Let

$$2x + 3y = w \implies 2dx + 3dy = dw \implies 2dx = dw - 3dy.$$

So,

$$(w-1)(dw - 3dy) + 2(w+2)dy = 0 \implies$$

$$(w-1)dw - 3(w-1)dy + 2wdy + 4dy = 0 \implies$$

$$(w-1)dw - 3wdy + 3dy + 2wdy + 4dy = 0 \implies$$

$$(w-1)dw - (w-7)dy = 0 \implies \frac{w-1}{w-7}dw - dy = 0$$

$$\frac{w}{w-7}dw - \frac{1}{w-7}dw - dy = 0 \implies w - y + c + 6\ln|w-7| = 0$$

Pull back the original variables to the last solution we have

$$2x + 2y + c + 6\ln|2x + 3y - 7| = 0,$$

which is the general solution where c is an arbitrary constant.

But $y(1) = 3$, so $c = -8 - 6\ln 4$. Hence the particular solution is

$$x + y - 4 = -3\ln[\frac{1}{4}(2x + 3y - 7)].$$

Homework 8. *Solve the following differential equations:*

1) $(y - 2)dx - (x - y - 1)dy = 0.$

2) $(x - 4y - 9)dx + (4x + y - 2)dy = 0.$

3) $(x + y - 1)dx + (2x + 2y + 1)dy = 0.$

2.5 Exact differential equations

Definition 12. *The total differential of a function $\phi(x, y)$ is denoted by $d\phi$ and defined as*

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy.$$

Definition 13. *If a function $\phi(x, y)$ exists such that $\frac{\partial \phi}{\partial x} = M(x, y)$ and $\frac{\partial \phi}{\partial y} = N(x, y)$, then the differential equation*

$$M(x, y)dx + N(x, y)dy = 0,$$

is said to be an exact differential equations and the general solution is of the form $\phi(x, y) = C$, where C is an arbitrary constant.

Test for exactness The differential equation

$$M(x, y)dx + N(x, y)dy = 0,$$

is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Theorem 2. *If M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous functions in x and y, then a necessary and sufficient condition that*

$$M(x, y)dx + N(x, y)dy = 0,$$

be exact equation is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Remark 9. *Theorem 2 means that:*
1) If $M(x, y)dx + N(x, y)dy = d\phi = 0$, (i.e. the equation is exact), then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (necessary).
2) If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M(x, y)dx + N(x, y)dy = d\phi$ (i.e. the equation is exact), or equivalently, ϕ exists such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (sufficient).

Proof. (Theorem 2) If

$$M(x, y)dx + N(x, y)dy = 0, \tag{2.15}$$

is exact, then from definition of exactness, ϕ exists such that

$$d\phi = M(x,y)dx + N(x,y)dy = 0,$$

$\frac{\partial \phi}{\partial x} = M \quad \Longrightarrow \quad \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y},$
and $\frac{\partial \phi}{\partial y} = N \quad \Longrightarrow \quad \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial N}{\partial x}.$
But from Calculus (Theorem: If the function $\phi(x,y)$ has continuos second derivatives, then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$), we have $\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$ provided these partial derivatives are continuous. Therefore, if (2.15) is an exact equation, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.16}$$

Thus for (2.15) to be exact, it is necessary that (2.16) be satisfied.
Let us now show that if condition (2.16) is holds, then (2.15) is an exact equation.
i.e. we have to show that $\exists$ a function $\phi(x,y)$ s.t. $\frac{\partial \phi}{\partial x} = M$, $\frac{\partial \phi}{\partial y} = N$.
Now $\exists$ a function ϕ s.t. $\frac{\partial \phi}{\partial x} = M$ (is trivial) $\Longrightarrow \phi = \int M dx + f(y)$ where $f(y)$ is an arbitrary function of y. Now we have to show that $\frac{\partial \phi}{\partial y} = N$.

i.e. $\exists$ a function $f(y)$ s.t.

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\Big[\int M dx + f(y)\Big],$$

or

$$\frac{\partial}{\partial y}\Big[\int M dx + f(y)\Big] = N,$$

or

$$f'(y) = N - \frac{\partial}{\partial y}\int M dx.$$

To show this, we only need to prove that

$$N - \frac{\partial}{\partial y}\int M dx \tag{2.17}$$

is a function of y alone. This will be true indeed, if the partial derivative w.r.t. x of (2.17) is zero.

$$\begin{aligned} \frac{\partial}{\partial x}\Big[N - \frac{\partial}{\partial y}\int M dx\Big] &= \frac{\partial}{\partial x}N - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int M dx] \\ &= \frac{\partial}{\partial x}N - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M dx] \\ &= \frac{\partial}{\partial x}N - \frac{\partial}{\partial y}M = 0, \quad \text{(because } \tfrac{\partial M}{\partial y} = \tfrac{\partial N}{\partial x}\text{)}. \end{aligned}$$

The sufficiency is therefore proved.

Example 15. *Solve the differential equation*

$$2xydx + (x^2 + \cos y)dy = 0. \tag{2.18}$$

Solution: Here $M(x, y) = 2xy$ and $N(x, y) = x^2 + \cos y$. Now $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$, then the differential equation (2.18) is exact. Thus, $\exists$ a function ϕ s.t. $\frac{\partial \phi}{\partial x} = M = 2xy$ and $\frac{\partial \phi}{\partial y} = N = x^2 + \cos y$. Now,

$$\frac{\partial \phi}{\partial x} = 2xy \implies \phi = \int 2xy\,dx + f(y) \implies \phi = x^2 y + f(y)$$

where $f(y)$ is an arbitrary function of y. Since

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N = x^2 + \cos y,$$

then

$$x^2 + f'(y) = x^2 + \cos y \quad \implies \quad f'(y) = \cos y \quad \implies \quad f(y) = \sin y.$$

Hence, $\phi = x^2 y + \sin y$ and the general solution is $\phi = x^2 y + \sin y = C$ where C is an arbitrary constant.

Example 16. *Solve*

$$(x + e^y)dx + (xe^y - e^{2y})dy = 0.$$

Solution: Let $M(x, y) = x + e^y$ and $N(x, y) = xe^y - e^{2y}$. Clearly,

$$\frac{\partial M}{\partial y} = e^y \quad \text{and} \quad \frac{\partial N}{\partial x} = e^y \Longrightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So the differential equation is exact. Then

$$\frac{\partial \phi}{\partial y} = N = xe^y - e^{2y} \Longrightarrow \phi = \int (xe^y - e^{2y})dy + g(x) = xe^y - \frac{1}{2}e^{2y} + g(x).$$

Since,

$$\frac{\partial \phi}{\partial x} = e^y + g'(x) = M = x + e^y \implies g'(x) = x \implies g(x) = \frac{x^2}{2}.$$

Therefore, $\phi = xe^y - \frac{1}{2}e^{2y} + \frac{x^2}{2}$ and the general solution is $\phi = k$ where k is an arbitrary constant.

Homework 9. *Solve the following differential equations:*

1) $(\cos x \cos y - \cot x)dx - \sin x \sin y dy = 0.$

2) $2xydx + (x^2 + 1)dy = 0.$

3) $\frac{dy}{dx} = -\frac{3x^2+4xy}{2x^2+2y}.$

4) $y' = (xy^2 - 1)/(1 - x^2y)$.

Remark 10. *Sometimes, exact differential equations can be solve by another method which is called* **"grouping of terms"** *or* **by inspection** *which can be explain in the examples below.*

Example 17. *Solve the following differential equations by* **"grouping of terms"***:*

1) $2xydx + (x^2 + \cos y)dy = 0$.

2) $(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)dy = 0$.

Remark 11. *For inspection method, look for*

1) $xdy + ydx = d(xy)$.

2) $\frac{xdy - ydx}{x^2} = d(\frac{y}{x})$.

3) $\frac{ydx - xdy}{y^2} = d(\frac{x}{y})$.

4) $\frac{xdy - ydx}{x^2 + y^2} = d(\tan^{-1}(\frac{y}{x}))$.

5) $\frac{d(x+y)}{x+y} = d\ln(x + y)$.

It may help to group terms of like degrees.

Solution:

1) We have seen that the first equation is exact in Example 8. Thus we can group term as follows:

$$(2xydx + x^2dy) + \cos ydy = 0 \implies d(x^2y) + d(\sin y) = 0.$$

because $d(x^2y) = 2xydx + x^2dy$ and $d(\sin y) = \cos y$. Then

$$d(x^2y + \sin y) = 0 \implies x^2y + \sin y = C,$$

is the general solution where C is an arbitrary constant.

2) Put $M = 2x^3 - xy^2 - 2y + 3$ and $N = -x^2y - 2x$. Now

$$\frac{\partial M}{\partial y} = -2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = -2xy \implies \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

So the differential equation is exact. We can group as follows:

$$(2x^3 + 3)dx - (xy^2dx + x^2ydy) - (2ydx + 2xdy) = 0,$$

and clearly, this implies that

$$d(\frac{1}{2}x^4+3x)-d(\frac{1}{2}x^2y^2)-2d(xy) = 0 \implies d(\frac{1}{2}x^4+3x-\frac{1}{2}x^2y^2-2xy) = 0,$$

So, $\frac{1}{2}x^4 + 3x - \frac{1}{2}x^2y^2 - 2xy = C$ is a general solution where C is an arbitrary constant.

2.6 Non-exact differential equations (Integrating factors)

When the differential equations

$$M(x, y)dx + N(x, y)dy = 0$$

is not exact differential equation, one can found a function $I(x, y)$ such that the differential equation

$$I(x, y)\, M(x, y)dx + I(x, y)\, N(x, y)dy = 0$$

is exact.The function $I(x, y)$ is called an *integrating factor*.

Definition 14. *A function $I(x, y)$ is said to be an integrating factor for $M(x, y) + N(x, y)y' = 0$, if $I\, Mdx + I\, Ndy = 0$ is an exact differential equation.*

Now, consider the case

$$M(x, y)dx + N(x, y)dy = 0, \tag{2.19}$$

is not exact. We multiply (2.19) by the integrating factor $I(x, y)$. Then from definition of an integrating factor, the new differential equation

$$I\,Mdx + I\,Ndy = 0,$$

is now exact so that

$$\begin{aligned}
\frac{\partial IM}{\partial y} = \frac{\partial IN}{\partial x} \quad &\Longrightarrow \quad I\frac{\partial M}{\partial y} + M\frac{\partial I}{\partial y} = I\frac{\partial N}{\partial x} + N\frac{\partial I}{\partial x} \\
&\Longrightarrow \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = N\frac{\frac{\partial I}{\partial x}}{I} - M\frac{\frac{\partial I}{\partial y}}{I} \\
&\Longrightarrow \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = N\frac{\partial}{\partial x}(\ln I) - M\frac{\partial}{\partial y}(\ln I)
\end{aligned} \tag{2.20}$$

There are many cases:

1) If I is a function of x alone, i.e. $I = I(x) \quad \Longrightarrow \quad \frac{\partial}{\partial y}(\ln I) = 0$ and substitute in (2.20), we have

$$\begin{aligned}
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = N\frac{\partial}{\partial x}(\ln I) \quad &\Longrightarrow \quad \frac{d}{dx}(\ln I(x)) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \\
&\Longrightarrow \quad d(\ln I) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}dx
\end{aligned} \tag{2.21}$$

If the coefficients of dx on the right hand side of (2.21) is a function of x alone (say $f(x)$), then we have

$$\ln I = \int f(x)dx \quad \Longrightarrow \quad I = e^{\int f(x)dx}.$$

2) I is a function of y alone, i.e. $I = I(y) \quad \Longrightarrow \quad \frac{\partial}{\partial x}(\ln I) = 0$

$$\begin{aligned} \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -M\frac{\partial}{\partial y}(\ln I) \quad &\Longrightarrow \quad \frac{d}{dy}(\ln I(y)) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \\ &\Longrightarrow \quad d(\ln I) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}dy \\ &\Longrightarrow \quad I = e^{\int g(y)dy}, \end{aligned}$$

where $g(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$.

Example 18. *Solve the differential equation*

$$ydx + (3 + 3x - y)dy = 0. \tag{2.22}$$

Solution: Here $M = y$ and $N = 3 + 3x - y$. Then $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 3$. Since $\frac{\partial M}{\partial y} = 1 \neq 3 = \frac{\partial N}{\partial x}$, then (2.22) is not exact differential equation.

Now, we compute

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{1-3}{3+3x-y} = -\frac{2}{3+3x-y},$$

which is not a function of x a lone. But

$$\frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) = \frac{3-1}{y} = \frac{2}{y},$$

is a function of y alone. Thus, the integrating factor is

$$I = e^{\int \frac{2}{y}dy} = e^{2\ln y} = e^{\ln y^2} = y^2 = g(y).$$

Multiplying both sides of (2.22) by y^2, we have

$$y^3dy + y^2(3 + 3x - y)dy = 0. \tag{2.23}$$

Now, let $M_1 = y^3$ and $N_1 = y^2(3 + 3x - y)$. Compute $\frac{\partial M_1}{\partial y} = 3y^2$ and $\frac{\partial N_1}{\partial x} = 3y^2$. Since $\frac{\partial M_1}{\partial y} = 3y^2 = \frac{\partial N_1}{\partial x}$, hence equation (2.23) is an exact differential equation. Therefore, ϕ exists such that $\frac{\partial \phi}{\partial x} = M_1$ and $\frac{\partial \phi}{\partial y} = N_1$. Since

$$\frac{\partial \phi}{\partial x} = M_1 = y^3 \quad \Longrightarrow \quad \phi = xy^3 + f(y).$$

where $f(y)$ is an arbitrary function in y. Now, since

$$\frac{\partial \phi}{\partial y} = 3xy^2 + f'(y) = N_1 = 3y^2 + 3xy^2 - y^3 \Longrightarrow f'(y) = 3y^2 - y^3 \Longrightarrow f(y) = y^3 - \frac{y^4}{4}$$

Thus,

$$\phi = xy^3 + y^3 - \frac{y^4}{4} = C \quad \Longrightarrow \quad xy^3 + y^3 - \frac{y^4}{4} = C,$$

is a general solution of (2.22) where C is an arbitrary constant.

Example 19. *Solve the differential equation*

$$(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0. \tag{2.24}$$

Let $M = 4xy + 3y^2 - x$ and $N = x(x + 2y)$. So, $\frac{\partial M}{\partial y} = 4x + 6y$ and $\frac{\partial N}{\partial x} = 2x + 2y$. Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then (2.24) is not exact differential equation.

We compute

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{2x + 4y}{x(x + 2y)} = \frac{2}{x} = f(x),$$

which is a function of x alone. Then, an integrating factor for equation (2.24) is

$$I = exp(\int f(x)dx) = e^{2\int \frac{dx}{x}} = e^{2\ln|x|} = x^2.$$

Multiplying (2.24) by integrating factor x^2, we have

$$(4x^3y + 3x^2y^2 - x^3)dx + x^3(x + 2y)dy = 0, \qquad (2.25)$$

which is now an exact differential equation because

$$\frac{\partial M_1}{\partial y} = 4x^3 + 6x^2y = \frac{\partial N_1}{\partial x}.$$

Therefore, $\exists$ a function ϕ such that $\frac{\partial \phi}{\partial x} = M_1$ and $\frac{\partial \phi}{\partial y} = N_1$. As we have

$$\frac{\partial \phi}{\partial x} = M_1 = 4x^3y + 3x^2y^2 - x^3 \implies \phi = x^4y + x^3y^2 - \frac{x^4}{4} + f(y),$$

where $f(y)$ is an arbitrary function in y. Now, since

$$\frac{\partial\phi}{\partial y} = x^4+2x^3y+f'(y) = N_1 = x^4+2x^3y \Longrightarrow f'(y) = 0 \Longrightarrow f(y) = C_1,$$

where C_1 is an arbitrary constant. Hence

$$\phi = x^4y + x^3y^2 - \frac{x^4}{4} + C_1 = C \implies x^3(4xy + 4y^2 - x) = K,$$

is a general solution where $K = 4(C - C_1)$ is an arbitrary constant.

Homework 10. *Solve the following differential equation:*

1) $(2y - 3x)dx + xdy = 0.$

2) $(x^2 + y^3 + 1)dx + x(x - 2y)dy = 0.$

3) $y(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0.$

4) $y(2xy + 1)dx + x(1 + 2xy - x^3y^3)dy = 0.$

5) $xdx + ydy + 4y^3(x^2 + y^2)dy = 0.$

6) $(y + x^3y + 2x^2)dx + (x + 4xy^4 + 8y^3)dy = 0.$

2.7 Linear Differential equations of first order

Definition 15. *Any differential equation of any order is said to be linear, if the dependent variable and all its derivatives which appear in the differential equation are of degree one and not product of each other.*
The general form is

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x), \quad (2.26)$$

where $a_0, a_1, \ldots, a_n$ and $f(x)$ are functions of the independent variable x.
If $f(x) = 0$ for all x, the equation (2.26) is called homogeneous linear differential equation. If $f(x) \neq 0$, then equation (2.26) is called non-homogeneous (inhomogeneous) linear differential equation.
If all $a_0, a_1, \ldots, a_n$ are constants, then (2.26) is called linear differential equation with constant coefficients. If at least one of $a_0, a_1, \ldots, a_n$ is a function of x, then (2.26) is called linear differential equation with variable coefficients.

2.7.1 First order linear differential equation

The general first order linear differential equation is

$$a_1(x)\frac{dy}{dx} + a_0(x)y = h(x), \tag{2.27}$$

whenever $a_1(x) \neq 0$ for all x in an interval I. We dividing the coefficients of the equation (2.27) by $a_1(x)$ and rewrite in the normal (standard) form as

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{2.28}$$

where $P(x) = \frac{a_0(x)}{a_1(x)}$ and $Q(x) = \frac{h(x)}{a_1(x)}$.
In the homogeneous case $Q(x) = 0$ on I, equation (2.28) can be solved separately, thus,

$$\frac{dy}{y} = -P(x)dx \Longrightarrow \ln y = -\int P(x)dx + C \Longrightarrow y_h = Ke^{-\int P(x)dx},$$

where $K = e^C$ is an arbitrary essential constant. The differential form of (2.28) is

$$dy + (P(x)y - Q(x))dx = 0.$$

Let $M(x,y) = P(x)y - Q(x)$ and $N(x,y) = 1$. Now

$$\frac{\partial M}{\partial y} = P(x) \neq 0 = \frac{\partial N}{\partial x}.$$

Compute

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{P(x) - 0}{1} = f(x),$$

which is a function of x alone. Thus there exists an integrating factor $I = I(x)$.

If such an I exists, we must have

$$\frac{\partial}{\partial x} I(x) = \frac{\partial}{\partial y}[I(x)P(x)y - I(x)Q(x)].$$

Since $I = I(x)$ is a function only of variable x, so $\frac{\partial}{\partial x} I(x) = \frac{d}{dx} I(x)$. So

$$\frac{\partial}{\partial y}[I(x)P(x)y - I(x)Q(x)] = I(x)P(x)$$

and I must satisfy the differential equation

$$\frac{dI}{dx} = I(x)P(x).$$

Therefore,

$$\frac{dI}{I} = P(x)dx \Longrightarrow I = e^{\int P(x)dx},$$

is an integrating factor of (2.28) When both sides of (2.28) multiply by I, we obtain

$$\begin{aligned} e^{\int P(x)dx}\Big(\frac{dy}{dx}+P(x)y=Q(x)\Big) &\Longrightarrow e^{\int P(x)dx}\frac{dy}{dx}+e^{\int P(x)dx}P(x)y=e^{\int P(x)dx}\,Q(x)\\ &\Longrightarrow \frac{d}{dx}\Big(e^{\int P(x)dx}y\Big)=e^{\int P(x)dx}\,Q(x)\\ &\Longrightarrow ye^{\int P(x)dx}=\int e^{\int P(x)dx}\,Q(x)dx\\ &\Longrightarrow y_p=\frac{\int e^{\int P(x)dx}\,Q(x)dx}{e^{\int P(x)dx}} \end{aligned}$$

is a particular solution of (2.28).

Since the general solution of (4.30) is obtained by summing y_h and y_p, so, we have

$$y=\frac{\int e^{\int P(x)dx}\,Q(x)dx+K}{e^{\int P(x)dx}}.$$

Example 20. *Solve the differential equation*

$$y'+xy=2x.$$

Solution: This is a first order linear differential equation with $P(x) = x$ and $Q(x) = 2x$, then the general solution is given by

$$y = \frac{\int e^{\int P(x)dx} Q(x)dx + K}{e^{\int P(x)dx}} \implies y = \frac{\int e^{\int xdx} 2xdx + K}{e^{\int xdx}}$$
$$\implies y = \frac{\int e^{x^2/2} 2xdx + K}{e^{x^2/2}}$$
$$y = e^{-x^2/2}(2e^{x^2/2} + K)$$
$$\implies y = 2 + Ke^{-x^2/2},$$

where K is an arbitrary constant.

Example 21. *Solve the differential equation*

$$2(y - 4x^2)dx + xdy = 0.$$

Solution: In standard (normal) form is

$$\frac{dy}{dx} + \frac{2}{x}y = 8x,$$

where $x \neq 0$. The integrating factor is

$$I = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2.$$

Multiply equation above by x^2, we have

$$x^2\frac{dy}{dx} + 2xy = 8x^3 \implies \frac{d}{dx}(x^2y) = 8x^3$$
$$\implies x^2y = 2x^4 + C$$
$$y = 2x^2 + \frac{C}{x^2}$$

is a general solution where C is an arbitrary constant.

Remark 12. *The differential equation of the form*

$$\frac{dx}{dy} + P(y)x = Q(y),$$

is a linear differential equation of the first order and its general solution is

$$x = \frac{\int e^{\int P(y)dy}\, Q(y)dy + K}{e^{\int P(y)dy}}.$$

Homework 11. *Solve the following differential equations (Find the general solution of the following):*

1) $y\frac{dx}{dy} + 2x = y^3$.

2) $x\frac{dy}{dx} + y = x$.

3) $y' + \tan(x)\, y = \cos^2(x)$, *over the interval* $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

4) $3xy' - y = \ln(x) + 1$, $x > 0$ *satisfying* $y(1) = -2$.

2.8 Equations reducible to linear Differential equations of first order

Sometimes equations which are not linear can be reduced to the linear form by suitable transformations. One such equations is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

which is known as Bernoulli equation.

2.8.1 Bernoulli equations

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \tag{2.29}$$

is called a Bernoulli differential equation.

If $n = 1$, the Bernoulli equation is separable; while $n = 0$, it is a linear differential equation of first order.

When $n \neq 0$ and $n \neq 1$, the substitution $v = y^{1-n}$ reduces equation

(2.29) to a first order linear differential equation. Since $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

From equation (2.29), we have

$$y^{-n}\,\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

Finally,

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x),$$

which is a first order linear differential equation in the variables v, x and the general solution can be given by

$$v = y^{1-n} = \frac{\int e^{\int(1-n)P(x)dx}\,(1-n)Q(x)dx + K}{e^{\int(1-n)P(x)dx}}.$$

where K is an arbitrary constant.

Example 22. *Solve*

$$\frac{dy}{dx} - y = e^{-x}y^2. \tag{2.30}$$

Solution: This differential equation is a **Bernoulli** differential equation with $P(x) = -1$, $Q(x) = e^{-x}$ and $n = 2$. Let

$$v = y^{1-n} = y^{1-2} = y^{-1} \quad \Longrightarrow \quad \frac{dv}{dx} = -y^{-2}\,\frac{dy}{dx},$$

but from equation (2.30), we have $\frac{dy}{dx} = e^{-x}y^2 + y$, so

$$\frac{dv}{dx} = -y^{-2}(e^{-x}y^2 + y) = -e^{-x} - y^{-1} \quad \Longrightarrow \quad \frac{dv}{dx} + v = -e^{-x},$$

which is a linear differential equation of order one. Here for this linear differential equation $P(x) = 1$ and $Q(x) = -e^{-x}$. The general solution is then given by

$$\begin{aligned} v = y^{-1} &= \frac{\int e^{\int P(x)dx}\, Q(x)dx + K}{e^{\int P(x)dx}} = \frac{\int e^{\int dx}\,(-e^{-x})dx + K}{e^{\int dx}} \\ &= \frac{\int e^{x}\,(-e^{-x})dx + K}{e^{x}} = \frac{-\int dx + K}{e^{x}} \\ &= \frac{-x + K}{e^{x}} \quad \Longrightarrow \quad y = \frac{e^{x}}{K - x} \end{aligned}$$

where K is an arbitrary essential constant.

Homework 12. *Solve the following differential equations:*

1) $y(6y^2 - x - 1)dx + 2xdy = 0.$

2) $\frac{dy}{dx} + y = (xy)^2.$

3) $xy - \frac{dy}{dx} = y^3 e^{-x^3}.$

2.8.2 Riccati equations

The nonlinear equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^2 + R(x), \tag{2.31}$$

is known as a *Riccati* equation. If $R(x) \equiv 0$, then the Riccati equation becomes a special case of the Bernoulli equation; if $Q(x) \equiv 0$, then equation (2.31) is a linear first order differential equation.

If y_1 is a known function that satisfies (2.31), then the substitution

$$y = y_1 + \frac{1}{u}, \tag{2.32}$$

transform (2.31) into a first order linear differential equation that is linear in u. Here $u = u(x)$ is an unknown function to be determined. Differentiating equation (2.32), we have

$$y' = y_1' - \frac{1}{u^2}u',$$

and substitute in (2.31), we get

$$y_1' - (\frac{1}{u^2})u' + P(x)(y_1 + \frac{1}{u}) = Q(y_1^2 + \frac{2y_1}{u} + \frac{1}{u^2}) + R.$$

Since y_1 is a solution, then it satisfies equation (2.31), so

$$y_1' + Py_1 = Qy_1^2 + R.$$

Now,

$$\begin{aligned}
&y_1' - (\frac{1}{u^2})u' + Py_1 + P\,\frac{1}{u} = Qy_1^2 + Q\,\frac{2y_1}{u} + Q\,\frac{1}{u^2} + R,\\
\Longrightarrow &-(\frac{1}{u^2})u' + (y_1' + Py_1) + P\,\frac{1}{u} = Qy_1^2 + Q\,\frac{2y_1}{u} + Q\,\frac{1}{u^2} + R'\\
\Longrightarrow &-(\frac{1}{u^2})u' + (Qy_1^2 + R) + P\,\frac{1}{u} = Qy_1^2 + Q\,\frac{2y_1}{u} + Q\,\frac{1}{u^2} + R,\\
\Longrightarrow &-(\frac{1}{u^2})u' + P\,\frac{1}{u} = -Qy_1^2 - R + Qy_1^2 + Q\,\frac{2y_1}{u} + Q\,\frac{1}{u^2} + R,\\
\Longrightarrow &-(\frac{1}{u^2})u' = (2y_1Q - P)\frac{1}{u} + Q\,\frac{1}{u^2},\\
\Longrightarrow &\frac{du}{dx} + (2y_1(x)Q(x) - P(x))u = -Q(x),
\end{aligned}$$

and this is a first order linear differential equation in variables x and u. After solving this linear equation, use $u^{-1} = y - y_1 (y = y_1 + \frac{1}{u})$.

Example 23. *Solve*

$$y' - (\frac{1}{x})y = 1 - (\frac{1}{x^2})y^2, \quad x > 0. \tag{2.33}$$

Solution: It is a Riccati equation in which $P(x) = -\frac{1}{x}$, $Q(x) = -\frac{1}{x^2}$ and $R(x) = 1$. It is easy to see that $y_1 = x$ is a solution of (2.33). Let

$$y = y_1 + \frac{1}{u} = x + \frac{1}{u} \quad \Longrightarrow \quad \frac{du}{dx} + (2x(-\frac{1}{x^2}) - (-\frac{1}{x}))u = -(-\frac{1}{x^2})$$

$$\Longrightarrow \frac{du}{dx} - (\frac{1}{x})u = \frac{1}{x^2}, \tag{2.34}$$

which is a first order linear differential equation and clearly the integrateing factor is

$$I = e^{\int -\frac{dx}{x}} = e^{-\ln x} = x^{-1}.$$

Multiplying both sides of (2.34) by integrating factor $I = x^{-1}$, we have

$$\begin{aligned} x^{-1}\frac{du}{dx} - x^{-1}(\frac{1}{x})u = x^{-1}\frac{1}{x^2} \quad &\Longrightarrow \quad \frac{d}{dx}(\frac{u}{x}) = \frac{1}{x^3} \\ &\Longrightarrow \quad \frac{u}{x} = \int \frac{1}{x^3}dx = -\frac{1}{2x^2} + C = \frac{2x^2C - 1}{2x^2} \\ &\Longrightarrow \quad \frac{1}{u} = \frac{2x}{2x^2C - 1}. \end{aligned}$$

Since

$$y = x + \frac{1}{u} \quad \Longrightarrow \quad \frac{1}{u} = y - x \quad \Longrightarrow \quad y = x + \frac{2x}{2x^2C - 1},$$

is the general solution where C is an arbitrary constant.

Homework 13. *Solve the Riccati equation*

$$\frac{dy}{dx} = x^3(y - x)^2 + \frac{y}{x},$$

if $y_1(x) = x$ is a particular solution of it.

2.9 Substitution suggested by the equation

The basic idea is always to look at the form of the equation. See if it suggests anything. Do not be afraid to play around with the equation and see if you can make it simpler.

Example 24. *Solve* $\quad (x + 2y - 1)dx + 3(x + 2y)dy = 0.$

Solution: Let $v = x + 2y \implies dv = dx + 2dy \implies dx = dv - 2dy$, so equation above becomes

$$(v-1)(dv - 2dy) + (3v)dy = 0 \quad \implies \quad (v-1)dv + (v+2)dy = 0,$$

which is a separable equation, then

$$(\frac{v-1}{v+2})dv + dy = 0 \implies (1 - \frac{3}{v+2})dv + dy = 0 \implies v - 3\ln|v+2| + y + c = 0.$$

Since $v = x + 2y$, then

$$x + 3y + c = 3\ln|x + 2y + 2|,$$

is the general solution where c is an arbitrary constant.

Example 25. *Solve* $\quad (1 + 3x\sin y)dx - x^2\cos y dy = 0.$

Solution: Let $u = \sin y$, then $du = \cos y\, dy$ and substitute in equation above, we have

$$(1 + 3x\,u)dx - x^2 du = 0 \quad \implies \quad \frac{du}{dx} - \frac{3}{x}u = \frac{1}{x^2},$$

which is a first order linear differential equation in variables u and x with $P(x) = -\frac{3}{x}$ and $Q(x) = \frac{1}{x^2}$. The general solution is given by

$$u = \sin y = \frac{\int e^{-\int \frac{3}{x}dx}(\frac{1}{x^2})dx + K}{e^{-\int \frac{3}{x}dx}}$$

Evaluate the integrals and simplify, you should see that

$$4x \sin y = cx^4 - 1,$$

where c is an arbitrary constant.

Homework 14. *Solve the following differential equations:*

1) $\frac{dy}{dx} = -\frac{x^2+2xy+y^2}{1+(x+y)^2}$.

2) $\frac{dy}{dx} - (3x - 2y)^3 = 0$.

2.10 Simultaneous first order differential equations

We study differential equations containing one independent variable, but with two or more dependent variable.

A system of ordinary differential equations which contain one independent variable and the number of dependent variables equal the number of differential equations is called simultaneous differential equations. The general form of simultaneous equations of two dependent variables x, y is

$$\frac{dy}{dt} = f(x, y, t),$$
$$\frac{dx}{dt} = g(x, y, t).$$

Example 26. *Solve the simultaneous differential equations*

$$\frac{dy}{dt} = \frac{t}{x}, \tag{2.35}$$
$$t\frac{dx}{dt} + 2x = x^2 t. \tag{2.36}$$

Solution: First we solve (2.36), because it is contains only two variables x, t, so

$$\frac{dx}{dt} + \frac{2}{t}x = x^2 \tag{2.37}$$

is a bernoulli equation with $P(t) = \frac{2}{t}$, $Q(t) = 1$ and $n = 2$. Let $z = x^{1-n} = x^{1-2} = x^{-1}$, then $\frac{dz}{dt} = -\frac{1}{x^2}\frac{dx}{dt}$. But from (2.37) we have $\frac{dx}{dt} = x^2 - \frac{2}{t}x$, so

$$\frac{dz}{dt} = -\frac{1}{x^2}(x^2 - \frac{2}{t}x) = -1 + \frac{2}{x\,t} = -1 + \frac{2}{t}z \implies \frac{dz}{dt} - \frac{2}{t}z = -1.$$

is a linear first order differential equation with $P(t) = -\frac{2}{t}$ and $Q(t) = -1$, so

$$z = \frac{\int e^{\int P(t)dt} Q(t)dt + C_1}{e^{\int P(t)dt}} = \frac{\int e^{\int -\frac{2}{t}dt} (-1)dt + C_1}{e^{\int -\frac{2}{t}dt}}$$
$$\Longrightarrow \ z = t^2(\int -\frac{1}{t^2}dt + C_1) = t^2(\frac{1}{t} + C_1) = t + C_1t^2.$$

Since $x = \frac{1}{z}$, then $x = \frac{1}{t+C_1t^2}$ and substitute in (2.35), we have

$$\frac{dy}{dt} = \frac{t}{\frac{1}{t+C_1t^2}} = (t^2 + C_1t^3) \Longrightarrow y = \frac{t^3}{3} + \frac{C_1}{4}t^4 + C_2$$

which is a general solution where C_1 and C_2 are arbitrary essential equations.

Example 27. *Solve the simultaneous differential equations*

$$\frac{dx}{dt} + t\frac{dy}{dt} = t, \tag{2.38}$$
$$\frac{dx}{dt} - \frac{dy}{dt} = t - 1. \tag{2.39}$$

Solution: Multiplying equation (2.39) by t and adding with equation (2.38), yields

$$
\begin{aligned}
\frac{dx}{dt} + t\frac{dy}{dt} &= \quad t, \\
t\frac{dx}{dt} - t\frac{dy}{dt} &= \quad t^2 - t.
\end{aligned}
$$

$$(t+1)\frac{dx}{dt} = t^2$$

$$
\begin{aligned}
(t+1)\frac{dx}{dt} = t^2 \implies \frac{dx}{dt} = \frac{t^2}{t+1} \implies x = \int \frac{t^2}{t+1}dt \\
\implies x = \int (t - 1 + \frac{1}{t+1})dt = \frac{t^2}{2} - t + \ln(t+1) + C_1,
\end{aligned}
$$

and substitute in (2.39), we have

$$
\begin{aligned}
t - 1 + \frac{1}{t+1} + t\frac{dy}{dt} = t \implies t\frac{dy}{dt} = 1 - \frac{1}{t+1} = \frac{t}{t+1} \\
\implies \frac{dy}{dt} = \frac{1}{t+1} \implies y = \ln|t+1| + C_2.
\end{aligned}
$$

Note that C_1 and C_2 are arbitrary constants.

Homework 15. *Solve the following differential equations simultaneously:*

1) $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$.

2) $\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = x - y.$

2.11 Applications of First Order Equations

In this section we discuss applications of first order differential equations to problems of mathematics, economy, chemistry and others.

Example 28. *The slope of a curve at any point is the reciprocal of twice the ordinate at the point and it passes through the point (4, 3). Formulate the differential equation and hence find the equation of the curve.*

Solution: Slope of the curve at any point $P(x, y)$ is the slope of the tangent at $P(x, y)$, so

$$\frac{dy}{dx} = \frac{1}{2y}$$

which is a separable differential equation and we have

$$2ydy = dx \implies y^2 = x + c$$

where c is a constant. Since the curve passes through $(4, 3)$, we have

$$9 = 4 + c \implies c = 5$$

Therefore, the equation of the curve is $y^2 = x + 5$.

2.11.1 Exponential Growth and Decay

There are many situations where the rate of change of some quantity x is proportional to the amount of that quantity, that is, $\frac{dx}{dt} = kx$ for some constant k. The general solution is then $x = Ae^{kt}$, for some constant A.

Example 29. *The half-life of radium is 1600 years, that means, it takes 1600 years for half of any quantity to decay. If a sample initially contains 50 g, how long will it be until it contains 45 g?*

Solution: Let $r(t)$ be the amount of radium present at time t in years. Then

$$\frac{dr}{dt} \propto r \implies \frac{dr}{dt} = kr.$$

clearly is a separable differential equation and easily can seen that

$$r(t) = r_0 e^{kt}.$$

At $t = 0$, $r = 50$, so quickly have

$$r(t) = 50e^{kt}.$$

Solving for t gives $t = \frac{\ln(\frac{r}{50})}{k}$. With $r(1600) = 25$, we have $25 = 50e^{1600k}$. Therefore,

$$1600k = \ln(\frac{1}{2}) = -\ln(2).$$

This gives $k = -\frac{\ln(2)}{1600}$. When $r = 45$, we have

$$t = \frac{\ln(r/50)}{k} = \frac{\ln(45/50)}{-\ln(2)/1600} \approx 1600 \times 0.152 = 243.2$$

Example 30. *Suppose the population of a certain country was 23 million in 1990 and 27 million in 1995. Estimate the population in 2000.*

Solution: Consider $P(t)$ represents the size of the population, in millions, t years. So

$$\frac{dP}{dt} \propto P \implies \frac{dP}{dt} = kP.$$

Obviously this equation is a separable equation and can easily seen its solution is of the form

$$P(t) = c\,e^{kt},$$

where k is a parameter and c is an arbitrary constant. Since initially (i.e. $t = 0$) the population was 23 million, so

$$23 = c\, e^{k(0)} \implies c = 23.$$

Now we have

$$P(t) = 23\, e^{kt}.$$

To find k, we note that

$$27 = P(5) = 23\, e^{5k} \implies k = \frac{1}{5}\ln(\frac{27}{23}) = 0.0321,$$

where we have rounded to four decimal places. Hence

$$P(t) = 23\, e^{0.0321t}.$$

The model would predict a population in 2000

$$P(10) = 23\, e^{0.0321(10)} \approx 31.7\, million.$$

Homework 16. *Formulate the following and solve them:*

1) The slope at any point (x, y) of a curve is $\frac{y}{x}$ and it passes through the point $(2, 3)$. Find the equation of the curve.

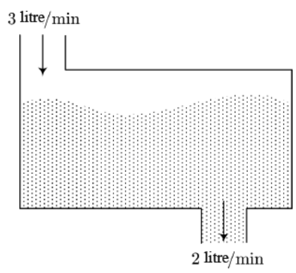
2) During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When 60 grams of A are present, and after 1 hour only 10 grams of A remain unconverted. How much of A is present after 2 hours?

3) Suppose that a petri dish initially contains 3000 bacteria and that 12 minutes later there are 3500 bacteria.
a) Find a formula for the bacteria population t hours (not minutes) after the initial measurement.
b) Predict the bacteria population in 4 hours.

4) Let $N(t)$ be the number of people at time t. Assume that the land is intrinsically capable of supporting L people and that the rate of increase is proportional to both N and $L - N$.

2.11.2 Water Tanks

Example 31. *A tank contains a salt water solution consisting initially of 20 kg of salt dissolved into 10 ℓ of water. Fresh water is being poured* *into the tank at a rate of 3 ℓ/min and the solution (kept uniform by stirring) is flowing out at 2 ℓ/min. Figure below shows this setup. Find the amount of salt in the tank after 5 minutes.*



Fresh water is being poured into the tank as the well-mixed solution is flowing out.

Solution: Let $Q(t)$ denotes the amount of salt (in kilogram) in the tank at a time t (in minutes). The volume of water at time t is

$$10 + 3t - 2t = 10 + t.$$

The concentration at time t is given by

$$\frac{\text{amount of salt}}{\text{volume}} = \frac{Q}{10+t}, \quad \text{kg per litre.}$$

Hence,

$$\frac{dQ}{dt} = -(\text{rate at which salt is leaving}) = -\frac{Q}{10+t} \cdot 2 = -\frac{2Q}{10+t}.$$

Now the solution of the problem is the solution of

$$\frac{dQ}{dt} = -\frac{2Q}{10+t}.$$

evaluated at $t = 5$. We see that it is simply a separable equation. Then

$$\frac{dQ}{Q} = -\frac{2dt}{10+t} \Longrightarrow \int \frac{dQ}{Q} = -2\int \frac{dt}{10+t} \Longrightarrow \ln(|Q|) = -2\ln(|10+t|)+C.$$

It is easy to see that

$$|Q(t)| = A|10+t|^{-2},$$

where $A = e^C$ is a constant. But $Q \geq 0$ (we cannot have a negative amount of salt) and $t \geq 0$ (we do not visit the past), so we remove absolute value signs, giving us

$$Q(t) = A(10+t)^{-2}.$$

Initially, i.e., at $t = 0$, we know that the tank contains 20 kg of salt. Thus, the initial condition is $Q(0) = 20$, and we have

$$Q(t) = 2000(10+t)^{-2} = \frac{2000}{(10+t)^2}. \quad \text{How?}$$

Finally, Evaluating at $t = 5$ gives $Q(5) = \frac{2000}{(15)^2} = \frac{2000}{225} = \frac{80}{9} \approx 8.89$. Therefore, after 5 minutes, the tank will contain approximately 8.89 kg of salt.

Table 2.1: Given: $M(x,y)dx + N(x,y)dy = 0$

When	An integrating factor is
$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = f(x)$	$I = e^{\int f(x)dx}$
$\frac{1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = -g(y)$	$I = e^{\int g(y)dy}$
M and N are homogeneous function of the same degree	$I = \frac{1}{M\,x+N\,y}$
$M = yf(xy)$, $N = xg(xy)$, $f(xy) \neq g(xy)$	$I = \frac{1}{xy(f(xy)-g(xy))}$
$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N-M} = h(z)$ is a function of $x+y$	$I = e^{\int h(z)dz}$
$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{2x\,N-2y\,M} = k(w)$ is a function of $w = x^2 + y^2$	$I = e^{\int k(w)dw}$
$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{x\,N-y\,M} = F(xy)$ is a function of $u = xy$	$I = e^{\int F(u)du}$

Chapter 3

The equation is of first order and of second or higher degree

The general form is

$$f(x, y, p, p^2, \ldots, p^n) = 0, \tag{3.1}$$

where we use the symbol p, in place $\frac{dy}{dx}$ or y' and n is the degree of equation (3.1). That is, equation (3.1) may has the form

$$p^n + a_1(x, y)p^{n-1} + a_2(x, y)p^{n-2} + \ldots + a_{n-1}(x, y)p + a_n(x, y) = 0. \quad (3.2)$$

It may be possible, sometimes, to solve such equations by one of the procedures outlines below. In each case the problem is reduced to that of solving one or more equations of the first order and first degree.

3.1 Equations solvable for p

If equation (3.1) can be solved for p and can be written as

$$[p - q_1(x, y)][p - q_2(x, y)] \ldots [p - q_n(x, y)] = 0.$$

Set each factor equal to zero and solve the resulting n differential equations of first order and first degree

$$\frac{dy}{dx} = q_1(x, y), \quad \frac{dy}{dx} = q_2(x, y), \quad \ldots \quad , \quad \frac{dy}{dx} = q_n(x, y),$$

to obtain

$$f_1(x, y, c) = 0, \quad f_2(x, y, c) = 0, \quad \dots \quad , \quad f_n(x, y, c) = 0,$$

where c is an arbitrary constant. Hence the general solution of (3.1) is the product

$$f_1(x, y, c) \cdot f_2(x, y, c) \cdot \cdots \cdot f_n(x, y, c) = 0.$$

Example 32. *Solve*

$$xy(\frac{dy}{dx})^2 + (x^2 + y^2)\frac{dy}{dx} + xy = 0. \tag{3.3}$$

Solution: It is obvious that equation (3.3) is a first order differential equation and its degree is two. Let $p = \frac{dy}{dx}$, so

$$xyp^2 + (x^2 + y^2)p + xy = 0 \implies (xp + y)(yp + x) = 0$$

$$\implies \quad (xp + y) = 0 \quad \text{or} \quad (yp + x) = 0$$

If $xp+y = 0 \implies x\frac{dy}{dt}+y = 0 \implies \frac{dy}{y}+\frac{dx}{x} = 0 \implies \ln|y| + \ln|x| = \ln|c| \implies xy = c$. Now, if $yp + x = 0 \implies y\frac{dy}{dx} + x = 0 \implies ydy + xdx = 0 \implies \frac{1}{2}(x^2 + y^2) = c$.

The general is

$$(xy - c)\Big(\frac{1}{2}(x^2 + y^2) - c\Big) = 0,$$

where c is an arbitrary constant.

3.2 Equations solvable for y

If equation (3.1) is solved for y. Then y can be expressed as a function of x and p. i.e. $y = g(x, p)$. Now differentiating it w.r.t. x, we have

$$\frac{dy}{dx} = \frac{dg}{dx} + \frac{dg}{dp}\frac{dp}{dx} \implies p = \frac{dg}{dx} + \frac{dg}{dp}\frac{dp}{dx} = G(x, p, \frac{dp}{dx}),$$

which is a first order and first degree differential equation of variables x and p and its solution, say, $\phi(x, p, c) = 0$. The general solution is then given by eliminating p between $y = g(x, p)$ and $\phi(x, p, c) = 0$ or express x and y separately as function of the parameter p when the elimination of p is not practicable.

Example 33. *Solve*

$$16x^2 + 2p^2y - p^3x = 0. \tag{3.4}$$

Solution: Clearly, equation (3.4) is a first order and third degree differential equation and is solvable for y. We have

$$16x^2 + 2p^2y - p^3x = 0 \Longrightarrow 2p^2y = p^3x - 16x^2 \Longrightarrow y = \frac{p}{2}x - 8\frac{x^2}{p^2}.$$

Differentiating with respect to x, we get

$$\begin{aligned}
& \frac{dy}{dx} = p = \frac{p}{2} + \frac{x}{2}\frac{dp}{dx} - \frac{16x}{p^2} + \frac{16x^2}{p^3}\frac{dp}{dx} \\
\Longrightarrow \quad & p - \frac{p}{2} - \frac{x}{2}\frac{dp}{dx} + \frac{16x}{p^2} - \frac{16x^2}{p^3}\frac{dp}{dx} = 0 \\
\Longrightarrow \quad & (\frac{p}{2} + 16\frac{x}{p^2}) - \frac{x}{2}(1 + 32\frac{x}{p^3})\frac{dp}{dx} = 0 \\
\Longrightarrow \quad & \frac{p}{2}(1 + 32\frac{x}{p^3}) - \frac{x}{2}(1 + 32\frac{x}{p^3})\frac{dp}{dx} = 0 \\
\Longrightarrow \quad & (1 + 32\frac{x}{p^3})(\frac{p}{2} - \frac{x}{2}\frac{dp}{dx}) = 0 \\
\Longrightarrow \quad & (1 + 32\frac{x}{p^3}) = 0 \quad \text{or} \quad (\frac{p}{2} - \frac{x}{2}\frac{dp}{dx}) = 0
\end{aligned}$$

First, if

$$\frac{p}{2} - \frac{x}{2}\frac{dp}{dx} = 0 \Longrightarrow \frac{dp}{p} - \frac{dx}{x} = 0 \Longrightarrow \ln(p) - \ln(x) = c \Longrightarrow p = kx.$$

Substitute in (3.4), we obtain

$$16x^2 + 2k^2x^2y - 6k^3x^4 = 0,$$

is a general solution where $k = e^c$ is an arbitrary essential constants. Second, if

$$1 + 32\frac{x}{p^3} = 0 \implies p^3 = -32x \implies p = \sqrt[3]{-32x}.$$

Substitute in (3.4), we get the singular solution

$$16x^2 + 2(\sqrt[3]{-32x})^2 y + 32x^2 = 0.$$

3.3 Equations soluble for x

Equations that can be solved for x, i.e. such that they may be written in the form

$$x = h(y, p), \tag{3.5}$$

can be reduced to first degree equations in p by differentiating both sides with respect to y, so that

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial h}{\partial y} + \frac{\partial h}{\partial p}\frac{\partial p}{\partial y} = H(y, p, \frac{dp}{dy}),$$

is a first order and first degree differential equation of variables y and p and its solution, say, $\phi(y,p,c)=0$. Obtain the general solution by eliminating p between $x=h(y,p)$ and $\phi(y,p,c)=0$.

Example 34. *Solve*

$$3px - y^2p^2 - y = 0. \tag{3.6}$$

Solution: We have $3px - y^2p^2 - y = 0$ then $3x = \frac{y}{p} + y^2p$. Differentiating with respect to y, we get

$$\begin{aligned}
& 3\frac{dx}{dy} = \frac{3}{p} = \frac{p - y\frac{dp}{dy}}{p^2} + 2yp + y^2\frac{dp}{dy} \\
\Longrightarrow\quad & \frac{3}{p} - \frac{1}{p} + \frac{y}{p^2}\frac{dp}{dy} - 2yp - y^2\frac{dp}{dy} = 0 \\
\Longrightarrow\quad & (\frac{2}{p} - 2yp) + (\frac{y}{p^2} - y^2)\frac{dp}{dy} = 0 \\
\Longrightarrow\quad & 2(\frac{1}{p} - yp) + \frac{y}{p}(\frac{1}{p} - yp)\frac{dp}{dy} = 0 \\
\Longrightarrow\quad & (\frac{1}{p} - yp)(2 + \frac{y}{p}\frac{dp}{dy}) = 0 \\
\Longrightarrow\quad & \frac{1}{p} - yp = 0 \quad \text{or} \quad 2 + \frac{y}{p}\frac{dp}{dy} = 0.
\end{aligned}$$

If

$$2 + \frac{y}{p}\frac{dp}{dy} = 0 \Longrightarrow py^2 = c \Longrightarrow p = \frac{c}{y^2}$$

and substitute in (3.6) we get the general solution

$$3(\frac{c}{y^2})x - y^2(\frac{c}{y^2})^2 - y = 0 \implies y^3 = c(3x - c),$$

where c is an arbitrary essential constant.

If

$$\frac{1}{p} - yp = 0 \implies \frac{1}{p} = yp \implies p^2 = \frac{1}{y} \implies p = \mp\sqrt{\frac{1}{y}}$$

and substitute in (3.6) we get the singular solutuio solution

$$3(\sqrt{\frac{1}{y}})x - y^2(\sqrt{\frac{1}{y}})^2 - y = 0 \implies y^{\frac{3}{2}} = 3x.$$

Homework 17. *1)* $(x^3 + x^2 + x + 1)p^2 - (3x^2 + 2x + 1)yp + 2xy^2 = 0.$

2) $y + x\frac{dy}{dx} - x^4(\frac{dy}{dx})^2 = 0.$

3) $y = y^2(y')^3 + 2y'x.$

Chapter 4

Linear differential equations with constant coefficients

4.1 Linear differential equations

The general n-th order linear differential equations

Definition 16. *A linear differential equation of order n has the form*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x), \quad (4.1)$$

where $a_0(x), a_1(x), \dots, a_n(x)$ and $F(x)$ depending only on x and not y.

Remark 13. *1. An n-th order differential equation which is not of the form* (4.1) *is called nonlinear.*

2. If $n = 1$, equation (4.1) *is called a linear first order equation.*

3. If $n = 2$, equation (4.1) *becomes a second order linear differential equation.*

4. If the coefficients $a_0, a_1, \dots, a_n$ are constants, we call the equation (4.1)*, a linear differential equation with constant coefficients.*

5. If at least one of the coefficients $a_0(x), a_1(x), \dots, a_n(x)$ is a function of x, equation (4.1) *is called a linear differential equation with variable coefficients.*

6. We use the symbols $D, D^2, \dots$ to indicate the operator of taking the first, second, … derivatives. Thus, $Dy = \frac{dy}{dx}$

7. If $F(x) = 0$, equation (4.1) *is called a linear homogeneous differential equation. Otherwise, it is called non homogeneous (inhomogeneous, the complementary or reduced) equation.*

Examples

1) $y'' + y = x^2$, is a linear inhomogeneous differential equation with constant coefficients and it is of second order and first degree.

2) $\frac{d^5y}{dx^5} + \frac{d3y}{dx^3} + \frac{dy}{dx} = 0$, is a linear homogeneous differential equation of fifth order and first degree.

3) $3x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 3y\frac{dy}{dx} + x^2y = e^x$, in a nonlinear non-homogeneous third order and first degree differential equation with variable coefficients.

4) $5y'' - 2(y')^3 - 8y = 0$, in a nonlinear homogeneous second order and first degree differential equation.

Remark 14. *We now prove that if y_1 and y_2 are solutions of the homogeneous equation* (4.1)*, and if c_1 and c_2 are constants, then*

$$y = c_1y_1 + c_2y_2 \tag{4.2}$$

is also a solution of homogeneous equation (4.1)*.*

Since y_1 and y_2 are solutions of the homogeneous equation (4.1)*, then*

$$a_0(x)\frac{d^ny_1}{dx^n} + a_1(x)\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_1}{dx} + a_n(x)y_1 = 0 \tag{4.3}$$

$$a_0(x)\frac{d^ny_2}{dx^n} + a_1(x)\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_2}{dx} + a_n(x)y_2 = 0. \tag{4.4}$$

Multiplying equation (4.3) by c_1 and equation (4.4) by c_2 and adding the result, we have

$$a_0(x)(c_1y_1^{(n)}+c_2y_2^{(n)})+\cdots+a_{n-1}(x)(c_1y_1'+c_2y_2')+a_n(x)(c_1y_1+c_2y_2)=0 \tag{4.5}$$

Since

$$y = c_1y_1 + c_2y_2 \implies y' = c_1y_1' + c_2y_2',\ \ldots,\ y^n = c_1y_1^n + c_2y_2^n.$$

So equation (4.5) becomes

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0.$$

Thus y is also a solution for homogeneous equation (4.1).

Remark 15. *The expression in equation* (4.2) *is called a linear combination of the functions* y_1 *and* y_2*.*

Theorem 3. *Any linear combination of solutions of a linear homogeneous differential equation is also a solution.*

4.2 Linear dependence

Given the functions $f_1, f_2, \ldots, f_n$, and if constants $c_1, c_2, \ldots, c_n$, not all zero, exists such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \tag{4.6}$$

identically in some interval $a \leq x \leq b$, then the functions are said to be linearly dependent. If no such relations exists, the functions are said to be linearly independent. That is, the functions $f_1, f_2, \ldots, f_n$ are linearly independent when equation (4.6) implies that $c_1 = c_2 = \cdots = c_n = 0$.

Remark 16. *If the function are linearly dependent, then at least one of them is a linear combination of the others.*

Example 35. *Show that e^x and e^{2x} are linearly independent.*

solution: Suppose that there exists c_1, c_2 such that

$$c_1 e^x + c_2 e^{2x} = 0. \tag{4.7}$$

Differentiating equation (4.7) w.r.t. x, we get

$$c_1 e^x + 2c_2 e^{2x} = 0. \tag{4.8}$$

Now subtract equation (4.7) from equation (4.8), we have $c_2e^{2x} = 0$. Since $e^{2x} > 0$ for all x, then $c_2 = 0$ and substitute in equation (4.13), obtaining $c_1e^x = 0 \Longrightarrow c_1 = 0$. Hence, $c_1 = c_2 = 0$ which implies that e^x and e^{2x} are linearly independent.

The Wronskian:

Definition 17. *The Wronskian of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ is denoted by $W[f_1, f_2, \ldots, f_n]$ and defined as the determinant*

$$W[f_1, f_2, \ldots, f_n] = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Example 36. *Show that the solutions $\sin x$ and $\cos x$ of $\frac{d^2y}{dx^2} + y = 0$ are linearly independent.*

Solution: We calculate the Wronskian

$$W[\sin x, \cos x] = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

Remark 17. *Two functions are linearly dependent on an interval I if and only if one of the functions is a constant multiple of the other function.*

Theorem 4. *If, on the interval $a \leq x \leq b$ $a_0(x) \neq 0$, $a_0, a_1, \ldots, a_n$ are continuous, and $y_1, y_2, \ldots, y_n$ are solutions of the equation*

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0,$$

then a necessary and sufficient condition that $y_1, \ldots, y_n$ be linearly independent is the nanvanishing of the Wronskian of $y_1, \ldots, y_n$ on the interval $a \leq x \leq b$.

Remark 18. *Note that the nonvanishing of the Wronskian is a sufficient condition that the functions be linearly independent.*
The non vanishing of the Wronskian on an interval is not a necessary condition for linear independence. The Wronskian may vanish even when the functions are linearly independent.

Proof: [**Theorem 4**] Suppose that $W[y_1, \ldots, y_n] \neq 0$ for all $a \leq x \leq b$. We have to prove that solutions $y_1, \ldots, y_n$ are linearly independent on $a \leq x \leq b$. We prove by contradiction.
Let $y_1, \ldots, y_n$ be linearly dependent on $a \leq x \leq b$. So by definition

of liner dependence there exist constants $b_1, \dots, b_n$, not all zero, such that $b_1 y_1(x) + \cdots + b_n y_n(x) = 0$. Then for every $a \le x \le b$, we have

$$\begin{aligned} b_1 y_1(x) + \cdots + b_n y_n(x) &= 0 \\ b_1 y_1'(x) + \cdots + b_n y_n'(x) &= 0 \\ &\vdots \\ b_1 y_1^{(n-1)}(x) + \cdots + b_n y_n^{(n-1)}(x) &= 0 \end{aligned} \tag{4.9}$$

We rewrite the system (4.9) in the matrix notation at $x = x_0$

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) & \dots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \dots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \dots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.10}$$

Since $b_1, \dots, b_n$, not all zero, then system (4.10) has nontrivial solution iff

$$\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} = 0 \quad \Longrightarrow \quad W[y_1, \dots, y_n] = 0,$$

which is a contradiction of our assumption that $W[y_1, \ldots, y_n] \neq 0$. Thus, $y_1, \ldots, y_n$ are linearly independent on $a \leq x \leq b$.

Suppose now that $y_1, \ldots, y_n$ are linearly independent on $a \leq x \leq b$. We have to prove that $W[y_1, \ldots, y_n] \neq 0$ for all $a \leq x \leq b$. Now from the definition of linearly independent we have

$$c_1 y_1(x) + \cdots + c_n y_n(x) = 0 \quad \text{iff} \quad c_1 = c_2 = \cdots = c_n = 0$$

We differentiate equation above $(n-1)$ times. So we get a system of equations

$$\begin{aligned} c_1 y_1(x) + \cdots + c_n y_n(x) &= 0 \\ c_1 y_1'(x) + \cdots + c_n y_n'(x) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x) + \cdots + c_n y_n^{(n-1)}(x) &= 0 \end{aligned} \tag{4.11}$$

$$\Longrightarrow \begin{pmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.12}$$

Since the system (4.12) has only zero (trivial) solution $c_1 = c_2 = \cdots = c_n = 0$, then we must have

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0 \quad \Longrightarrow \quad W[y_1, \ldots, y_n] \neq 0,$$

for all $a \leq x \leq b$.

Homework 18. *Can you give an example that two function are linearly independent even that their Wronskian is zero?*

Theorem 5. *Let $y_1, \ldots, y_n$ be solutions to the n-th order homogeneous linear differential equation*

$$a_n y^{(n)} + a_{(n-1)} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

on an interval I, and suppose that $W[y_1, \ldots, y_n] = 0$ is identically zero on I. Then $y_1, \ldots, y_n$ are linearly dependent on I.

Proof: Let x_0 be any point in I, and consider the system of linear equations

$$\begin{aligned} c_1 y_1(x_0) + \cdots + c_n y_n(x_0) &= 0 \\ c_1 y_1'(x_0) + \cdots + c_n y_n'(x_0) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) &= 0 \end{aligned} \tag{4.13}$$

in the unknowns $c_1, \ldots, c_n$. Since the Wronskian $y_1, \ldots, y_n$ vanishes identically on I, then the determinant of (4.13) is zero and the system has nontrivial solutions $c_1, \ldots, c_n$. Thus, $y_1, \ldots, y_n$ are linearly dependent.

Homework 19. *Prove that the set of solutions y_1 and y_2 is linearly depend if and only if the Wronskian $W[y_1, y_2] = 0$. Hint: Use Remark 17.*

Remark 19. *There are very interesting and important relationships between the Wronskian for a linear differential equation and the coefficients in the equation.*

Consider the second order differential equation of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0. \tag{4.14}$$

Let y_1 and y_2 be solutions of (4.14), then these solutions satisfy (4.14)

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0, \tag{4.15}$$

and

$$a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0. \tag{4.16}$$

Multiplying equation (4.15) by $(-y_2)$ and equation (4.16) by (y_1), and adding the result, we have

$$a_0(x)\big(y_1y_2'' - y_2y_1''\big) + a_1(x)\big(y_1y_2' - y_2y_1'\big) = 0. \tag{4.17}$$

Since $W[y_1, y_2] = y_1y_2' - y_2y_1'$, then

$$\frac{dW}{dx} = \frac{d(y_1y_2' - y_2y_1')}{dx} = y_1y_2'' - y_2y_1''.$$

Substitutes $W[y_1, y_2]$ and $\frac{dW}{dx}$ in (4.17), we obtain

$$a_0\frac{dW}{dx} + a_1W = 0 \quad \Longrightarrow \quad W = Ce^{-\int \frac{a_1}{a_0}dx}.$$

4.3 Differential operators

Let $\frac{dy}{dx} = Dy$. The symbol D is said to be a differentiation operator as it transform a differentiable function into another function. So, $D = \frac{d}{dx}$ denotes differentiation with respect to independent variable, say x, $D^2 = \frac{d^2}{dx^2}$ differentiation twice with respect to x and containing this process, we have $D^n = \frac{d^n}{dx^n}$ and $D^n y = \frac{d^n y}{dx^n}$, for n is positive integer. We define an n-th order differential operator to be

$$L = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x).$$

Note that $L\{\alpha f(x) + \beta g(x)\} = \alpha L\{f(x)\} + \beta L\{g(x)\}$.

Homogeneous linear differential equations with constant coefficients

The general form of homogeneous linear differential equations with constant coefficients is

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0, \tag{4.18}$$

where a_i's are constants for $i = 0, \ldots, n$. So, equation (4.18) can be written in the operator notation

$$(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = 0 \quad \Longrightarrow \quad F(D)y = 0$$

where

$$F(D) = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n,$$

which is called characteristic polynomial. If $F(D) = 0$, then it is called characteristic equation.

Properties of operator D:

1. $D^n + D^m = D^m + D^n$.
2. $D^nD^m = D^mD^n = D^{n+m}$.

3. $(D^n + D^m)f(x) = D^n f(x) + D^m f(x)$.

4. $D^n(f(x) + g(x)) = D^n f(x) + D^n g(x)$.

5. $(D - a)(D - b) = (D - b)(D - a)$, a, b are constants.

We now describe and illustrate how one can solve second order differential equation via an example and then in general.

Example 37. *Solve* $y'' + y' - 6y = 0$.

Solution: Let $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$, then

$$(D^2 + D - 6)y = 0 \quad \Longrightarrow \quad (D - 2)(D + 3)y = 0.$$

Let

$$(D + 3)y = u \tag{4.19}$$

then

$$(D-2)u = 0 \quad \Longrightarrow \quad \frac{du}{dx} - 2u = 0 \quad \Longrightarrow \quad \frac{du}{dx} = 2u \quad \Longrightarrow \quad \frac{du}{u} = 2dx$$

$$\Longrightarrow \quad \ln|u| = 2x + c \quad \Longrightarrow \quad u = ke^{2x}, k = e^c$$

Substitutes in (4.19), we have

$$(D + 3)y = u = ke^{2x} \quad \Longrightarrow \quad \frac{dy}{dx} + 3y = ke^{2x}$$

which is a linear differential equation of first order with $P(x) = 3$ and $Q(x) = ke^{2x}$. The general solution is given by

$$y = \frac{\int e^{\int 3dx}\, k\, e^{2x} dx + C_1}{e^{\int 3dx}} = \frac{k \int e^{5x} dx + C_1}{e^{3x}} = \frac{k}{5} e^{2x} + C_1 e^{-3x} = Ae^{2x} + Be^{-3x}.$$

where $A = \frac{k}{5}$ and $B = C_1$ are arbitrary constants.

Homework 20. *1)* $y'' - \frac{1}{2}y' - \frac{1}{2}y = 0.$
2) $y'' + y' - 2y = 0.$

Homogeneous linear differential equations of second order with constant coefficients

The general is

$$y'' + ay' + by = 0, \tag{4.20}$$

where a, b are constants. Then using The operator D, equation (4.20) can be written

$$(D^2 + aD + b)y = 0 \quad \Longrightarrow \quad (D - \alpha_1)(D - \alpha_2)y = 0,$$

where $-(\alpha_1 + \alpha_2) = a$ and $\alpha_1 \alpha_2 = b$.

Let

$$(D - \alpha_2)y = u \tag{4.21}$$

and then substitute in equation (4.20), we have

$$(D-\alpha_1)u=0 \implies \frac{du}{dx}-\alpha_1 u=0 \implies \frac{du}{dx}=\alpha_1 u \implies \ln|u|=\alpha_1 x+C_1$$

$$\implies u=K_1 e^{\alpha_1 x}, \quad \text{where } K_1=e^{C_1}.$$

Substitute in (4.21), we get

$$(D-\alpha_2)y=u=K_1 e^{\alpha_1 x} \implies \frac{dy}{dx}-\alpha_2 y=K_1 e^{\alpha_1 x},$$

is a first order linear differential equation with $P(x)=-\alpha_2$ and $Q(x)=K_1 e^{\alpha_1 x}$. So,

$$y=\frac{\int e^{\int -\alpha_2 dx}\,(K_1\,e^{\alpha_1 x})\,dx+K_2}{e^{\int -\alpha_2 dx}}=\frac{K_1\int e^{(\alpha_1-\alpha_2)x}\;dx+K_2}{e^{-\alpha_2 x}}.$$

There are three cases which depends on the nature of α_1 and α_2.

Case 1: If the roots α_1 and α_2 are real distinct (unequal), i.e. $\alpha_1 \neq \alpha_2 \in \mathbb{R}$ (if $a^2-4b>0$), then

$$y=\frac{\frac{K_1}{\alpha_1-\alpha_2}\,e^{(\alpha_1-\alpha_2)x}\;+K_2}{e^{-\alpha_2 x}}=e^{\alpha_2 x}\Big(\frac{K_1}{\alpha_1-\alpha_2}\,e^{(\alpha_1-\alpha_2)x}\Big)+K_2\,e^{\alpha_2 x}$$

$$\implies y=A\,e^{\alpha_1 x}+B\,e^{\alpha_2 x}, \text{ where } A=\frac{K_1}{\alpha_1-\alpha_2} \text{ and } B=K_2.$$

Case 2: If the roots α_1 and α_2 are real equal (repeated). i.e. $\alpha_1 = \alpha_2 = \alpha$ (if $a^2 - 4b = 0$), then

$$y = \frac{K_1 \int e^{(0)}\, dx + K_2}{e^{-\alpha x}} = K_2\, e^{\alpha x} + K_1\, x\, e^{\alpha x} = A\, e^{\alpha x} + B\, x\, e^{\alpha x},$$

where $A = K_2$ and $B = K_1$.

Case 3: If the roots α_1 and α_2 are complex. Let $\alpha_1 = a + ib$ and $\alpha_2 = a - ib$ where $a, b \in \mathbb{R}$, $b \neq 0$ and $i^2 = -1$ (when $a^2 - 4b < 0$). From Case 1, we have

$$y = A\, e^{\alpha_1 x} + B\, e^{\alpha_2 x}$$

So,

$$y = A\, e^{(a+ib)x} + B\, e^{(a-ib)x} = e^{ax}(A\, e^{ibx} + B\, e^{-ibx}).$$

By Euler's formula, we have

$$e^{i\theta} = \cos\theta + i \sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i \sin\theta.$$

Hence,

$$\begin{aligned} y &= e^{ax}\big(A(\cos bx + i\sin bx) + B(\cos bx - i\sin bx)\big) \\ &= e^{ax}\big((A + B\cos bx + i(A - B)\sin bx\big) \\ &= e^{ax}\big(C_1\cos bx + C_2\sin bx\big) \end{aligned}$$

is a general solution where $C_1 = A + B$ and $C_2 = i(A - B)$.

Example 38. *Solve the following differential equation:*
1) $y'' - y' - 2y = 0,$ *2)* $y'' - 2y' + y = 0,$ *3)* $y'' + 2y' + 2y = 0.$

Solution

1) Clearly, the differential equation is a homogeneous second order linear differential equation with constant coefficients. In the operator notation, this equations becomes $(D^2 - D - 2)y = 0$. So, the characteristic (auxiliary) equation is

$$\alpha^2 - \alpha - 2 = 0 \implies (\alpha - 2)(\alpha + 1) = 0 \implies \alpha_1 = 2 \quad \text{and} \quad \alpha_2 = -1.$$

Since the roots are real and distinct (unequal), the the general solution is

$$y = Ae^{\alpha_1 x} + Be^{\alpha_2 x} = Ae^{2x} + Be^{-x},$$

where A and B are arbitrary constants.

2) The characteristic equation is

$$\alpha^2 - 2\alpha + 1 = 0 \implies (\alpha - 1)(\alpha - 1) = 0 \implies \alpha = 1$$

which is a double root and the general solution is given by

$$y = Ae^{\alpha x} + Bxe^{\alpha x} = Ae^{x} + Bxe^{x},$$

where A and B are arbitrary constants.

3) In this example, the characteristic equation is given by

$$\alpha^2 + 2\alpha + 2 = 0 \implies \alpha_{1,2} = \frac{-2 \mp \sqrt{4-8}}{2} = -1 \mp i$$

$$\implies \alpha_1 = -1 + i \quad \text{and} \quad \alpha_2 = -1 - i.$$

The roots are complex and clearly $a = -1$ and $b = 1$, so the general solution is

$$y = e^{-x}(C_1 \cos x + C_2 \sin x),$$

where C_1 and C_2 are arbitrary constants.

Remark 20. *The general solution of a homogeneous linear differential* *equation with constant coefficients, is also known as a complementary function (solution).*

Homework 21. *Solve the following differential equation:*

1) $y'' + y = 0.$

2) $y'' - 4y' + 4y = 0.$

3) $y'' - 7y' = 0.$

4) $y'' - 2\sqrt{2}y' + 2y = 0.$

5) $4y'' + 4y' + y = 0.$

Homogeneous linear differential equations with constant coefficients of arbitrary order

Theorem 6. *Let*

$$y^{(n)} + a_{(n-1)}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{4.22}$$

be an n-th order homogeneous linear differential equation with constant real coefficients. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be the roots of its characteristic polynomial, and suppose that*

$$f(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_0 = (D - \alpha_1)(D - \alpha_2)\cdots(D - \alpha_n)$$

1) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are real distinct (unequal) numbers* ($\alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_n$ *and* $\alpha_i \in \mathbb{R}$*, for i=1,...,n), then the functions*

$$e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}$$

are linearly independent and the general solution of equation (4.22) *is*

$$y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \ldots + C_n e^{\alpha_n x},$$

where C_i *are arbitrary constants.*

2) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are real equal numbers (repeated)* ($\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$ *and* $\alpha_i = \alpha \in \mathbb{R}$*, for i=1,...,n), then the functions*

$$e^{\alpha x}, xe^{\alpha x}, \ldots, x^{n-1} e^{\alpha x}$$

are linearly independent and the general solution of equation (4.22) *is*

$$y = C_1 e^{\alpha_1 x} + C_2 x e^{\alpha_2 x} + \ldots + C_n x^{n-1} e^{\alpha_n x},$$

where C_i *are arbitrary constants.*

3) If the roots are complex. Let $\alpha = a \mp ib$ (k times), where n=2k, $a, b \in \mathbb{R}, b \neq 0$, then the functions

$$e^{ax}\sin(bx), xe^{ax}\sin(bx), \dots, x^{n-1}e^{ax}\sin(bx)$$

$$e^{ax}\cos(bx), xe^{ax}\cos(bx), \dots, x^{n-1}e^{ax}\cos(bx)$$

are linearly independent and the general solution of equation (4.22) *is*

$$y = e^{ax}\big(C_1\cos(bx) + C_2\sin(bx)\big) + xe^{ax}\big(C_3\cos(bx) + C_4\sin(bx)\big) + \cdots$$

$$+e^{ax}x^{n-1}\big(C_{n-1}\cos(bx) + C_n\sin(bx)\big)$$

where C_i are arbitrary constants.

Example 39. *Solve the differential equation*

$$y''' - 6y'' + 11y' - 6y = 0.$$

Solution:

$$y''' - 6y'' + 11y' - 6y = 0 \quad \Longrightarrow \quad (D^3 - 6D^2 + 11D - 6)y = 0,$$

then the characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0 \implies (\alpha - 1)(\alpha^2 - 5\alpha + 6) = 0$$

$$\implies (\alpha - 1))(\alpha - 2))(\alpha - 3)) = 0 \implies \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3.$$

Since all roots are real and distinct, so the general solution is given by

$$y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + C_3 e^{\alpha_3 x} = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x},$$

where C_1, C_2 and C_3 are arbitrary constants.

Example 40. *Find the general solution of the differential equation*

$$y^{(4)} + 2y'' + y = 0. \tag{4.23}$$

Solution: Equation (4.23) has auxiliary equation

$$\alpha^4 + 2\alpha^2 + 1 = 0 \implies (\alpha^2 + 1)(\alpha^2 + 1) = 0 \implies (\alpha^2 + 1)^2 = 0$$

$$\implies \alpha = \mp i, \mp i \implies a = 0, b = 1.$$

The roots are repeated complex numbers and clearly it is purely imaginary. So the general solution is

$$y = e^{ax}\big(C_1\cos(bx) + C_2\sin(bx)\big) + xe^{ax}\big(C_3\cos(bx) + C_4\sin(bx)\big)$$

$$\implies y = C_1\cos(x) + C_2\sin(x) + x\big(C_3\cos(x) + C_4\sin(x)\big),$$

where C_i, for $i = 1, 2, 3, 4$ are arbitrary constants.

Example 41. *Solve the equation*

$$y^{(7)} - 2y^{(5)} + y^{(3)} = 0.$$

Solution: In operator notation this equation becomes

$$(D^7 - 2D^5 + D^3)y = 0,$$

then the characteristic equation is

$$\alpha^7 - 2\alpha^5 + \alpha^3 = 0 \implies \alpha^3(\alpha^4 - 2\alpha^2 + 1) = 0 \implies \alpha^3(\alpha^2 - 1)(\alpha^2 - 1) = 0$$

$$\implies \alpha^3(\alpha+1)^2(\alpha-1)^2 = 0 \implies \alpha = 0, 0, 0, 1, 1, -1, -1.$$

In this case, the general form is given by

$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^x + C_5 x e^x + C_6 e^{-x} + C_7 x e^{-x},$$

where C_i, for $i = 1, \ldots, 7$ are arbitrary constants.

Example 42. *Solve the equation*

$$y^{(5)} + 3y^{(4)} + 4y^{(3)} - 4y' - 4y = 0.$$

Solution: The characteristic equation is

$$\alpha^5 + 3\alpha^4 + 4\alpha^3 - 4\alpha - 4 = (\alpha - 1)(\alpha^2 + 2\alpha + 2)^2 = 0.$$

Clearly, the roots are $\alpha_1 = 1$, $\alpha_{2,3} = -1 + i$ and $\alpha_{4,5} = -1 - i$ $(a = -1, b = 1)$. The general solution, in this case, is

$$y = C_1 e^x + e^{-x}(C_2 \cos x + C_3 \sin x) + x e^{-x}(C_4 \cos x + C_5 \sin x),$$

where C_i, for $i = 1, \ldots, 5$ are arbitrary constants.

Homework 22. *Solve the following differential equations:*

1) $y^{(6)} - y^{(5)} + 2y^{(4)} - 2y''' + y'' - y' = 0.$

2) $(D^3 + 1)y = 0.$

3) $(D^3 + 2D^2 - 5D - 6)y = 0$.

4) $(D^4 + 4D)y = 0$.

5) $(D^5 - 5D^4 + 12D^3 - 16D^2 + 12D - 4)y = 0$.

6) $y^{(5)} - y^{(4)} + 4y' - 4y = 0$.

Example 43. *Find a linear differential equation that has* e^{2x} *and* xe^{-3x} *among its solutions.*

Solution: Clearly, the roots are $\alpha_1 = 2$ and $\alpha_2 = -3$. Note that $\alpha_2 = -3$ is repeated root, so the characteristic polynomial is given be

$$f(D) = (D-2)(D+3)^2 \quad \Longrightarrow \quad (D-2)(D+3)^2 y = 0.$$

Thus, the linear differential equation is

$$y''' + 4y'' - 3y' - 18y = 0.$$

Homework 23. *The equation*

$$(D^3 + aD^2 + bD + c)y = 0,$$

where a*,* b *and* c *are constants, has a solution*

$$y = C_1 e^{-x} + e^{-2x}(C_2 \sin 4x + C_3 \cos 4x).$$

Determine the values of a, b and c.

Properties of the operator D

We know that the characteristic polynomial is

$$f(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

where a_i are constants for $i = 1, \ldots, n$.

1) If b is a constant, then $f(D)\{e^{bx}\} = f(b)e^{bx}$.

Proof: Since

$$D\{e^{bx}\} = \frac{d}{dx}(e^{bx}) = be^{bx},$$

$$D^2\{e^{bx}\} = b^2 e^{bx},$$

$$\vdots$$

$$D^n\{e^{bx}\} = b^n e^{bx}.$$

So,

$$
\begin{aligned}
f(D)\{e^{bx}\} =&(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)\{e^{bx}\}, \\
=&a_n D^n\{e^{bx}\} + a_{n-1} D^{n-1}\{e^{bx}\} + \cdots + a_1 D\{e^{bx}\} + a_0\{e^{bx}\}, \\
=&a_n b^n\, e^{bx} + a_{n-1} b^{n-1}\, e^{bx} + \cdots + a_1 b\, e^{bx} + a_0\, e^{bx}, \\
=&(a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0) e^{bx}, \\
=&f(b) e^{bx}.
\end{aligned}
$$

Example 44. *Evaluate*

$$(D^2 + 3D + 2)e^{3x}.$$

Solution: $f(D) = D^2 + 3D + 2$ and $b = 3$, so, $f(3) = 9 + 9 + 2 = 20$. Hence,

$$(D^2 + 3D + 2)e^{3x} = 20\, e^{3x}.$$

2) $f(D^2)\{\cos(bx)\} = f(-b^2)\cos(bx)$, where b is a constant.

Proof: We have

$$f(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0,$$

then

$$f(D^2) = a_n D^{2n} + a_{n-1} D^{2(n-1)} + \cdots + a_1 D^2 + a_0.$$

Since

$$\begin{aligned}
D(\cos bx) &= \frac{d}{dx}(\cos bx) = -b \sin bx, \\
D^2(\cos bx) &= D(D(\cos bx)) = D(-b \sin bx) = -b^2 \cos bx, \\
D^3(\cos bx) &= b^3 \sin bx \\
D^4(\cos bx) &= b^4 \cos bx = (-b^2)^2 \cos bx.
\end{aligned}$$

By mathematical induction, let

$$D^{2k}(\cos bx) = (-b^2)^k \cos bx, \quad k \in \mathbb{Z}^+.$$

Now,

$$\begin{aligned}
D^{2(k+1)}(\cos bx) =& D^2 D^{2k}(\cos bx) = D^2((-b^2)^k \cos bx) \\
=& (-b^2)^k D^2(\cos bx) = (-b^2)^k(-b^2) \cos bx \\
=& (-b^2)^{k+1} \cos bx.
\end{aligned}$$

$$D^{2n}(\cos bx) = (-b^2)^n \cos bx, \quad n = 1, 2, \ldots.$$

$$
\begin{aligned}
f(D^2)\{\cos(bx)\} =&(a_n D^{2n} + \cdots + a_1 D^2 + a_0)\{\cos(bx)\}\\
=&a_n(-b^2)^n \cos bx + \cdots + a_1(-b^2)\cos bx + a_0 \cos bx\\
=&f(-b^2)\cos bx.
\end{aligned}
$$

3) $f(D^2)\{\sin(bx)\} = f(-b^2)\sin(bx)$, where b is a constant.

Proof: H.W.

Theorem 7. *If g is a function of x, then*

$$f(D)\{e^{bx}g(x)\} = e^{bx}f(D+b)\{g(x)\}.$$

Proof: Since

$$f(D) = a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0,$$

so,

$$
\begin{aligned}
f(D)\{e^{bx}g(x)\} &= (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)\{e^{bx}g(x)\}\\
&= a_nD^n\{e^{bx}g(x)\} + a_{n-1}D^{n-1}\{e^{bx}g(x)\} + \cdots\\
&\quad + a_1D\{e^{bx}g(x)\} + a_0\{e^{bx}g(x)\}
\end{aligned}
$$

Now,

$$
\begin{aligned}
D\{e^{bx}g(x)\} &= \frac{d}{dx}\{e^{bx}g(x)\} = e^{bx}\frac{d}{dx}(g(x)) + be^{bx}g(x) = e^{bx}D\{g(x)\} + be^{bx}g(x)\\
&= e^{bx}[D\{g(x)\} + bg(x)] = e^{bx}(D+b)g(x).
\end{aligned}
$$

and

$$
\begin{aligned}
D^2\{e^{bx}g(x)\} &= D[D\{e^{bx}g(x)\}] = \frac{d}{dx}[e^{bx}\frac{d}{dx}(g(x)) + be^{bx}g(x)]\\
&= e^{bx}\frac{d^2}{dx^2}(g(x)) + be^{bx}\frac{d}{dx}(g(x)) + be^{bx}\frac{d}{dx}(g(x)) + b^2e^{bx}g(x)\\
&= e^{bx}[D^2(g(x) + 2bD(g(x)) + b^2g(x))]\\
&= e^{bx}(D+b)^2g(x).
\end{aligned}
$$

and so on

$$
D^n\{e^{bx}g(x)\} = e^{bx}(D+b)^ng(x).
$$

Hence,

$$\begin{aligned} f(D)\{e^{bx}g(x)\} &= a_n e^{bx}(D+b)^n g(x) + a_{n-1}e^{bx}(D+b)^{n-1}g(x) + \cdots \\ &+ a_1 e^{bx}(D+b)g(x) + a_0 e^{bx}g(x) \\ &= e^{bx}[a_n(D+b)^n + a_{n-1}(D+b)^{n-1} + \cdots + a_1(D+b) + a_0]g(x) \\ &= e^{bx}f(D+b)\{g(x)\}. \end{aligned}$$

4.4 General solution of a non-homogeneous differential equations

Let y_p be any particular solution of the differential equation

$$b_0 y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1}y' + b_n y = R(x), \qquad (4.24)$$

and let y_c be a solution of the corresponding homogeneous equation (4.24)

$$b_0 y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1}y' + b_n y = 0. \qquad (4.25)$$

Then, $y = y_c + y_p$ is a general solution of (4.24). Now,

$$\begin{aligned} b_0 y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1} y' + b_n y &= (b_0 y_c^{(n)} + b_1 y_c^{(n-1)} + \cdots + b_{n-1} y_c' + b_n y_c) \\ &\quad + (b_0 y_p^{(n)} + b_1 y_p^{(n-1)} + \cdots + b_{n-1} y_p' + b_n y_p) \\ &= 0 + R(x) = R(x). \end{aligned}$$

If $y_1, y_2, \ldots, y_n$ are linearly independent solutions of (4.25), then

$$y_c = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

in which C_i's are arbitrary constants, is a general solution of (4.25) and it is called the *complementary function* (solution) for equation (4.24). The general solution of (4.24), is the sum of the complementary function and any particular solution.

The Operator $\frac{1}{f(D)}$ (Inverse of $f(D)$): To find a particular solution of

$$f(D)y = R(x),$$

it is natural to right

$$y_p = \frac{1}{f(D)}\{R(x)\}.$$

Remark 21. *Note that* $f(D) \cdot \frac{1}{f(D)}\{R(x)\} = R(x)$.

There are several cases to find the particular solution of

$$f(D)y = R(x).$$

Case 1: If $R(x) = e^{ax}$, a is a constant and we have $f(D)\{e^{ax}\} = f(a)e^{ax}$.

Case i: When $f(a) \neq 0$ and

$$f(D)\{\frac{e^{ax}}{f(a)}\} = \frac{f(a)}{f(a)}e^{ax} = e^{ax},$$

then

$$\frac{1}{f(D)}\{e^{ax}\} = \frac{1}{f(a)}e^{ax}.$$

Now,

$$f(D)y = e^{ax}, \tag{4.26}$$

then

$$y_p = \frac{1}{f(D)}\{e^{ax}\} = \frac{1}{f(a)}e^{ax},$$

which is a particular solution of the equation (4.26).

Example 45. *Solve the equation* $(D^2 + 1)y = e^{2x}$.

Solution: First we should find a complementary solution, i.e. a general solution of the corresponding homogeneous equation. Here, the roots of the characteristic equation is

$$\alpha^2 + 1 = 0 \implies \alpha = \mp i.$$

Then the complementary function is

$$y_c = C_1 \cos x + C_2 \sin x.$$

Now to find the particular solution, we have $f(D) = D^2 + 1$ and $a = 2$, then $f(a) = a^2 + 1 = 4 + 1 = 5 \neq 0$, so,

$$y_p = \frac{1}{D^2 + 1} e^{2x} = \frac{e^{2x}}{5}.$$

Thus, the general solution is

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x + \frac{e^{2x}}{5},$$

where C_1 and C_2 are arbitrary constants.

Case ii**:** When $f(a) = 0$, then $f(D)$ contains the factor $(D - a)$. Suppose that this factor occurs precisely k times in $f(D)$, that is,

$f(D) = \phi(D)(D-a)^k$; $\phi(a) \neq 0$, $k = 1, 2, \ldots, n$ (n is the order of the differential equation). Now

$$f(D)y = e^{ax} \implies y = \frac{1}{f(D)}\{e^{ax}\} = \frac{1}{(D-a)^k\phi(D)}\{e^{ax}\},$$

Then by Theorem 7, we have

$$f(D)\{e^{bx}g(x)\} = e^{bx}f(D+b)\{g(x)\}.$$

So,

$$y = e^{ax}\frac{1}{(D+a-a)^k\phi(D+a)}\{1\} = e^{ax}\frac{1}{(D)^k\phi(D+a)}\{1\} \quad \text{(Note, g(x)=1).}$$

Since,

$$\phi(D+a)\{1\} = \phi(a) \Longleftrightarrow \frac{1}{\phi(D+a)}\{1\} = \frac{1}{\phi(a)},$$

then,

$$y = e^{ax}\frac{1}{D^k\phi(a)}\{1\} = \frac{e^{ax}}{\phi(a)}\frac{1}{D^k}\{1\} = \frac{e^{ax}}{\phi(a)}\frac{x^k}{k!}$$

Case 2 If $R(x) = \sin ax$ or $R(x) = \cos ax$, where a is a constant. To find the particular solution for $f(D)y = R(x)$, there exists two types:

case i : If ai is not a root of the characteristic equation $f(D) = 0$, $i = \sqrt{-1}$, we use the following properties:

1) $f(D^2)\{\sin ax\} = f(-a^2)\{\sin ax\}$.

2) $f(D^2)\{\cos ax\} = f(-a^2)\{\cos ax\}$.

Example 46. *Solve* $(D^2 + 3D + 2)y = \cos 2x$.

Solution: The characterise equation is

$$\alpha^2 + 3\alpha + 2 = 0 \implies (\alpha + 2)(\alpha + 1) = 0 \implies \alpha_1 = -1, \ \alpha_2 = -2$$

then the complementary function (solution) is

$$y_c = C_1 e^{-x} + C_2 e^{-2x}.$$

To find the particular solution y_p, we can see that

$$y_p = \frac{1}{D^2 + 3D + 2}\{\cos 2x\} = \frac{1}{-(2)^2 + 3D + 2}\{\cos 2x\} = \frac{1}{3D - 2}\{\cos 2x\}$$

$$= \frac{3D + 2}{(3D - 2)(3D + 2)}\{\cos 2x\} = \frac{3D + 2}{9D^2 - 4}\{\cos 2x\} = \frac{3D + 2}{-36 - 4}\{\cos 2x\}$$

$$= \frac{3D+2}{-40}\{\cos 2x\} = \frac{3}{20}\{\sin 2x\} - \frac{1}{20}\{\cos 2x\}.$$

Finally, the general solution of the non-homogeneous system is

$$y = y_c + y_p = C_1 e^{-x} + C_2 e^{-2x} + \frac{3}{20}\{\sin 2x\} - \frac{1}{20}\{\cos 2x\}.$$

where C_1 and C_2 are arbitrary essential constants.

case ii : If ai is a root of the characteristic equation $f(D) = 0$, then by Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can find the particular solution from

$$f(D) = e^{iax}. \tag{4.27}$$

Since,

$$e^{iax} = \cos ax + i\sin ax,$$

then the particular solution of $f(D) = \cos ax$ is the real part of the particular solution of equation (4.27), and the particular solution of $f(D) = \sin ax$ is the imaginary part of the particular solution of equation (4.27).

Example 47. *Solve*

$$(D^2 + 9)y = \sin 3x. \tag{4.28}$$

Solution: In this example, the characterise equation is

$$\alpha^2 + 9 = 0 \implies \alpha = \mp 3i \implies a = 0, b = 3,$$

then the complementary function (solution) is

$$y_c = C_1 \cos 3x + C_2 \sin 3x.$$

To find the particular solution y_p, we can see that

$$y_p = \frac{1}{D^2 + 9}\{\sin 3x\} = \frac{1}{D^2 + 9}\{e^{3ix}\}.$$

But note that $f(3i) = 0$. Therefore,

$$y_p = \frac{1}{(D - 3i)(D + 3i)}\{e^{3ix}\} = \frac{1}{6i} x e^{3ix} = \frac{1(-i)}{6i(-i)} x e^{3ix} = \frac{-i}{6} x e^{3ix}.$$

$$\implies y_p = \frac{1(-i)}{6i(-i)} x(\cos 3x + i \sin 3x) = \frac{x}{6} \sin 3x - \frac{x}{6} i \cos 3x.$$

Hence the particular solution pf equation (4.28) is then

$$y_p = -\frac{x}{6} \cos 3x.$$

Therefore, the general solution is

$$y = y_c + y_p = C_1 \cos 3x + C_2 \sin 3x - \frac{x}{6} \cos 3x,$$

where C_1 and C_2 are arbitrary essential constants.

Case 3: IF $R(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, $n \in \mathbb{Z}^+$, is a polynomial of degree n in x, then to find a particular solution of

$$f(D)y = R(x) \implies y_p = \frac{1}{f(D)}\{R(x)\},$$

we use the ordinary long division. Since

$$\frac{1}{1-D} = 1 + D + D^2 + \cdots + D^n + \cdots$$

then

$$\frac{1}{1-f(D)} = 1 + f(D) + (f(D))^2 + \cdots .$$

Example 48. *Find the particular solution of*

$$(D^2 - 3D + 5)y = x^2 - 1.$$

Solutions: Since

$$(D^2-3D+5)y=x^2-1 \implies y_p=\frac{1}{D^2-3D+5}\{x^2-1\}$$
$$\implies y_p=\frac{1}{5[1-(\frac{3}{5}D-\frac{D^2}{5})]}\{x^2-1\}$$
$$\implies y_p=\frac{1}{5}(1+(\frac{3}{5}D-\frac{D^2}{5})+(\frac{3}{5}D-\frac{D^2}{5})^2+\cdots)\{x^2-1\}$$
$$\implies y_p=\frac{1}{5}(1+\frac{3}{5}D-\frac{D^2}{5}+\frac{9}{25}D^2-\frac{6}{50}D^3+\cdots)\{x^2-1\}$$
$$\implies y_p=\frac{1}{5}(x^2-1+\frac{3}{5}(2x)-\frac{2}{5}+\frac{9}{25}(2)+0\cdots)=\frac{x^2}{5}+\frac{6x}{5}-\frac{7}{25}.$$

is a particular solution.

Homework 24. *Solve the following differential equations:*

1) $(2D^2+2D+3)y=x^2+2x-1.$

2) $(D^3-2D+4)y=x^4+3x^2-5x+2.$

Remark 22. *If $R(x)$ is a polynomial of degree $n \in \mathbb{Z}^+$ in x, then to find a particular solution of $f(D)y=R(x)$, we suppose that the particular solution is a polynomial of the same degree as $R(x)$ and we must find its coefficients.*

Example 49. *Solve $y''-2y'-3y=1-x^2$.*

Solution: Since

$$y'' - 2y' - 3y = 1 - x^2 \implies (D^2 - 2D - 3)y = 1 - x^2$$

$$\alpha^2 - 2\alpha - 3 = 0 \implies (\alpha - 3)(\alpha + 1) = 0 \implies \alpha_1 = 3 \text{ and } \alpha_2 = -1.$$

So,

$$y_c = C_1 e^{3x} + C_2 e^{-x}.$$

To find the particular solution y_p, let the $y_p = Ax^2 + Bx + C$, so $y_p' = 2Ax + B$ and $y_p'' = 2A$. Substitutes in the original differential equation, we have

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$

$$\implies 2A - 4Ax - 2B - 3Ax^2 - 3Bx - 3C = 1 - x^2$$

$$\implies (2A - 2B - 3C) - (4A + 3B)x - 3Ax^2 = 1 - x^2$$

It is easy to see that $A = \frac{1}{3}$ $B = -\frac{4}{9}$ and $C = \frac{5}{27}$. So the particular solution is

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

Case 4: If $R(x) = e^{ax}Q(x)$, where $Q(x) = \sin(bx)$, $Q(x) = \cos(bx)$ or $Q(x)$ is a polynomial in x and a, b are constants. To find y_p, we use the theorem

$$f(D)\{e^{ax}Q(x)\} = e^{ax}f(D+a)Q(x),$$

so,

$$\frac{1}{f(D)}\{e^{ax}Q(x)\} = e^{ax}\frac{1}{f(D+a)}\{Q(x)\},$$

then this case transform to the second case or third case.

Example 50. *Solve* $(D^2 - 2D)y = e^x \sin x$.

Solutions: The characteristic equation is

$$\alpha^2 - 2\alpha = 0 \implies \alpha(\alpha - 2) = 0 \implies \alpha_1 = 0 \text{ and } \alpha_2 = 2.$$

So, $y_c = C_1 + C_2 e^{2x}$.

To find the particular solution y_p,

$$y_p = \frac{1}{D^2 - 2D}\{e^x \sin x\} = e^x \frac{1}{(D+1)^2 - 2(D+1)}\{\sin x\}$$

$$\implies y_p = e^x \frac{1}{D^2 - 1}\{\sin x\} = e^x \frac{\sin x}{-(1)^2 - 1} = e^x \frac{\sin x}{-2} = -e^x \frac{\sin x}{2}.$$

Homework 25. *Solve* $(D^2 - 2D + 2)y = e^x \sin x$.

Example 51. *Solve* $(D^2 + D - 2)y = x\, e^x$.

Solutions: The characteristic equation is

$$\alpha^2 + \alpha - 2 = 0 \implies (\alpha + 2)(\alpha - 1) = 0 \implies \alpha_1 = 1 \text{ and } \alpha_2 = -2.$$

So, $y_c = C_1 e^x + C_2 e^{-2x}$.

To find the particular solution y_p,

$$y_p = \frac{1}{D^2 + D - 2}\{x\, e^x\} = e^x \frac{1}{(D+1)^2 + (D+1) - 2}\{x\}$$

$$\implies y_p = \frac{e^x}{D^2 + 3D}\{x\} = \frac{e^x}{3D(1 - (-\frac{D}{3}))} = \frac{e^x}{3D}(1 - \frac{D}{3} + \frac{D^2}{9} + \cdots)\{x\}$$

$$\implies y_p = \frac{e^x}{3D}(x - \frac{1}{3}) = \frac{e^x}{3}(\frac{x^2}{2} - \frac{1}{3}x).$$

Case 5): If $R(x) = \sin ax\, p(x)$ or $R(x) = \cos ax\, p(x)$, where $p(x)$ is a polynomial in x. We use Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

To find the particular solution for

$$f(D)y = R(x),$$

first we find the particular solution for

$$f(D)y = e^{iax}p(x), \tag{4.29}$$

which is a forth case. Then the particular solution of

$$f(D)y = \sin ax\, p(x)$$

is the imaginary part of the particular solution of (4.29) and the particular solution of

$$f(D)y = \cos ax\, p(x)$$

is the real part of the particular solution of (4.29).

Example 52. *Solve* $\quad y'' - 3y = (x^2 - 1)\sin 2x.$

Solution: Clearly, the characteristic equation is

$$\alpha^2 - 3 \implies \alpha = \mp\sqrt{3},$$

so the complementary function is

$$y_c = C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x}.$$

We will now find the particular solution.

$$(D^2 - 3)y = (x^2 - 1)\sin 2x \implies (D^2 - 3)y = (x^2 - 1)e^{2ix}$$

$$\implies y_p = \frac{1}{D^2 - 3}\{(x^2 - 1)e^{2ix}\} \implies y_p = e^{2ix}\frac{1}{(D + 2i)^2 - 3}\{x^2 - 1\}$$

$$\implies y_p = e^{2ix}\frac{1}{D^2 + 4iD - 7}\{x^2 - 1\} = -\frac{e^{2ix}}{7}\frac{1}{[1 - (\frac{D^2+4iD}{7})]}\{x^2 - 1\}$$

$$\implies \quad = -\frac{e^{2ix}}{7}[1 + (\frac{D^2 + 4iD}{7}) + (\frac{D^2 + 4iD}{7})^2 + \cdots]\{x^2 - 1\}$$

$$\implies \quad = -\frac{e^{2ix}}{7}[1 + \frac{4i}{7}D + \frac{D^2}{7} - \frac{16}{49}D^2 + \cdots]\{x^2 - 1\}$$

$$\implies \quad = -\frac{e^{2ix}}{7}[x^2 - 1 + \frac{4i}{7}(2x) + \frac{2}{7} - \frac{32}{49} + 0]$$

$$\implies \quad = -\frac{e^{2ix}}{7}[x^2 + \frac{8i}{7}(x) - \frac{32 + 49 - 14}{49}]$$

$$\implies \quad = -\frac{1}{7}(\cos 2x + i\sin 2x)(x^2 + \frac{8i}{7}x - \frac{67}{49})$$

$$\implies \quad = -\frac{1}{7}\cos 2x(x^2 - \frac{67}{49}) + \frac{8}{49}x\ \sin 2x - \frac{i}{7}\sin 2x(x^2 - \frac{67}{49}).$$

So, y_p for the original equation is

$$-\frac{1}{7}\sin 2x(x^2 - \frac{67}{49}).$$

The general solution is

$$y = y_c + y_p = C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x} - \frac{1}{7}\sin 2x(x^2 - \frac{67}{49}),$$

where C_1 and $C - 2$ are arbitrary constants.

4.5 Variation of parameters

Suppose we have a constant coefficient second order equation

$$y'' + ay' + by = g(x), \tag{4.30}$$

where $g(x)$ is a continuous function. Let

$$y_c = C_1 y_1(x) + C_2 y_2(x),$$

where y_1 and y_2 are linearly independent solutions, denotes the complementary solution to the corresponding homogeneous equation.
We now vary the parameters C_1 and C_2 and replace them by functions $v_1(x)$ and $v_2(x)$. We propose that the particular solution is of the form

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x).$$

To solve the nonhomogeneous (4.30), we must determine the functions $v_1(x)$ and $v_2(x)$. Now,

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2',$$

and let

$$v_1' y_1 + v_2' y_2 = 0, \tag{4.31}$$

for prevent any second derivatives of v_1 and v_2 from arising. Thus,

$$y_p' = v_1 y_1' + v_2 y_2' \implies y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''.$$

Substitute the expression for y_p and its derivatives into equation (4.30), we have

$$v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' + a(v_1y_1' + v_2y_2') + b(v_1y_1 + v_2y_2) = g(x)$$

$$\Longrightarrow\ v_1(y_1'' + ay_1' + by_1) + v_2(y_2'' + ay_2' + by_2) + v_1'y_1' + v_2'y_2' = g(x).$$

Since y_1 and y_2 are solutions of the corresponding homogeneous equation (4.30), then we have

$$y_1'' + ay_1' + by_1 = 0 \quad \text{and} \quad y_2'' + ay_2' + by_2 = 0.$$

So,

$$v_1'y_1' + v_2'y_2' = g(x) \tag{4.32}$$

Now, from (4.31) and (4.32), we can find v_1' and v_2' by Cramer's rule:

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}.$$

Since

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0,$$

as y_1 and y_2 are linearly independent solutions, then

$$v_1' = -\frac{y_2}{W[y_1, y_2]} g(x) \implies v_1 = -\int \frac{y_2}{W[y_1, y_2]} g(x) dx$$

and

$$v_2' = \frac{y_1}{W[y_1, y_2]} g(x) \implies v_2 = \int \frac{y_1}{W[y_1, y_2]} g(x) dx$$

Example 53. *Find the general solution to the equation*

$$y'' + y = \tan x.$$

Solution: The solution of the homogeneous equation

$$y'' + y = 0,$$

is given by

$$y_c = C_1 \cos x + C_2 \sin x.$$

We now suppose that the particular solution is of the form

$$y_p = v_1(x) \cos x + v_2(x) \sin x,$$

where $v_1(x)$ and $v_2(x)$ are unknown functions of x. So, by variation of parameters method, we have

$$v_1 = -\int \frac{y_2}{W[y_1, y_2]} g(x)dx \quad \text{and} \quad v_2 = \int \frac{y_1}{W[y_1, y_2]} g(x)dx.$$

Clearly,

$$W[y_1, y_2] = W[\cos x, \sin x] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0.$$

Now,

$$\begin{aligned} v_1 &= -\int \frac{\sin x}{1} \tan dx = -\int \frac{\sin^2 x}{\cos x} dx = -\int (\sec x - \cos x)dx \\ &= -\ln|\sec x + \tan x| + \sin x, \end{aligned}$$

and

$$v_2 = \int \frac{\cos x}{1} \tan dx = \int \sin x dx = -\cos x.$$

Thus,

$$y_p = (-\ln|\sec x + \tan x| + \sin x)\cos x - \cos x \sin x,$$

and the general solution is

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x + (-\ln|\sec x + \tan x| + \sin x)\cos x - \cos x \sin x,$$

where C_1 and C_2 are arbitrary constants.

Homework 26. *Solve the following nonhonogeneous equations:*
1) $y'' - 3y' + 2y = \frac{e^{3x}}{e^x+1}$.
2) $y'' + 2y' + y = e^{-x}\ln x$.

4.6 Reduction of orders

Consider the linear differential equation of order n with constant coefficients of the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = H(x), \qquad (4.33)$$

where $a_i, i = 0, \ldots, n$ are real numbers. Let $\alpha_1, \ldots, \alpha_n$ be n roots of the characteristic equation, so, equation (4.33), can be written as

$$(D - \alpha_1)(D - \alpha_2)\cdots(D - \alpha_n)y = H(x) \qquad (4.34)$$

We now assume that

$$(D-\alpha_2)\cdots(D-\alpha_n)y = u_1, \tag{4.35}$$

therefore equation (4.34) becomes

$$(D-\alpha_1)u_1 = H(x) \quad \Longrightarrow \quad \frac{du_1}{dx} - \alpha_1 u_1 = H(x),$$

which is a first order linear differential equation. So,

$$u_1 = \frac{\int e^{-\int \alpha_1 dx} H(x)dx + C_1}{e^{-\int \alpha_1 dx}} = e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\, H(x)dx + C_1\Big).$$

Substitutes in equation (4.35), we have

$$(D-\alpha_2)\cdots(D-\alpha_n)y = e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\, H(x)dx + C_1\Big).$$

Let

$$(D-\alpha_3)\cdots(D-\alpha_n)y = u_2,$$

So,

$$(D-\alpha_2)u_2 = e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\, H(x)dx + C_1\Big)$$

$$\Longrightarrow \frac{du_2}{dx} - \alpha_2 u_2 = e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\, H(x)dx + C_1\Big)$$

which is a first order linear differential equation and

$$u_2 = \frac{\int e^{-\int \alpha_2 dx}\Big[e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\ H(x)dx + C_1\Big] + C_2}{e^{-\int \alpha_2 dx}}$$

$$= e^{\alpha_2 x}\Big[e^{(\alpha_1-\alpha_2)x}\Big(\int e^{-\alpha_1 x}\ H(x)dx + C_1 + C_2\Big)\Big]$$

Continuing in this way, we get

$$y = e^{\alpha_n}\Big[e^{(\alpha_n-\alpha_{2-1})x}\cdots\Big(\int e^{-\alpha_1 x}\ H(x)dx + C_1\Big) + \cdots + C_n\Big]$$

is a general solution of (4.33).

4.7 Linear differential equations with variable coefficients

4.7.1 The Cauchy and Legendre linear equations

The linear Cauchy equation is of the form

$$p_0 x^n \frac{d^n y}{dx^n} + p_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}x\frac{dy}{dx} + p_n y = Q(x), \qquad (4.36)$$

in which $p_0, p_1, \cdots, p_n$ are constants, and the Legendre linear equation is of the form

$$p_0(ax+b)^n\frac{d^ny}{dx^n}+p_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_{n-1}(ax+b)\frac{dy}{dx}+p_ny = Q(x), \tag{4.37}$$

of which equation (4.36) is a special case of equation (4.37) ($a = 1$, $b = 0$). These equations may be reduced to a linear differential equation with constant coefficients by properly transformation of the independent variable.

4.7.2 Solving the Legendre linear equation

Let $ax + b = e^z$, then $z = \ln(ax + b)$ and $\frac{dz}{dx} = \frac{a}{ax+b}$, so,

$$Dy = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{a}{ax+b}\frac{dy}{dz}$$

and

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = aD_1y.$$

Similarly,

$$D^2y = \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{a}{ax+b}\frac{dy}{dz}) = -\frac{a^2}{(ax+b)^2}\frac{dy}{dz}+\frac{a}{ax+b}\frac{d}{dx}(\frac{dy}{dz})$$

$$= -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\frac{d}{dz}(\frac{dy}{dz})\frac{dz}{dx}$$

$$= -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a^2}{(ax+b)^2}\frac{d^2y}{dz^2}$$

and

$$(ax+b)^2D^2y = -a^2\frac{dy}{dz} + a^2\frac{d^2y}{dz^2} = a^2(D_1^2 - D_1)y = a^2D_1(D_1 - 1)y.$$

Continuing in this way, we get

$$(ax+b)^nD^ny = a^nD_1(D_1 - 1)(D_1 - 2)\cdots(D_1 - n + 1)y.$$

After making these replacements, equation (4.37) becomes

$$[p_0a^nD_1(D_1-1)(D_1-2)\cdots(D_1-n+1)+p_1D_1(D_1-1)(D_1-2)\cdots(D_1-n+2)$$

$$+\cdots + p_{n-1}aD_1 + p_n]y = Q(\frac{e^z - b}{a}),$$

is a linear differential equations with constant coefficients.

Example 54. *Solve*

$$(x^3D^3 + 2xD - 2)y = x^2 \ln x + 3x. \qquad (4.38)$$

Solution: The transformation $x = e^z$ reduces the equation as follows

$$xDy = D_1 y \quad \text{and} \quad x^3 D^3 y = D_1(D_1 - 1)(D_1 - 2)y.$$

Substitutes into equation (4.38), we have

$$[D_1(D_1 - 1)(D_1 - 2) + 2D_1 - 2]y = ze^{2z} + 3e^z$$

$$\Longrightarrow \; (D_1^3 - 3D_1^2 + 4D_1 - 2)y = ze^{2z} + 3e^z,$$

which is a third order differential equation with constant coefficients. To find the complementary solution, clearly, the characteristic equation is

$$\alpha^3 - 3\alpha^2 + 4\alpha - 2 = 0 \;\Longrightarrow\; (\alpha - 1)(\alpha^2 - 2\alpha + 2) = 0,$$

so, $\alpha_1 = 1$, $\alpha_{2,3} = 1 \mp i$ and the complementary function is

$$y_c = C_1 e^z + e^z(C_2 \cos z + C_3 \sin z).$$

The particular solution is

$$y_p = \frac{1}{D_1^3 - 3D_1^2 + 4D_1 - 2}\{ze^{2z} + 3e^z\}$$

$$\implies y_p = e^{2z}\frac{1}{(D_1+2)^3-3(D_1+2)^2+4(D_1+2)-2}\{z\}$$

$$+\frac{3}{D_1^3-3D_1^2+4D_1-2}\{e^z\}$$

$$\implies y_p = e^{2z}\frac{1}{D_1^3+3D_1^2+4D_1+2}\{z\}+\frac{3}{(D_1-1)(D_1^2-2D_1+2)}\{e^z\}$$

$$=\frac{e^{2z}}{2[1-(-\frac{D_1^3}{2}-\frac{3}{2}D_1^2-2D_1)]}\{z\}+3e^z\frac{z}{1!}$$

$$=\frac{ze^{2z}}{2}+\frac{e^{2z}}{2}(-2)+3ze^z=\frac{ze^{2z}}{2}-e^{2z}+3ze^z.$$

Therefore, the general solution is

$$y = y_c + y_p = C_1x + x(C_2\cos\ln x + C_2\sin\ln x) + \frac{x^2\ln x}{2} - x^2 + 3x\ln x,$$

where C_1 and C_2 are arbitrary constants.

Homework 27. *Solve the following differential equations:*

1) $(x^2D^2 - xD + 4)y = \cos\ln x + x\sin\ln x.$

2) $[(3x+2)^2D^2 + 3(3x+2)D - 36]y = 3x^2 + 4x + 1.$

Example 55. *Find the general solution of*

$$(x+2)^2\frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x + 4.$$

Solution: Let $x+2=e^z$, so the differential equation above transform to

$$[D_1(D_1-1)-D_1+1]y=3e^z-2 \implies (D_1-1)^2y=3e^z-2,$$

so the complementary function is

$$y_c=C_1e^z+C_2ze^z.$$

In this example,

$$y_p=\frac{1}{(D_1-1)^2}\{3e^z-2\}=3e^z\frac{z^2}{2!}-2\frac{1}{(D_1-1)^2}\{e^0\}=3e^z\frac{z}{2}-2.$$

Thus, the general solution is

$$y=y_c+y_p.$$

4.8 Non-linear differential equations with variable coefficients

In this section various types of higher order differential equations with variable coefficients will be considered. There is no general procedure comparable to that for linear equations. However, for the types treated here, the procedure consists in obtaining from the given equation another of lower order. We consider the following types.

4.8.1 Dependent variable missing (absent)

If the equation is free of y, the independent variable, that is of the form

$$f(y^{(n)}, y^{(n-1)}, \dots, y'', y', x) = 0,$$

the substitution

$$\frac{dy}{dx} = y' = p, \frac{dy^2}{dx^2} = \frac{dp}{dx}, \dots$$

will reduces the order by one.

Example 56. *Solve*

$$y'' + (y')^2 + 1 = 0. \tag{4.39}$$

Solution: Clearly equation (4.39) is a non-linear differential equation such that the dependent variable y is absent. So, let $y' = p$ and $y'' = \frac{dp}{dx}$ and equation (4.39) becomes

$$\frac{dp}{dx} + p^2 + 1 = 0 \Longrightarrow \frac{dp}{p^2+1} = -dx \Longrightarrow \tan^{-1} p = -x + c_1 \Longrightarrow p = \tan(c_1 - x).$$

Since $p = \frac{dy}{dx}$, then

$$\frac{dy}{dx} = \tan(c_1 - x) = \frac{\sin(c_1 - x)}{\cos(c_1 - x)} \Longrightarrow y = \int \frac{\sin(c_1 - x)}{\cos(c_1 - x)} dx \Longrightarrow y = \ln|cox(c_1 - x)| +$$

is a general solution where c_1 and c_2 are arbitrary essential constants.

4.8.2 Independent variable missing (absent)

Suppose we have the equation

$$f(y^{(n)}, y^{(n-1)}, \dots, y'', y') = 0,$$

which the independent variable x is missing. Then the substitution

$$y' = p \quad \text{and} \quad y'' = \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

will reduced the order of the differential equation bye one.

Example 57. *Solve*

$$yy'' - (y')^2 = y^2 \ln(y).$$

Solution: The independent variable x is absent, so, let $y' = p$ and $y'' = p\frac{dp}{dy}$, then the differential equation above takes the form

$$yp\frac{dp}{dy} - p^2 = y^2 \ln y \implies \frac{dp}{dy} - \frac{p}{y} = y \ln y \, p^{-1}$$

is a Bernoulli differential equation in variables p and y with $n = -1$.

Let

$$z = p^{1-n} = p^2 \implies \frac{dz}{dy} = 2p\frac{dp}{dy} = 2p(y \ln y \, p^{-1} + \frac{p}{y})$$

$$\implies \frac{dz}{dy} = 2y \ln y + 2\frac{p^2}{y} = 2y \ln y + 2\frac{z}{y} \implies \frac{dz}{dy} - 2\frac{z}{y} = 2y \ln y$$

which is a first order linear differential equation. Hence

$$z = p^2 = \frac{\int e^{\int -2\frac{dy}{y}}(2y \ln y)dy + c_1}{e^{\int -2\frac{dy}{y}}} = \frac{\int e^{-2\ln y}(2y \ln y)dy + c_1}{e^{-2\ln y}}$$

$$= y^2\left(\int 2\left(\frac{\ln y}{y}\right)dy + c_1\right) = y^2\left((\ln y)^2 + c_1\right).$$

Now,

$$p = \pm\sqrt{y^2\left((\ln y)^2 + c_1\right)} = y\sqrt{(\ln y)^2 + c_1} \implies \frac{dy}{y\sqrt{(\ln y)^2 + c_1}} = dx$$

$$\implies \ln(\ln y + \sqrt{(\ln y)^2 + c_1}) = x + k \implies \ln y + \sqrt{(\ln y)^2 + c_1} = c_2 e^x$$

$$\implies \sqrt{(\ln y)^2 + c_1} = c_2 e^x - \ln y \implies c_1 = c_2^2 e^{2x} - 2c_2 e^x \ln y$$

$$\implies \ln y = Ae^x + Be^{-x},$$

where A and B are constants.

4.9 Second order linear differential equations with variable coefficients

Consider a second order equation

$$y'' + \alpha(x)y' + \beta(x)y = \gamma(x), \tag{4.40}$$

where α, β and γ are functions of x.

If $y = u(x)$ is a solution of the corresponding homogeneous equation (4.40), that is,

$$y'' + \alpha(x)y' + \beta(x)y = 0,$$

then

$$u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = 0. \tag{4.41}$$

Now, let $y(x) = u(x)v(x)$ is a solution of the differential equation (4.40), where $v(x)$ is a function of x, so

$$y' = u'v + uv' \quad \text{and} \quad y'' = u''v + 2u'v' + uv'',$$

and substitutes in equation(4.40), we have

$$u''v + 2u'v' + uv'' + \alpha(x)(u'v + uv') + \beta(x)uv = \gamma(x)$$

$$\Longrightarrow (u'' + \alpha(x)u' + \beta(x)u)v + 2u'v' + uv'' + \alpha(x)uv' = \gamma(x).$$

So, by equation (4.41), we get

$$uv'' + (2u' + \alpha(x)u)v' = \gamma(x) \tag{4.42}$$

which is a second order differential equation of variable v and x and since in equation (4.42), the dependent variable is not appeared, so, it can be solved by letting $v' = p$ and $v'' = \frac{dp}{dx} = p'$, then (4.42) becomes

$$u\frac{dp}{dx} + (2u' + \alpha(x)u)p = \gamma(x),$$

which is a first order linear differential equation in variable p and x. Then we have the following theorem.

Theorem 8. *If $y = u(x)$ is a solution of a second order homogeneous differential equation*

$$y'' + \alpha(x)y' + \beta(x)y = 0,$$

then, the substitution $y(x) = u(x)v(x)$ reduces the differential equation

$$y'' + \alpha(x)y' + \beta(x)y = \gamma(x),$$

to a linear differential equation of first order.

Example 58. *If $y = x$ is a solution of the corresponding homogeneous equation*

$$y'' + 3x^2y' - 3xy = 5x^3. \tag{4.43}$$

Solution: Let $y = vx$ be a solution of equation (4.43), then

$$y' = v'x + v \quad \text{and} \quad y'' = 2v' + xv''.$$

So, equation (4.43) becomes

$$2v' + xv'' + 3x^2(xv' + v) - 3x^2v = 5x^3 \Longrightarrow xv'' + (2 + 3x^3)v' = 5x^3.$$

Now, let $v' = p$ and $v'' = p'$, so we have

$$x\frac{dp}{dx} + (2 + 3x^3)p = 5x^3,$$

which is a first order linear differential equation in variables p and x.

$$\frac{dp}{dx} + \frac{2 + 3x^3}{x}p = 5x^2$$

$$\therefore\ p = \frac{\int e^{\int (\frac{2+3x^3}{x})dx}\, 5x^2 dx + C_2}{e^{\int (\frac{2+3x^3}{x})dx}}.$$

Since, $p = v'$, then $v = \int p\, dx$. Thus, the general solution is

$$y = C_1 x + vx,$$

where C_1 and C_2 are arbitrary constants.

4.10 How one can find the particular solution of a homogeneous differential equation with variable coefficints

Consider a second order differential equation of the form

$$y'' + P(x)y' + Q(x)y = 0. \tag{4.44}$$

To find a particular solution of equation (4.44), there are several cases:

Case 1: If $y = x$ is a particular solution of the equation (4.44), then $y' = 1$ and $y'' = 0$. Substitutes into equation (4.44), we have

$$P(x) + xQ(x) = 0.$$

Then, if $P(x)+xQ(x) = 0$, so $y = x$ is a particular solution of equation (4.44).

Example 59. *Solve*

$$(D^2 - \frac{3}{x}D + \frac{3}{x^2})y = 2x - 1. \tag{4.45}$$

Solution: Here, $P(x) + xQ(x) = -\frac{3}{x} + x\frac{3}{x^2} = 0$, so, $y = x$ is a particular solution of equation (4.45). Thus, the transformation $y = xv$ reduces the equation (4.45) to a linear first order differential equation. Now,

$$Dy = x\frac{dv}{dx} + v \quad \text{and} \quad D^2y = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substitutes in equation (4.45), we have

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 3\frac{dv}{dx} - \frac{3}{x}v + \frac{3}{x}v = x\frac{d^2v}{dx^2} - \frac{dv}{dx} = 2x - 1$$

$$\Longrightarrow \frac{d^2v}{dx^2} - \frac{1}{x}\frac{dv}{dx} = \frac{2x-1}{x}.$$

Let, $\frac{dv}{dx} = p$ and $\frac{d^2v}{dx^2} = p'$, so

$$\frac{dp}{dx} - \frac{1}{x}p = \frac{2x-1}{x},$$

which is a linear first order differential equation.

$$\therefore p = \frac{\int e^{-\int \frac{1}{x}dx}\,(2 - \frac{1}{x})dx + C_1}{e^{-\int \frac{1}{x}dx}} = x(\int (\frac{2}{x} - \frac{1}{x^2})dx + C_1)$$

$$\Longrightarrow p = x(2\ln|x| + \frac{1}{x} + C_1).$$

But,

$$p = \frac{dv}{dx} = 2x \ln |x| + 1 + C_1 x$$

$$\Longrightarrow v = \frac{y}{x} = \int (2x \ln |x| + 1 + C_1 x) dx = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + x + \frac{C_1}{2}x^2 + C_2$$

$$\Longrightarrow y = \frac{1}{2}x^3 \ln(x) - \frac{1}{4}x^3 + x^2 + \frac{C_1}{2}x^3 + C_2 x$$

is a general solution where C_1 and C_2 are arbitrary constant.

Homework 28. *Solve*

$$x^2(x + 1\frac{d^2y}{dx^2}) - x(2 + 4x + x^2)\frac{dy}{dx} + (2 + 4x + x^2)y = -x^4 - 2x^3.$$

Case 2: Find a condition that $y = ax + b$ is a particular solution of equation (4.44).

Case 3: If $y = e^{ax}$ is a solution of the differential equation (4.44), where a is any constant. Then, $y' = ae^{ax}$ and $y'' = a^2e^{ax}$. Substitutes in equation (4.44), we have

$$a^2e^{ax} + P(x)ae^{ax} + Q(x)e^{ax} = 0 \quad \Longrightarrow \quad e^{ax}\big(a^2 + aP(x) + Q(x)\big) = 0.$$

Since, $e^{ax} \neq 0$, then $a^2+aP(x)+Q(x) = 0$. So, if $a^2 + aP(x) + Q(x) = 0$, then $y = e^{ax}$ is a particular solution of equation (4.44).

Example 60. *Solve*

$$(1 + x)y'' + (4x + 5)y' + (4x + 6)y = e^{-2x}. \qquad (4.46)$$

Solution: Clearly,

$$y'' + \frac{4x + 5}{1 + x}y' + \frac{4x + 6}{1 + x}y = \frac{e^{-2x}}{1 + x},$$

So, $P(x) = \frac{4x+5}{1+x}$ and $Q(x) = \frac{4x+6}{1+x}$. Now,

1) $P(x) + xQ(x) = \frac{4x+5}{1+x} + \frac{x(4x+6)}{1+x} = \frac{4x+5+x(4x+6)}{1+x} = \frac{4x^2+10x+5}{1+x} \neq 0$. So, $y = x$ is not a particular solution of the corresponding homogeneous equation (4.46).

2) $a^2 + aP(x) + Q(x)\big) = a^2 + \frac{4ax+5a}{1+x} + \frac{4x+6}{1+x} = \frac{a^2(1+x)+4ax+5a+4x+6}{1+x} = 0$

$\Longrightarrow a^2+a^2x+4ax+5a+4x+6 = 0 \Longrightarrow (a^2+4a+4)x+a^2+5a+6 = 0$

$\Longrightarrow (a + 2)^2 = 0$ and $(a + 2)(a + 3) = 0 \Longrightarrow a + 2 = 0 \Longrightarrow a = -2$.

Thus, $y = e^{-2x}$ is a particular solution of the corresponding homogeneous equation (4.46).

Let $y = e^{-2x}v(x)$ be a solution of the differential equation (4.46), then

$$y' = -2e^{-2x}v + e^{-2x}v'$$

and

$$y'' = 4e^{-2x}v - 4e^{-2x}v' + e^{-2x}v''.$$

Substitutes in (4.46), we have

$$(1+x)e^{-2x}(4v-4v'+v'')+e^{-2x}(4x+5)(-2v+v')+e^{-2x}(4x+6)v = e^{-2x}$$

$$(1+x)v'' + (-4(1+x) + (4x+5))v' = 1 \implies (1+x)v'' + v' = 1.$$

Note that the dependent variable v is not appear, so, let $p = v'$ and $\frac{dp}{dx} = v''$, then

$$(1+x)\frac{dp}{dx} = 1 - p \Longrightarrow \frac{dp}{1-p} = \frac{dx}{1+x} \implies \frac{1}{(1-p)} = C_1(1+x)$$

$\implies p = 1 - \frac{1}{C_1(1+x)}$. Since,

$$p = v' = \frac{dv}{dx} \Longrightarrow v = \int [1 - \frac{1}{C_1(1+x)}]dx \Longrightarrow v = x - \frac{1}{C_1}\ln(1+x) + C_2.$$

Then,

$$y = e^{-2x} + e^{-2x}(x - \frac{1}{C_1}\ln(1+x) + C_2)$$

is a general solution where C_1 and C_2 are arbitrary constants.

Case 4: Let $y = u(x)v(x)$ be a solution of

$$y'' + P(x)y' + Q(x)y = f(x). \tag{4.47}$$

So,

$$y' = uv' + u'v \quad \text{and} \quad y'' = uv'' + 2u'v' + u''v.$$

Substitutes in (4.47), we get

$$uv'' + 2u'v' + u''v + P(x)(uv' + u'v) + Q(x)uv = f(x)$$

$$\Longrightarrow uv'' + (2u' + P(x)u)v' + (u'' + P(x)u' + Q(x)u)v = f(x) \tag{4.48}$$

If u is chosen so that,

$$2u' + P(x)u = 0 \Longrightarrow 2\frac{du}{dx} + P(x)u = 0 \Longrightarrow \frac{du}{u} + \frac{1}{2}P(x)dx = 0$$

$$\Longrightarrow \ln u = -\frac{1}{2}\int P(x)dx \Longrightarrow u = e^{-\frac{1}{2}\int P(x)dx}$$

$$\Longrightarrow u' = -\frac{1}{2}P(x)e^{-\frac{1}{2}\int P(x)dx} = -\frac{1}{2}P(x)u$$

$$\Longrightarrow u'' = -\frac{1}{2}P'(x)u - \frac{1}{2}P(x)u' = -\frac{1}{2}P'(x)u + \frac{1}{4}P^2(x)u$$

Substitute in

$$u'' + P(x)u' + Q(x)u = -\frac{1}{2}P'u + \frac{1}{4}P^2u - \frac{1}{2}P^2 + Qu.$$

So, equation (4.48), becomes

$$v'' + (\frac{u'' + Pu' + Qu}{u})v = \frac{f(x)}{u} \tag{4.49}$$

If $\frac{u''+Pu'+Qu}{u}$ is a constant, then,

$$\frac{-\frac{1}{2}P'u - \frac{1}{4}P^2u + Qu}{u} = -\frac{1}{2}P' - \frac{1}{4}P^2 + Q = C$$

where C is a constant.

Then, equation (4.49) becomes

$$v'' + C\,v = \frac{f(x)}{u},$$

which is a second order differential equation with constant coefficients and $u = e^{-\frac{1}{2}\int P(x)dx}$.

Now, if

$$-\frac{1}{2}P' - \frac{1}{4}P^2 + Q = \frac{C}{x^2},$$

where C is a constant, then

$$v'' + \frac{C}{x^2}v = \frac{f(x)}{u} \implies x^2v'' + Cv = \frac{x^2}{u}f(x),$$

which is a Cauchy equation.

Example 61. *Solve*

$$y'' - 4xy' + 4x^2y = xe^{x^2}. \tag{4.50}$$

Solution: Here, in this example, $P(x) = -4x$ and $Q(x) = 4x^2$.

1) Since, $P(x) + xQ(x) = -4x + 4x^3 \neq 0$, so, $y = x$, is not a particular solution of the corresponding homogeneous equation equation (4.50).

2) Since there is no number a such that $a^2 + a(-4x) + 4x^2 = 0$.

3) Note $P'(x) = -4$, $P^2 = 16x^2$, then

$$-\frac{1}{2}P' - \frac{1}{4}P^2 + Q = -\frac{1}{2}(-4) - \frac{1}{4}16x^2 + 4x^2 = 2 = constant = C.$$

So,

$$u = e^{-\frac{1}{2}\int P(x)dx} = e^{-\frac{1}{2}\int(-4x)dx} = e^{x^2}.$$

Now, let $y = e^{x^2} v$ is a solution of the differential equation (4.50), so,

$$y' = 2xe^{x^2}v + e^{x^2}v' \quad \text{and} \quad y'' = 4x^2e^{x^2}v + 4xe^{x^2}v + e^{x^2}v'' + 2e^{x^2}v.$$

Substitute in equation (4.50), we have

$$v'' + C\,v = \frac{f(x)}{u} = \frac{xe^{x^2}}{e^{x^2}} = x$$

so, we have

$$v'' + 2v = x$$

which is a second order differential equation with constant coefficients. The characteristic equation is $\alpha^2 + 2 = 0 \implies \alpha_{1,2} = \mp i\sqrt{2}$, therefore,

$$v_c = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x.$$

To find the particular solution,

$$(D^2+2)v = x \implies v_p = \frac{1}{D^2+2}\{x\} = \frac{1}{2} = \frac{1}{[1-(-\frac{D^2}{2})]}\{x\}$$

$$\implies v_p = \frac{1}{2}[1 - \frac{D^2}{2} + (\frac{D^2}{2})^2 + \cdots]\{x\} = \frac{1}{2}[x+0] = \frac{1}{2}x.$$

$$\therefore v = v_c + v_p = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x + \frac{1}{2}x.$$

Hence,

$$y = e^{x^2} + uv = e^{x^2} + e^{x^2}(C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x + \frac{1}{2}x),$$

is a general solution where C_1 and C_2 are arbitrary constants.

Homework 29. *1)* $y'' - 2xy' + (x^2 + 2)y = e^{\frac{1}{2}(x^2+2x)}$.
2) $(1 + x)^2 y'' + (x + 1)(x - 2)y' + (2 - x)y = 0$.

4.11 Applications of Second Order Differential Equations

Chapter 5

The Laplace transformation (Laplace's transform and its application to differential equations)

5.1 Laplace Transformation

Pierre Simon de Laplace (1749-1827) was a French Mathematician who made many discoveries in mathematical physics. His last words were reported to be "*What we know is very slight; what we don't know is immense*"

Definition 18. *The Laplace transform of the function* $f(t)$, $0 \leq t \leq \infty$ *(*$t \geq 0$*) is the function* $F(s) = L\{f(t)\}$ *defined by*

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)dt. \tag{5.1}$$

Example 62. *With* $f(t) = 1, t \geq 0$*, the definition of the Laplace transform* (5.1)*, gives*

$$L\{1\} = \lim_{b\to\infty} \int_0^b e^{-st}dt = \lim_{b\to\infty} [-\frac{1}{s}e^{-st}]_0^b = \lim_{b\to\infty} [-\frac{1}{s}e^{-sb} + \frac{1}{s}] = \frac{1}{s}, \quad s > 0.$$

Remark 23. *It is good practice to specify the domain of the Laplace transform. The limit we computed in the example above, would not exists if* $s < 0$*, for then* $\frac{1}{s}e^{-bs}$ *would become unbounded as* $b \to \infty$*. Hence,* $L\{1\}$ *is defined only for* $s > 0$*.*

Homework 30. *Find the Laplace transform of* $f(t) = e^{2t}$ *and specify its domain.*

Definition 19 (Sectional or piecewise continuity)**.** *A function* $f(t)$ *is called sectional continuous or piecewise continuous in an interval* $\alpha \leq t \leq \beta$*, if the interval can be subdivided into a finite number of* *intervals in each of which the function is continuous and has finite right and left hand limits.*

Definition 20 (Functions of exponential order). *If real constants $M > 0$ and λ exists such that for all $t > N$ (N is a number)*

$$|e^{-\lambda t} f(t)| < M \quad or \quad |f(t)| < M e^{\lambda t}.$$

We say that $f(t)$ is a function of exponential order λ as $t \to \infty$ or, briefly, is of exponential order.

Theorem 9 (Sufficient condition for existence of the Laplace transform). *If $f(t)$ is sectionally continuous in every finite interval $0 \leq t \leq N$ and of exponential order λ for $t > N$, then its Laplace transform $F(s)$ exists for all $s > \lambda$.*

Theorem 10 (Linearity of the Laplace transform). *If a and b are constants, then*

$$L\{af(t) + bg(t)\} = a\, L\{f(t)\} + b\, L\{g(t)\},$$

for all s such that the Laplace transforms of the functions f and g both exists.

Proof. The proof of this theorem follows immediately from the linearity of the operation of taking limits of integration

$$\begin{aligned}
L\{af(t)+bg(t)\} &= \int_0^\infty e^{-st}\big(af(t)+bg(t)\big)dt \\
&= \lim_{c\to\infty}\int_0^c e^{-st}\big(af(t)+bg(t)\big)dt \\
&= a\,\big(\lim_{c\to\infty}\int_0^c e^{-st}f(t)dt\big) + b\,\big(\lim_{c\to\infty}\int_0^c e^{-st}g(t)dt\big) \\
&= a\,L\{f(t)\} + b\,L\{g(t)\}.
\end{aligned}$$

5.2 Laplace transforms of some elementary functions

Theorem 11. *(a)* $L\{k\} = \frac{k}{s}$, $\quad s > 0$ *for any constant* k.

(b) $L\{e^{at}\} = \frac{1}{s-a}$, $\quad s > a$.

(c) $L\{t^n\} = \frac{n!}{s^{n+1}}$, $\quad s > 0$, $n = 1, 2, \ldots$.

(d) $L\{\sin kt\} = \frac{k}{s^2+k^2}$, $\quad s > 0$.

(e) $L\{\cos kt\} = \frac{s}{s^2+k^2}$, $\quad s > 0$.

(f) $L\{\sinh kt\} = \frac{k}{s^2-k^2}$, $\quad s > k$.

(g) $L\{\cosh kt\} = \frac{s}{s^2-k^2}$, $\quad s > k$.

Proof.

(a) From definition of the Laplace transform, we have

$$L\{k\} = \int_0^\infty e^{-st}\, k\, dt = \lim_{b\to\infty} \int_0^b e^{-st}\, k\, dt = k \lim_{b\to\infty} \int_0^b e^{-st}\, dt$$
$$= -\frac{k}{s} \lim_{b\to\infty} [e^{-st}]_0^b = -\frac{k}{s}(0-1) = \frac{k}{s}, \quad s > 0.$$

Therefore,

$$L\{k\} = \frac{k}{s}, \quad s > 0.$$

If $s < 0$, then the integral does not converge and the Laplace transform is not defined.

Homework 31. *Prove* $(b), \dots, (g)$.

Homework 32. *Prove* (c) *by using Gamma function* $\Gamma(x)$ *such that*

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \quad \textit{and} \quad \Gamma(n+1) = n! \quad \textit{for } n \textit{ is positive integer.}$$

Example 63. *1)* $L\{\sin 3t\} = \frac{3}{s^2+9}, \quad s > 0.$

2) $L\{3e^{2t} + 2\sin^2 3t\} = L\{3e^{2t} + 2(\frac{1-\cos 6t}{2})\} = L\{3e^{2t}\} + L\{1\} - L\{\cos 6t\} = \frac{3}{s-2} + \frac{1}{s} - \frac{s}{s^2+36} = \frac{3s^2+144s-72}{s(s-2)(s^2+36)}, \quad for\ s > 0.$

Homework 33. *Find the following Laplace transforms:*

1) $L\{\sin t \cos t\}$.

2) $L\{t^{1/2}\}$.

Theorem 12. *If a is any real number, then*

$$L\{e^{at}f(t)\} = F(s-a),$$

where $F(s) = L\{f(t)\}$.

Proof. The proof follows directly from the definition of the Laplace transform

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st}\, e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a).$$

Notation: $L\{e^{at}f(t)\} = L\{f(t)\}|_{s\to s-a}$.

This property is known as shifting property.

Example 64. *Find $L\{e^{-2t}t^3\}$.*

Solution: By Theorem above, we have

$$L\{e^{-2t}t^3\} = L\{t^3\}|_{s\to s-(-2)} = \frac{3!}{s^4}|_{s\to s+2} = \frac{3!}{(s+2)^4}.$$

Homework 34. *Find $L\{e^{2t}\cos t \sin t\}$.*

Theorem 13. *If $L\{f(t)\} = F(s)$, then $L\{f(kt) = \frac{1}{k}F(\frac{s}{k})\}$, where k is a constant*

Proof. Since we have

$$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt = F(s),$$

then

$$L\{f(kt)\} = \int_0^\infty e^{-st}\, f(kt)\, dt.$$

Now, let $u = kt$, then $t = \frac{u}{k}$ and $dt = \frac{du}{k}$, so,

$$L\{f(kt)\} = \int_0^\infty e^{-st}\, f(kt)\, dt = \int_0^\infty e^{-\frac{u}{k}s}\, f(u)\, \frac{du}{k}$$

$$= \frac{1}{k}\int_0^\infty e^{-\frac{s}{k}u}\, f(u)\, du = \frac{1}{k}L\{f(\frac{s}{k})\} = \frac{1}{k}F(\frac{s}{k}).$$

5.3 Laplace transform of a derivation

$L\{f'(t)\} = sL\{f(t)\} - f(0).$

Proof. From the definition of the Laplace transformation, we have

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt.$$

We now use integration by parts.

$$\text{Let} \quad u = e^{-st} \quad \Longrightarrow \quad du = -se^{-st}dt$$

$$dv = f'(t)dt \quad \Longrightarrow \quad v = f(t)$$

So,

$$L\{f'(t)\} = [e^{-st}f(t)]_0^\infty + s\int_0^\infty e^{-st}f(t)dt = sL\{f(t)\} - f(0).$$

Because, the integrated term $e^{-st}f(t)$ approached to zero (when $s > 0$) as $t \to \infty$, and its value at the lower limit $t = 0$ contributes $-f(0)$. An extension of these ideas to equations of order two can easily made by letting the function $g(t) = f'(t)$, then

$$L\{f''(t)\} = L\{g'(t)\} = sL\{g(t)\} - g(0)$$

$$= sL\{f'(t) - f'(0)\}$$

$$= s[sL\{f(t)\} - f(0)] - f'(0)$$

$$= s^2L\{f(t)\} - sf(0) - f'(0).$$

A repetition of this calculation gives

$$L\{f'''(t)\} = sL\{f''(t)\} - f''(0) = s^3L\{f(t)\} - s^2f(0) - sf'(0) - f''(0).$$

After finitely many such steps, we obtain the following extensionn

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

5.4 Inverse Laplace Transform

Definition 21. *If $L\{f(t)\} = F(s)$, then $f(t)$ is the inverse Laplace transform of $F(s)$ and is written $f(t) = L^{-1}\big(F(s)\big)$.*

Example 65. *Evaluate the following inverse Laplace transforms:*
1) $L^{-1}\{\frac{1}{s^4}\}$. 2) $L^{-1}\{\frac{15}{s^2+4s+13}\}$.

Solution:

1) $L^{-1}\{\frac{1}{s^4}\} = \frac{3!}{3!}L^{-1}\{\frac{1}{s^4}\} = \frac{1}{3!}L^{-1}\{\frac{3!}{s^4}\} = \frac{t^3}{6}$.

2) First complete the square in the denominator

$$L^{-1}\{\frac{15}{s^2+4s+13} = L^{-1}\{\frac{15}{(s+2)^2+9}\}.$$

Since, we know that $L^{-1}\{\frac{k}{s^2+k^2} = \sin kt\}$, we proceed as follows

$$L^{-1}\{\frac{15}{(s+2)^2+9}\} = L^{-1}\{\frac{5\cdot 3}{(s+2)^2+9}\} = 5L^{-1}\{\frac{3}{(s+2)^2+9}\}$$

$$= 5e^{-2t}L^{-1}\{\frac{3}{s^2+9}\} = 5e^{-2t}\sin 3t.$$

Homework 35.

Theorem 14. *If c_1 and c_2 are constants, then*

$$L^{-1}\{c_1F_1(s)+c_2F_2(s)\}=c_1L^{-1}\{F(s)\}+c_2L^{-1}\{F_2(s)\}.$$

Theorem 15. *Prove that*

$$L^{-1}\{F(s-a)\}=e^{at}L^{-1}\{F(s)\}.,$$

where a is a constant.

Proof. From

$$F(s)=\int_0^\infty e^{-st}f(t)dt=L\{f(t)\},$$

we obtain

$$F(s-a)=\int_0^\infty e^{-(s-a)t}f(t)dt=\int_0^\infty e^{-st}\big(e^{at}f(t)\big)dt=L\{e^{at}f(t)\}$$

Since, $f(t)=L^{-1}\big(F(s)\big)$, then

$$L^{-1}\{F(s-a)\}=e^{at}f(t)=e^{at}L^{-1}\{F(s)\}.$$

Example 66. *Evaluate*

$$L^{-1}\{\frac{s+1}{s^2+6s+25}\}.$$

Solution:

$$\begin{aligned}
L^{-1}\{\frac{s+1}{s^2+6s+25}\} &= L^{-1}\{\frac{s+1}{(s+3)^2+16}\} = e^{-3t}L^{-1}\{\frac{s-2}{s^2+16}\} \\
&= e^{-3t}\big[L^{-1}\{\frac{s}{s^2+16}\} - L^{-1}\{\frac{2}{s^2+16}\}\big] \\
&= e^{-3t}\big[\cos 4t - \frac{1}{2}\sin 4t\big].
\end{aligned}$$

Example 67. *Evaluate*

$$L^{-1}\{\frac{3s-1}{s(s-1)}\}.$$

Solution: Using partial fraction decomposition, we have

$$\frac{3s-1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = \frac{A(s-1)+Bs}{s(s-1)} = \frac{(A+B)s-A}{s(s-1)}$$

$$\Longrightarrow A = 1 \quad \text{and} \quad A+B=3 \Longrightarrow B = 3-1 = 2.$$

Thus,

$$L^{-1}\{\frac{3s-1}{s(s-1)}\} = L^{-1}\{\frac{1}{s} + \frac{2}{s-1}\} = L^{-1}\{\frac{1}{s}\} + Ł^{-1}\{\frac{2}{s-1}\} = 1 + 2e^{t}.$$

Homework 36. *Evaluate the following Laplace transforms:*

1. $L^{-1}\{\frac{1}{(s+1)(s+3)(2s-1)}\}$.

2. $L^{-1}\{\frac{s-4}{(s+1)(s^2+4)}\}$.

Example 68. *Evaluate* $L^{-1}\{\frac{5}{(s-1)^3}\}$.

Solution:

$$L^{-1}\{\frac{5}{(s-1)^3}\} = 5e^{t}L^{-1}\{\frac{1}{s^3}\} = \frac{5e^t}{2!}L^{-1}\{\frac{2!}{s^3}\} = \frac{5e^t}{2!}t^2.$$

5.5 Initial Value Problems

Let $y = y(x)$ be a solution of a differential equation satisfying

$$y(x_0) = y_0. \tag{5.2}$$

Equation (5.2) is called an initial condition of differential equation. A differential equation together with an initial condition is called initial value problem. Therefore,

$$y' - x = 1, \quad y(0) = 0,$$

is an example of initial value problem.

5.6 Transformation of Initial Value Problems

We now discuss the application of Laplace transform to solve a linear differential equation with a constant coefficients such as

$$ay''(t) + by'(t) + cy(t) = f(t),$$

with given initial condition $y(0) = y_0$ and $y'(0) = y'_0$.
We must using the following procedure:

1. Take Laplace transform to both sides of the differential equation.

2. Substitute the initial conditions.

3. We take inverse Laplace transform.

Example 69. *Solve the initial value problem*

$$y'' + y = 1; \quad y(0) = 2, \quad y'(0) = 0. \tag{5.3}$$

Solution: First, take the Laplace transform to both sides of equation (5.3) and substitute the initial conditions, we have

$$L\{y'' + y\} = L\{1\} \implies s^2L\{y\} - sy(0) - y'(0) + L\{y\} = L\{1\}$$

$$\implies s^2L\{y\} - 2s + L\{y\} = \frac{1}{s} \implies L\{y\}(s^2+1) = \frac{1}{s} + 2s = \frac{1+2s^2}{s}$$

$$\implies L\{y\} = \frac{1+2s^2}{s(s^2+1)} \implies y = L^{-1}\{\frac{1+2s^2}{s(s^2+1)}\}.$$

Use the partial fraction decomposition to solve this inverse Laplace transform.

$$\frac{1+2s^2}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{As^2 + A + Bs^2 + Cs}{s(s^2+1)} = \frac{(A+B)s^2 + Cs + A}{s(s^2+1)}.$$

Clearly, $A = 1$, $C = 0$ and $A + B = 2 \implies B = 1$. Thus,

$$y = L^{-1}\{\frac{1}{s} + \frac{s}{s^2+1}\} = L^{-1}\{\frac{1}{s}\} + L^{-1}\{\frac{s}{s^2+1}\}$$

$$\implies y = 1 + \cos t.$$

Example 70. *Solve*

$$y'' - 5y' + 6y = 2e^{-t} \tag{5.4}$$

subject to the initial condition $y(0) = 2$ *and* $y'(0) = 1$.

Solution: Again, take the Laplace transform to both sides of equation (5.4) and substitute the initial conditions, we have

$$L\{y'' - 5y' + 6y\} = L\{2e^{-t}\}$$

$$\Longrightarrow s^2L\{y\} - sy(0) - y'(0) - 5[sL\{y\} - y(0)] + 6L\{y\} = L\{2e^{-t}\} = \frac{2}{s+1}$$

$$\Longrightarrow s^2L\{y\} - 1 - 5sL\{y\} + 6L\{y\} = \frac{2}{s+1} \Longrightarrow L\{y\}(s^2 - 5s + 6) = \frac{2}{s+1} + 1$$

$$\Longrightarrow L\{y\} = \frac{2}{(s+1)(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}$$

Now,

$$\frac{2}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}.$$

So,

$$\frac{2}{(s+1)(s-2)(s-3)} = \frac{A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)}{(s+1)(s-2)(s-3)}$$

It is easy to see that $A = \frac{1}{6}$, $B = \frac{1}{2}$ and $C = -\frac{2}{3}$. We also do the same procedure for the second fraction. Finally we get,

$$L\{y\} = \frac{\frac{1}{6}}{s+1} + \frac{\frac{1}{2}}{s-3} - \frac{\frac{2}{3}}{s-2} + \frac{1}{s-3} - \frac{1}{s-2} = \frac{\frac{1}{6}}{s+1} + \frac{\frac{3}{2}}{s-3} - \frac{\frac{5}{3}}{s-2}.$$

Therefore,

$$y = L^{-1}\{\frac{1}{6(s+1)} + \frac{3}{2}\frac{1}{s-3} - \frac{5}{3}\frac{1}{s-2}\} = \frac{1}{6}e^{-t} + \frac{3}{2}e^{3t} - \frac{5}{3}e^{2t}.$$

Homework 37. *1. Solve*

$$y'' + y = 4te^t,$$

subject to the initial condition $y(0) = -2$ *and* $y'(0) = 0$.

2. Solve the initial value problem

$$x'' - x' - 6x = 0; \quad x(0) = 2, \quad x'(0) = -1.$$

3. Solve the initial value problem

$$y'' + 4y = \sin 3t; \quad y(0) = y'(0) = 0.$$

5.7 Derivative of the Laplace transforms

Theorem 16. *If $f(t)$ is a piecewise continuous for $t \geq 0$ and of exponential order for some $c > 0$ and if $F(s) = L\{f(t)\}$, then*

$$\frac{d}{ds}F(s) = -L\{t\, f(t)\}.$$

Example 71. *Evaluate $L\{t\, e^{at}\}$.*

Solution: From theorem above (Theorem 8), we have

$$L\{t\, e^{at}\} = -\frac{d}{ds}F(s) = -\frac{d}{ds}L\{e^{at}\} = \frac{d}{ds}(\frac{1}{s-a}) = \frac{1}{(s-a)^2}.$$

Homework 38. *Evaluate the following:*

1. $L\{t^2\, e^{at}\}$.

2. $L\{t\, e^{2t}\, \cos 3t\}$.

3. $L\{\sin t + t\, \cos t\}$, *without using the linearity property.*

Chapter 6

The power series method

6.1 Power Series Method (Power Series Solutions)

Definition 22. *A function $f(x)$ is said to be analytic at x_0, if it can be represented by a Taylor series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \tag{6.1}$$

which converges for all x in some open interval containing x_0. If

$x_0 = 0$, then the sees in (6.1) is the Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n.$$

6.2 Maclaurin series expansion of some elementary functions

(1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad -\infty < x < \infty.$

(2) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad -\infty < x < \infty.$

(3) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad -\infty < x < \infty.$

(4) $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}, \qquad -\infty < x < \infty.$

(5) $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \qquad -\infty < x < \infty.$

(6) $\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \qquad -1 < x < 1 \ \big(|x| < 1\big).$

(7) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1$ (Geometric series).

(8) $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-3)}{3!}x^3 + \cdots,$ (Binomial series). If α is nonnegative integer, then the binomial series is converges for all x. Otherwise, $|x| < 1$ converges and $|x| > 1$ diverges.

Thus, if the Taylor series of the function f converges to $f(x)$ for all x in some open interval containing x_0, then we say that the function $f(x)$ is analytic at x_0. For example, every polynomial is analytic everywhere and every rational function is analytic whenever its denominator is nonzero. For instance, $\tan x = \frac{\sin x}{\cos x}$ is analytic at $x_0 = 0$.

6.3 Solutions around ordinary points

For purpose of discussion, it is useful to place the second order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{6.2}$$

in the standard form

$$y'' + P(x)y' + Q(x)y = 0, \tag{6.3}$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$, $a_2(x) \neq 0$.

Definition 23. *A point $x = x_0$ is an ordinary point of equation* (6.3)*, if both $P(x)$ and $Q(x)$ are analytic at x_0; that is, if both $P(x)$ and $Q(x)$ has a Taylor series expansion about $x = x_0$. A point that is not an ordinary point is called a singular point of the equation.*

Example 72. *The differentia equation*

$$(1-x^2)y'' - 6xy' - 4y = 0,$$

has an ordinary points at $x = 0$. *The points* $x = 1$ *and* $x = -1$ *are singular points of the equation.*

Example 73. *Singular points need not be real numbers. The equation*

$$(x^2+4)y'' + 2xy' - 12y = 0,$$

has singular points at $x = \mp 2i$. *The point* $x = 0$, *is an ordinary point.*

Example 74. *Solve the equation*

$$y'' + 4y = 0, \tag{6.4}$$

near the ordinary point $x = 0$.

Solution: Since $x = 0$ is an ordinary point, then the series solution is

$$y = \sum_{n=0}^{\infty} a_n x^n \implies y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1} \implies y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}.$$

We now substitute, y, y' and y'' in the equation (**??**), we have

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\implies \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + 4\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\implies \sum_{n=2}^{\infty} [n(n-1)\, a_n + 4a_{n-2}] x^{n-2} = 0$$

$$\implies n(n-1)\, a_n + 4a_{n-2} = 0 \implies a_n = -\frac{4}{n(n-1)} a_{n-2}, \quad n \geq 2$$

$$a_2 = -\frac{4}{2\cdot 1} a_0, \qquad a_3 = -\frac{4}{3\cdot 2} a_1$$

$$a_4 = -\frac{4}{4\cdot 3} a_2, \qquad a_5 = -\frac{4}{5\cdot 4} a_3$$

$$\vdots \qquad\qquad \vdots$$

$$a_{2n} = -\frac{4}{2n\cdot(2n-1)} a_{2n-2}, \qquad a_{2n+1} = -\frac{4}{(2n+1)\cdot 2n} a_{2n-1}$$

Now,

$$a_2 \cdot a_4 \cdot \cdots \cdot a_{2n} = \frac{(-1)^n 4^n}{(2n)!} a_0 \cdot a_2 \cdot \cdots \cdot a_{2n-2},$$

which simplify to

$$a_{2n} = \frac{(-1)^n 4^n}{(2n)!} a_0, \quad n \geq 1.$$

Similarly,

$$a_{2n+1} = \frac{(-1)^n 4^n}{(2n+1)!} a_1, \quad n \geq 1.$$

Since,

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

so,

$$y = a_0 + \sum_{n=1}^{\infty} a_{2n} x^{2n} + a_1 x + \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}$$

$$= a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n}\right] + a_1 \left[x + \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{(2n+1)!} x^{2n+1}\right]$$

$$= a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right] + \frac{1}{2} a_1 \left[2x + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}\right]$$

$$y = a_0 \cos 2x + \frac{1}{2} a_1 \sin 2x,$$

is the general solution where a_0 and a_1 are arbitrary constants.

Example 75. *Find the general solution in powers in x of*

$$(x^2 - 4)y'' + 3xy' + y = 0. \tag{6.5}$$

Then find the particular solution with $y(0) = 4$ and $y'(0) = 1$.

Solution: The only singular points of equation (6.5) are ∓ 2. Since, $x_0 = 0$ is an ordinary point of (6.5), then

$$y = \sum_{n=0}^{\infty} c_n x^n \implies y' = \sum_{n=1}^{\infty} n\, c_n x^{n-1} \implies y'' = \sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2}.$$

Substitutes in equation (**??**), yields

$$(x^2-4)\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} + 3x\sum_{n=1}^{\infty} n\, c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\implies \sum_{n=2}^{\infty} n(n-1)\, c_n x^n - 4\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} + 3\sum_{n=1}^{\infty} n\, c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} n(n-1)\, c_n x^n - 4\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n + 3\sum_{n=0}^{\infty} n\, c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} \Big[(n^2+2n+1)c_n - 4(n+2)(n+1)c_{n+2}\Big] x^n = 0$$

$$\implies (n+1)^2 c_n - 4(n+2)(n+1)c_{n+2} = 0$$

$$\implies c_{n+2} = \frac{n+1}{4(n+2)} c_n, \quad \text{for } n \geq 0.$$

$$\text{For } n = 0 \implies c_2 = \frac{c_0}{4 \cdot 2}$$

$$\text{For } n = 2 \implies c_4 = \frac{3\, c_2}{4 \cdot 4} = \frac{1 \cdot 3\, c_0}{4^2 \cdot 2 \cdot 4}$$

$$\text{For } n = 4 \implies c_6 = \frac{5\, c_4}{4 \cdot 6} = \frac{1 \cdot 3 \cdot 5\, c_0}{4^3 \cdot 2 \cdot 4 \cdot 6}$$

Continuing in this way, we evidently would find that

$$c_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\, c_0}{4^n \cdot 2 \cdot 4 \cdots 2n}$$

Since,

$$(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+1)!}{2^n \cdot n!}$$

and

$$2 \cdot \cdots 2n = (2n)!! = 2^n \cdot n!$$

Thus,

$$c_{2n} = \frac{(2n-1)!!}{2^{3n} n!} c_0.$$

$$\text{For } n = 1 \implies c_3 = \frac{2\, c_1}{4 \cdot 3}$$

$$\text{For } n = 3 \implies c_5 = \frac{4\, c_3}{4 \cdot 5} = \frac{2 \cdot 4\, c_1}{4^2 \cdot 3 \cdot 5}$$

$$\text{For } n = 5 \implies c_7 = \frac{6\, c_5}{4 \cdot 7} = \frac{2 \cdot 4 \cdot 6\, c_1}{4^3 \cdot 3 \cdot 5 \cdot 7}$$

So,

$$c_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{4^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n+1)} c_1 = \frac{2^n\, n!}{2^{2n}(2n+1)!!} c_1 = \frac{n!}{2^n (2n+1)!!} c_1.$$

$$y(x) = c_0 + \sum_{n=1}^{\infty} c_{2n} x^{2n} + c_1 x + \sum_{n=1}^{\infty} c_{2n+1} x^{2n+1}$$

$$= c_0\Big[1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^{3n} n!} x^{2n}\Big] + c_1\Big[1 + \sum_{n=1}^{\infty} \frac{(n)!!}{2^n (2n+1)!!} x^{2n}\Big]$$

$$\implies y(x) = c_0(1 + \frac{1}{8}x^2 + \frac{3}{128}x^4 + \cdots) + c_1(x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \cdots).$$

Since, $y(0) = 4$ and $y'(0) = 1$, then $c_0 = 4$ and $c_1 = 1$.

Thus,

$$y(x) = 4 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \cdots .$$

6.4 Frobenius series solution (Solutions near regular singular points)

Definition 24. *A singular point $x = x_0$ of the equation*

$$y'' + P(x)y' + Q(x)y = 0,$$

is said to be a regular singular point, if both terms $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at x_0. Otherwise, $x = x_0$ is an irregular singular point.

Example 76. *The point $x = 0$, is a singular point of the Euler-Cauchy equation*

$$x^2 y'' - xy' - 3y = 0. \qquad (6.6)$$

In this example, $P(x) = -\frac{1}{x}$ and $Q(x) = -\frac{3}{x^2}$.
Since, $xP(x) = x(-\frac{1}{x}) = -1$ and $x^2 Q(x) = x^2(-\frac{3}{x^2}) = -3$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point.

Example 77. *The points $x = 3$ and $x = -3$ are singular points of the equation*

$$(x^2 - 9)^2 y'' + (x - 3)y' + 2y = 0. \qquad (6.7)$$

Here, $P(x) = \frac{1}{(x+3)^2(x-3)}$ and $Q(x) = \frac{2}{(x+3)^2(x-3)^2}$.

Since, $(x-3)P(x) = \frac{1}{(x+3)^2}$ *and* $(x-3)^2 Q(x) = \frac{2}{(x+3)^2}$ *are both analytic at* $x = 3$*, so* $x = 3$ *is a regular singular point. But,* $x = -3$ *is an irregular singular point.*

Homework 39. *Classify the singular points for the following differential equations:*

1) $(x^2+1)y'' + (x+1)y' + 5y = 0.$

2) $x^4(x^2+1)(x-1)^2 y'' + 4x^3(x-1)y' + (x+1)y = 0.$

Theorem 17. *Assume that* $x = x_0$ *is a regular singular point of the differential equation*

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

Then, there is at least one series solution of the form

$$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}, \qquad (6.8)$$

where the number r is some real constant. The series will converge on some interval $0 < |x - x_0| < R$*. The series in* (6.8) *is known as a Frobenius series. We also assume that* $c_0 \neq 0$

Example 78. *Let us solve the Euler-Cauchy equation* (6.6) *by assuming a Frobenius series solution.*

Thus,

$$y = \sum_{n=0}^{\infty} c_n\, x^{n+r} \implies y' = \sum_{n=0}^{\infty} (n+r) c_n\, x^{n+r-1}$$

$$\implies y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n\, x^{n+r-2}.$$

Substitutes in equation (**??**), gives

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n\, x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) c_n\, x^{n+r-1} - 3 \sum_{n=0}^{\infty} c_n\, x^{n+r} = 0$$

$$\implies \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n\, x^{n+r} - \sum_{n=0}^{\infty} (n+r) c_n\, x^{n+r} - 3 \sum_{n=0}^{\infty} c_n\, x^{n+r} = 0.$$

If $n = 0$, we have

$$r(r-1)c_0 - rc_0 - 3c_0 = 0.$$

Since, $c_0 \neq 0$, then

$$r(r-1) - r - 3 = 0 \tag{6.9}$$

$$\implies r^2 - 2r - 3 = 0 \implies (r-3)(r+1) = 0 \implies r = 3 \quad \text{or} \quad r = -1.$$

Now,

$$\sum_{n=0}^{\infty} \Big[(n+r)(n+r-1) - (n+r) - 3 \Big] c_n\, x^{n+r} = 0$$

$$\Longrightarrow \Big[(n+r)(n+r-1)-(n+r)-3\Big]c_n = 0$$

$$\Longrightarrow \Big[n^2+nr-n+nr+r^2-r-n-r-3\Big]c_n = 0$$

$$\Longrightarrow \Big[(n+r)^2-2(n+r)-3\Big]c_n = 0$$

$$\Longrightarrow (n+r-3)(n+r+1)c_n = 0.$$

When $r=-1$, we have $(n-4)nc_n = 0$, so $c_n = 0$ if $n \neq 0$ or $n \neq 4$, where c_0 and c_4 are arbitrary. Substituting into the Frobenius series yields

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+r} = \sum_{n=0}^{\infty} c_n\, x^{n-1} = c_0 x^{-1} + c_4 x^3.$$

If $r=3$, then $n(n+4)c_n = 0$, so $c_n = 0$ if $n \neq 0$. Thus,

$$y_2 = \sum_{n=0}^{\infty} c_n\, x^{n+r} = \sum_{n=0}^{\infty} c_n\, x^{n+3} = c_0 x^3.$$

Therefore, the general solution is

$$y = C_1 x^{-1} + C_2 x^3,$$

where C_1 and C_2 are arbitrary constants.

Remark 24. *Equation* (6.9) *is called the indicate equation associate with Frobenius series solution.*

Theorem 18. *Assume that $x = 0$ is a regular singular point of the second order differential equation*

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \tag{6.10}$$

Suppose that $r_1 \geq r_2$ are two real roots to the indicated equation $p(r) = 0$.

1) If $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer, then there exists two linearly independent solutions to equation (4.4) *of the form*

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+r_1}, \quad c_0 \neq 0$$

and

$$y_2 = \sum_{n=0}^{\infty} c_n\, x^{n+r_2}, \quad c_0 \neq 0.$$

2) If $r_1 - r_2$ is a positive integer, then there exists two linearly independent solutions to equation (4.4) *of the form*

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+r_1}, \quad c_0 \neq 0$$

and

$$y_2 = Cy_1(x)\, \ln(x) + \sum_{n=1}^{\infty} b_n\, x^{n+r_2}, \quad b_0 \neq 0,$$

where C is a constant that could be zero.

3) If $r_1 = r_2$, then there exists two linearly independent solutions to equation (4.4) of the form

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+r_1}, \quad c_0 \neq 0$$

and

$$y_2 = y_1(x)\, \ln(x) + \sum_{n=1}^{\infty} b_n\, x^{n+r_1}, \quad b_0 \neq 0.$$