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Ordinary Differential Equations

LECTURE NOTES

Second Semester

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Ordinary Differential Equations

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Module Description

Module Aims

The aim of this module is to introduce the students to the basic theory of ordinary differential equations and give a competence in solving ordinary differential equations by using different methods of solution of differential equations.

General Description of the module

The subject of differential equations is a very important branch of applied mathematics. Many phenomena from physics, biology and engineering may be described using ordinary differential equations. They are also used to model the behaviour of systems in the natural world, and predict how these systems will behave in the further. For instance, exponential growth (the rate of change of a population is proportional to the size of the population) is expressed by the differential equation $dP/dt = kP$. Newton's Law of Gravitation (acceleration is inversely proportional to the square of distance) translates to the equation $y'' = -ky^2$. Many examples are found in the fields of physics, engineering, biology, chemistry and economics.

The traditional course in differential equations focused on the small number of differential equations for which exact solutions exist. However, the methods used by scientists today have changed dramatically due to computer (using different type of computational package like Maple, Mathematica, reduce, Singular, etc). Here we will cover almost all methods for solving every kind of ordinary differential equations.

Homework

Homework will be given at every lecture. You should start working on the homework problems for a section as soon as that section is covered in class. Although you are encouraged to consult with other students and seek help from tutor and me, homework should ultimately represent your own work. Answers unsupported by work will not receive credit. Not all problems may be graded. Homework should be neatly handwritten or typed, on one side of the page only. Copy the problem in its original form from the lecture (book) and provide the solution to the problem.

prerequisite

One must be familiar with the basic differential and integral calculus, which are the main contents of college level introductory Calculus course. Although the course does not require more more details in linear algebra, it will be very helpful if one has a little bit of knowledge on Linear Algebra such as the determinant of a square matrix, linear (in)dependence of vectors, and Cramers rule of solving a determined system of simultaneous linear algebraic equations.

Learning Objectives

- The student will learn to formulate ordinary differential equations (ODEs) and seek understanding of their solutions.

- The student will recognise basic types of differential equations which are solvable, and will understand the features of linear equations in particular.

- Students will be familiar to derive methods to solve ordinary differential equations.

Grades

Grades will be assigned on the basis of 100 points distributed as follows:

30 points midterm test.

10 points discussion.

60 points final examination.

Attendance

Class attendance is mandatory. Although I do not have a rigid policy, anyone who has missed lots of class and is doing poorly in the course should not except much sympathy from me. If you do miss a class, it is your responsibility to make up the material and make sure your homework is turned in on time.

Hours per week	Notice	Initial Warning	Last Warning
3	3	6	9

Contents

References

1) Differential Equations: a modelling approach. By *Frank R. Giordano and Maurice D. Weir*.

2) Elementary Differential Equations. By *Earl D. Rainvlle and Philip E. Bedient*.

3) Elementary Differential Equations with Linear Algebra. By *Ross L Finney and Donald R. Ostbery*.

4) Ordinary Differential Equations. By *Tyn Myint-V*.

5) Differential Equations and Boundary Value Problems. By *C. Henry Edward and David E. Penney*.

6) Applied Differential Equations. By *Murray R. Spoegel*.

7) Differential Equations. By *C. Ray Wylie*.

8) Schaum's Outline Series, Theory and problems of Differential Equations. By *Frank Ayres, JR. including 560 solved problems.*.

9) Schaum's: 2500 solved problem in Differential Equations. By *Richard Bronson*.

10) A first course in Differential Equations with Application.

11) Introduction to Differential Equations, Lecture notes. By Jeffrey R. Chasnov.

Chapter 1

Linear differential equations with constant coefficients

1.1 Linear differential equations

The general n-th order linear differential equations

Definition 1. *A linear differential equation of order n has the form*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x), \quad (1.1)$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $F(x)$ depending only on x and not y.

Remark 1. *1. An n-th order differential equation which is not of the form* (1.1) *is called nonlinear.*

2. If $n = 1$, equation (1.1) *is called a linear first order equation.*

3. If $n = 2$, equation (1.1) *becomes a second order linear differential equation.*

4. If the coefficients $a_0, a_1, \ldots, a_n$ are constants, we call the equation (1.1)*, a linear differential equation with constant coefficients.*

5. If at least one of the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ is a function of x, equation (1.1) *is called a linear differential equation with variable coefficients.*

6. We use the symbols $D, D^2, \ldots$ to indicate the operator of taking the first, second, $\ldots$ derivatives. Thus, $Dy = \frac{dy}{dx}$

7. If $F(x) = 0$, equation (1.1) *is called a linear homogeneous differential equation. Otherwise, it is called non homogeneous (inhomogeneous, the complementary or reduced) equation.*

Examples

1) $y'' + y = x^2$, is a linear inhomogeneous differential equation with constant coefficients and it is of second order and first degree.

2) $\frac{d^5y}{dx^5} + \frac{d3y}{dx^3} + \frac{dy}{dx} = 0$, is a linear homogeneous differential equation of fifth order and first degree.

3) $3x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 3y\frac{dy}{dx} + x^2y = e^x$, in a nonlinear non-homogeneous third order and first degree differential equation with variable coefficients.

4) $5y'' - 2(y')^3 - 8y = 0$, in a nonlinear homogeneous second order and first degree differential equation.

Remark 2. *We now prove that if y_1 and y_2 are solutions of the homogeneous equation* (1.1), *and if c_1 and c_2 are constants, then*

$$y = c_1y_1 + c_2y_2 \tag{1.2}$$

is also a solution of homogeneous equation (1.1).

Since y_1 and y_2 are solutions of the homogeneous equation (1.1), *then*

$$\begin{aligned} a_0(x)\frac{d^ny_1}{dx^n} + a_1(x)\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_1}{dx} + a_n(x)y_1 &= 0 \qquad (1.3) \\ a_0(x)\frac{d^ny_2}{dx^n} + a_1(x)\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_2}{dx} + a_n(x)y_2 &= 0. \qquad (1.4) \end{aligned}$$

Multiplying equation (1.3) by c_1 and equation (1.4) by c_2 and adding the result, we have

$$a_0(x)(c_1y_1^{(n)}+c_2y_2^{(n)})+\cdots+a_{n-1}(x)(c_1y_1'+c_2y_2')+a_n(x)(c_1y_1+c_2y_2)=0 \tag{1.5}$$

Since

$$y=c_1y_1+c_2y_2 \implies y'=c_1y_1'+c_2y_2',\ \ldots,\ y^n=c_1y_1^n+c_2y_2^n.$$

So equation (1.5) becomes

$$a_0(x)y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_{n-1}(x)y'+a_n(x)y=0.$$

Thus y is also a solution for homogeneous equation (1.1).

Remark 3. *The expression in equation* (1.2) *is called a linear combination of the functions* y_1 *and* y_2*.*

Theorem 1. *Any linear combination of solutions of a linear homogeneous differential equation is also a solution.*

1.2 Linear dependence

Given the functions $f_1, f_2, \ldots, f_n$, and if constants $c_1, c_2, \ldots, c_n$, not all zero, exists such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \tag{1.6}$$

identically in some interval $a \leq x \leq b$, then the functions are said to be linearly dependent. If no such relations exists, the functions are said to be linearly independent. That is, the functions $f_1, f_2, \ldots, f_n$ are linearly independent when equation (1.6) implies that $c_1 = c_2 = \cdots = c_n = 0$.

Remark 4. *If the function are linearly dependent, then at least one of them is a linear combination of the others.*

Example 1. *Show that* e^x *and* e^{2x} *are linearly independent.*

solution: Suppose that there exists c_1, c_2 such that

$$c_1 e^x + c_2 e^{2x} = 0. \tag{1.7}$$

Differentiating equation (1.7) w.r.t. x, we get

$$c_1 e^x + 2c_2 e^{2x} = 0. \tag{1.8}$$

Now subtract equation (1.7) from equation (1.8), we have $c_2e^{2x} = 0$. Since $e^{2x} > 0$ for all x, then $c_2 = 0$ and substitute in equation (1.13), obtaining $c_1e^x = 0 \Longrightarrow c_1 = 0$. Hence, $c_1 = c_2 = 0$ which implies that e^x and e^{2x} are linearly independent.

The Wronskian:

Definition 2. *The Wronskian of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ is denoted by $W[f_1, f_2, \ldots, f_n]$ and defined as the determinant*

$$W[f_1, f_2, \ldots, f_n] = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Example 2. *Show that the solutions $\sin x$ and $\cos x$ of $\frac{d^2y}{dx^2} + y = 0$ are linearly independent.*

Solution: We calculate the Wronskian

$$W[\sin x, \cos x] = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

Remark 5. *Two functions are linearly dependent on an interval I if and only if one of the functions is a constant multiple of the other function.*

Theorem 2. *If, on the interval $a \leq x \leq b$ $a_0(x) \neq 0$, $a_0, a_1, \ldots, a_n$ are continuous, and $y_1, y_2, \ldots, y_n$ are solutions of the equation*

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0,$$

then a necessary and sufficient condition that $y_1, \ldots, y_n$ be linearly independent is the nanvanishing of the Wronskian of $y_1, \ldots, y_n$ on the interval $a \leq x \leq b$.

Remark 6. *Note that the nonvanishing of the Wronskian is a sufficient condition that the functions be linearly independent.*
The non vanishing of the Wronskian on an interval is not a necessary condition for linear independence. The Wronskian may vanish even when the functions are linearly independent.

Proof: [**Theorem 2**] Suppose that $W[y_1, \ldots, y_n] \neq 0$ for all $a \leq x \leq b$. We have to prove that solutions $y_1, \ldots, y_n$ are linearly independent on $a \leq x \leq b$. We prove by contradiction.
Let $y_1, \ldots, y_n$ be linearly dependent on $a \leq x \leq b$. So by definition

of liner dependence there exist constants $b_1, \ldots, b_n$, not all zero, such that $b_1 y_1(x) + \cdots + b_n y_n(x) = 0$. Then for every $a \le x \le b$, we have

$$\begin{aligned} b_1 y_1(x) + \cdots + b_n y_n(x) &= 0 \\ b_1 y_1'(x) + \cdots + b_n y_n'(x) &= 0 \\ &\vdots \\ b_1 y_1^{(n-1)}(x) + \cdots + b_n y_n^{(n-1)}(x) &= 0 \end{aligned} \tag{1.9}$$

We rewrite the system (1.9) in the matrix notation at $x = x_0$

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1.10}$$

Since $b_1, \ldots, b_n$, not all zero, then system (1.10) has nontrivial solution iff

$$\begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} = 0 \quad \Longrightarrow \quad W[y_1, \ldots, y_n] = 0,$$

which is a contradiction of our assumption that $W[y_1, \dots, y_n] \neq 0$. Thus, $y_1, \dots, y_n$ are linearly independent on $a \leq x \leq b$.

Suppose now that $y_1, \dots, y_n$ are linearly independent on $a \leq x \leq b$. We have to prove that $W[y_1, \dots, y_n] \neq 0$ for all $a \leq x \leq b$. Now from the definition of linearly independent we have

$$c_1 y_1(x) + \cdots + c_n y_n(x) = 0 \quad \text{iff} \quad c_1 = c_2 = \cdots = c_n = 0$$

We differentiate equation above $(n-1)$ times. So we get a system of equations

$$\begin{aligned} c_1 y_1(x) + \cdots + c_n y_n(x) &= 0 \\ c_1 y_1'(x) + \cdots + c_n y_n'(x) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x) + \cdots + c_n y_n^{(n-1)}(x) &= 0 \end{aligned} \tag{1.11}$$

$$\Longrightarrow \begin{pmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1.12}$$

Since the system (1.12) has only zero (trivial) solution $c_1 = c_2 = \cdots = c_n = 0$, then we must have

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0 \quad \Longrightarrow \quad W[y_1, \ldots, y_n] \neq 0,$$

for all $a \leq x \leq b$.

Homework 1. *Can you give an example that two function are linearly independent even that their Wronskian is zero?*

Theorem 3. *Let $y_1, \ldots, y_n$ be solutions to the n-th order homogeneous linear differential equation*

$$a_n y^{(n)} + a_{(n-1)} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

on an interval I, and suppose that $W[y_1, \ldots, y_n] = 0$ is identically zero on I. Then $y_1, \ldots, y_n$ are linearly dependent on I.

Proof: Let x_0 be any point in I, and consider the system of linear equations

$$\begin{aligned} c_1y_1(x_0) + \cdots + c_ny_n(x_0) &= 0 \\ c_1y_1'(x_0) + \cdots + c_ny_n'(x_0) &= 0 \\ &\vdots \\ c_1y_1^{(n-1)}(x_0) + \cdots + c_ny_n^{(n-1)}(x_0) &= 0 \end{aligned} \tag{1.13}$$

in the unknowns $c_1, \ldots, c_n$. Since the Wronskian $y_1, \ldots, y_n$ vanishes identically on I, then the determinant of (1.13) is zero and the system has nontrivial solutions $c_1, \ldots, c_n$. Thus, $y_1, \ldots, y_n$ are linearly dependent.

Homework 2. *Prove that the set of solutions y_1 and y_2 is linearly depend if and only if the Wronskian $W[y_1, y_2] = 0$. Hint: Use Remark 5.*

Remark 7. *There are very interesting and important relationships between the Wronskian for a linear differential equation and the coefficients in the equation.*

Consider the second order differential equation of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0. \tag{1.14}$$

Let y_1 and y_2 be solutions of (1.14), then these solutions satisfy (1.14)

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0, \tag{1.15}$$

and

$$a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0. \tag{1.16}$$

Multiplying equation (1.15) by $(-y_2)$ and equation (1.16) by (y_1), and adding the result, we have

$$a_0(x)\big(y_1y_2'' - y_2y_1''\big) + a_1(x)\big(y_1y_2' - y_2y_1'\big) = 0. \tag{1.17}$$

Since $W[y_1, y_2] = y_1y_2' - y_2y_1'$, then

$$\frac{dW}{dx} = \frac{d(y_1y_2' - y_2y_1')}{dx} = y_1y_2'' - y_2y_1''.$$

Substitutes $W[y_1, y_2]$ and $\frac{dW}{dx}$ in (1.17), we obtain

$$a_0\frac{dW}{dx} + a_1W = 0 \quad \Longrightarrow \quad W = Ce^{-\int \frac{a_1}{a_0}dx}.$$

1.3 Differential operators

Let $\frac{dy}{dx} = Dy$. The symbol D is said to be a differentiation operator as it transform a differentiable function into another function. So, $D = \frac{d}{dx}$ denotes differentiation with respect to independent variable, say x, $D^2 = \frac{d^2}{dx^2}$ differentiation twice with respect to x and containing this process, we have $D^n = \frac{d^n}{dx^n}$ and $D^n y = \frac{d^n y}{dx^n}$, for n is positive integer. We define an n-th order differential operator to be

$$L = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x).$$

Note that $L\{\alpha f(x) + \beta g(x)\} = \alpha L\{f(x)\} + \beta L\{g(x)\}$.

Homogeneous linear differential equations with constant coefficients

The general form of homogeneous linear differential equations with constant coefficients is

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0, \tag{1.18}$$

where a_i's are constants for $i = 0, \ldots, n$. So, equation (1.18) can be written in the operator notation

$$(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = 0 \quad \Longrightarrow \quad F(D)y = 0$$

where

$$F(D) = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n,$$

which is called characteristic polynomial. If $F(D) = 0$, then it is called characteristic equation.

Properties of operator D:

1. $D^n + D^m = D^m + D^n$.

2. $D^nD^m = D^mD^n = D^{n+m}$.

3. $(D^n + D^m)f(x) = D^n f(x) + D^m f(x)$.

4. $D^n(f(x) + g(x)) = D^n f(x) + D^n g(x)$.

5. $(D - a)(D - b) = (D - b)(D - a)$, a, b are constants.

We now describe and illustrate how one can solve second order differential equation via an example and then in general.

Example 3. *Solve* $y'' + y' - 6y = 0$.

Solution: Let $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$, then

$$(D^2 + D - 6)y = 0 \quad \Longrightarrow \quad (D - 2)(D + 3)y = 0.$$

Let

$$(D + 3)y = u \tag{1.19}$$

then

$$(D-2)u = 0 \quad \Longrightarrow \quad \frac{du}{dx} - 2u = 0 \quad \Longrightarrow \quad \frac{du}{dx} = 2u \quad \Longrightarrow \quad \frac{du}{u} = 2dx$$

$$\Longrightarrow \quad \ln|u| = 2x + c \quad \Longrightarrow \quad u = ke^{2x}, k = e^c$$

Substitutes in (1.19), we have

$$(D + 3)y = u = ke^{2x} \quad \Longrightarrow \quad \frac{dy}{dx} + 3y = ke^{2x}$$

which is a linear differential equation of first order with $P(x) = 3$ and $Q(x) = ke^{2x}$. The general solution is given by

$$y = \frac{\int e^{\int 3dx}\, k\, e^{2x} dx + C_1}{e^{\int 3dx}} = \frac{k \int e^{5x} dx + C_1}{e^{3x}} = \frac{k}{5} e^{2x} + C_1 e^{-3x} = Ae^{2x} + Be^{-3x}.$$

where $A = \frac{k}{5}$ and $B = C_1$ are arbitrary constants.

Homework 3. *1)* $y'' - \frac{1}{2}y' - \frac{1}{2}y = 0.$
2) $y'' + y' - 2y = 0.$

Homogeneous linear differential equations of second order with constant coefficients

The general is

$$y'' + ay' + by = 0, \tag{1.20}$$

where a, b are constants. Then using The operator D, equation (1.20) can be written

$$(D^2 + aD + b)y = 0 \quad \Longrightarrow \quad (D - \alpha_1)(D - \alpha_2)y = 0,$$

where $-(\alpha_1 + \alpha_2) = a$ and $\alpha_1 \alpha_2 = b$.

Let

$$(D - \alpha_2)y = u \tag{1.21}$$

and then substitute in equation (1.20), we have

$$(D-\alpha_1)u = 0 \Longrightarrow \frac{du}{dx} - \alpha_1 u = 0 \Longrightarrow \frac{du}{dx} = \alpha_1 u \Longrightarrow \ln|u| = \alpha_1 x + C_1$$

$$\Longrightarrow u = K_1 e^{\alpha_1 x}, \quad \text{where } K_1 = e^{C_1}.$$

Substitute in (1.21), we get

$$(D-\alpha_2)y = u = K_1 e^{\alpha_1 x} \Longrightarrow \frac{dy}{dx} - \alpha_2 y = K_1 e^{\alpha_1 x},$$

is a first order linear differential equation with $P(x) = -\alpha_2$ and $Q(x) = K_1 e^{\alpha_1 x}$. So,

$$y = \frac{\int e^{\int -\alpha_2 dx} \, (K_1 \, e^{\alpha_1 x}) \, dx + K_2}{e^{\int -\alpha_2 dx}} = \frac{K_1 \int e^{(\alpha_1 - \alpha_2)x} \, dx + K_2}{e^{-\alpha_2 x}}.$$

There are three cases which depends on the nature of α_1 and α_2.

Case 1: If the roots α_1 and α_2 are real distinct (unequal), i.e. $\alpha_1 \neq \alpha_2 \in \mathbb{R}$ (if $a^2 - 4b > 0$), then

$$y = \frac{\frac{K_1}{\alpha_1 - \alpha_2} \, e^{(\alpha_1 - \alpha_2)x} + K_2}{e^{-\alpha_2 x}} = e^{\alpha_2 x}\Big(\frac{K_1}{\alpha_1 - \alpha_2} \, e^{(\alpha_1 - \alpha_2)x}\Big) + K_2 \, e^{\alpha_2 x}$$

$$\Longrightarrow y = A \, e^{\alpha_1 x} + B \, e^{\alpha_2 x}, \text{ where } A = \frac{K_1}{\alpha_1 - \alpha_2} \text{ and } B = K_2.$$

Case 2: If the roots α_1 and α_2 are real equal (repeated). i.e. $\alpha_1 = \alpha_2 = \alpha$ (if $a^2 - 4b = 0$), then

$$y = \frac{K_1 \int e^{(0)}\, dx + K_2}{e^{-\alpha x}} = K_2\, e^{\alpha x} + K_1\, x\, e^{\alpha x} = A\, e^{\alpha x} + B\, x\, e^{\alpha x},$$

where $A = K_2$ and $B = K_1$.

Case 3: If the roots α_1 and α_2 are complex. Let $\alpha_1 = a + ib$ and $\alpha_2 = a - ib$ where $a, b \in \mathbb{R}$, $b \neq 0$ and $i^2 = -1$ (when $a^2 - 4b < 0$). From Case 1, we have

$$y = A\, e^{\alpha_1 x} + B\, e^{\alpha_2 x}$$

So,

$$y = A\, e^{(a+ib)x} + B\, e^{(a-ib)x} = e^{ax}(A\, e^{ibx} + B\, e^{-ibx}).$$

By Euler's formula, we have

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta.$$

Hence,

$$\begin{aligned} y &= e^{ax}\big(A(\cos bx + i\sin bx) + B(\cos bx - i\sin bx)\big) \\ &= e^{ax}\big((A + B\cos bx + i(A-B)\sin bx\big) \\ &= e^{ax}\big(C_1\cos bx + C_2\sin bx\big) \end{aligned}$$

is a general solution where $C_1 = A + B$ and $C_2 = i(A - B)$.

Example 4. *Solve the following differential equation:*
1) $y'' - y' - 2y = 0,$ *2)* $y'' - 2y' + y = 0,$ *3)* $y'' + 2y' + 2y = 0.$

Solution

1) Clearly, the differential equation is a homogeneous second order linear differential equation with constant coefficients. In the operator notation, this equations becomes $(D^2 - D - 2)y = 0$. So, the characteristic (auxiliary) equation is

$$\alpha^2-\alpha-2 = 0 \implies (\alpha-2)(\alpha+1) = 0 \implies \alpha_1 = 2 \quad \text{and} \quad \alpha_2 = -1.$$

Since the roots are real and distinct (unequal), the the general solution is

$$y = Ae^{\alpha_1 x} + Be^{\alpha_2 x} = Ae^{2x} + Be^{-x},$$

where A and B are arbitrary constants.

2) The characteristic equation is

$$\alpha^2 - 2\alpha + 1 = 0 \implies (\alpha - 1)(\alpha - 1) = 0 \implies \alpha = 1$$

which is a double root and the general solution is given by

$$y = Ae^{\alpha x} + Bxe^{\alpha x} = Ae^{x} + Bxe^{x},$$

where A and B are arbitrary constants.

3) In this example, the characteristic equation is given by

$$\alpha^2 + 2\alpha + 2 = 0 \implies \alpha_{1,2} = \frac{-2 \mp \sqrt{4-8}}{2} = -1 \mp i$$

$$\implies \alpha_1 = -1 + i \quad \text{and} \quad \alpha_2 = -1 - i.$$

The roots are complex and clearly $a = -1$ and $b = 1$, so the general solution is

$$y = e^{-x}(C_1 \cos x + C_2 \sin x),$$

where C_1 and C_2 are arbitrary constants.

Remark 8. *The general solution of a homogeneous linear differential* *equation with constant coefficients, is also known as a complementary function (solution).*

Homework 4. *Solve the following differential equation:*

1) $y'' + y = 0.$

2) $y'' - 4y' + 4y = 0.$

3) $y'' - 7y' = 0.$

4) $y'' - 2\sqrt{2}y' + 2y = 0.$

5) $4y'' + 4y' + y = 0.$

Homogeneous linear differential equations with constant coefficients of arbitrary order

Theorem 4. *Let*

$$y^{(n)} + a_{(n-1)}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{1.22}$$

be an n-th order homogeneous linear differential equation with constant real coefficients. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be the roots of its characteristic polynomial, and suppose that*

$$f(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_0 = (D - \alpha_1)(D - \alpha_2)\cdots(D - \alpha_n)$$

1) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real distinct (unequal) numbers ($\alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_n$ and $\alpha_i \in \mathbb{R}$, for i=1,...,n), then the functions

$$e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}$$

are linearly independent and the general solution of equation (1.22) *is*

$$y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \ldots + C_n e^{\alpha_n x},$$

where C_i are arbitrary constants.

2) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real equal numbers (repeated) ($\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$ and $\alpha_i = \alpha \in \mathbb{R}$, for i=1,...,n), then the functions

$$e^{\alpha x}, xe^{\alpha x}, \ldots, x^{n-1} e^{\alpha x}$$

are linearly independent and the general solution of equation (1.22) *is*

$$y = C_1 e^{\alpha_1 x} + C_2 x e^{\alpha_2 x} + \ldots + C_n x^{n-1} e^{\alpha_n x},$$

where C_i are arbitrary constants.

3) If the roots are complex. Let $\alpha = a \mp ib$ (k times), where n=2k, $a, b \in \mathbb{R}, b \neq 0$, then the functions

$$e^{ax}\sin(bx), xe^{ax}\sin(bx), \ldots, x^{n-1}e^{ax}\sin(bx)$$

$$e^{ax}\cos(bx), xe^{ax}\cos(bx), \ldots, x^{n-1}e^{ax}\cos(bx)$$

are linearly independent and the general solution of equation (1.22) *is*

$$y = e^{ax}\big(C_1\cos(bx) + C_2\sin(bx)\big) + xe^{ax}\big(C_3\cos(bx) + C_4\sin(bx)\big) + \cdots$$

$$+e^{ax}x^{n-1}\big(C_{n-1}\cos(bx) + C_n\sin(bx)\big)$$

where C_i are arbitrary constants.

Example 5. *Solve the differential equation*

$$y''' - 6y'' + 11y' - 6y = 0.$$

Solution:

$$y''' - 6y'' + 11y' - 6y = 0 \quad \Longrightarrow \quad (D^3 - 6D^2 + 11D - 6)y = 0,$$

then the characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0 \implies (\alpha - 1)(\alpha^2 - 5\alpha + 6) = 0$$

$$\implies (\alpha - 1))(\alpha - 2))(\alpha - 3)) = 0 \implies \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3.$$

Since all roots are real and distinct, so the general solution is given by

$$y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + C_3 e^{\alpha_3 x} = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x},$$

where C_1, C_2 and C_3 are arbitrary constants.

Example 6. *Find the general solution of the differential equation*

$$y^{(4)} + 2y'' + y = 0. \tag{1.23}$$

Solution: Equation (1.23) has auxiliary equation

$$\alpha^4 + 2\alpha^2 + 1 = 0 \implies (\alpha^2 + 1)(\alpha^2 + 1) = 0 \implies (\alpha^2 + 1)^2 = 0$$

$$\implies \alpha = \mp i, \mp i \implies a = 0, b = 1.$$

The roots are repeated complex numbers and clearly it is purely imaginary. So the general solution is

$$y = e^{ax}\big(C_1\cos(bx) + C_2\sin(bx)\big) + xe^{ax}\big(C_3\cos(bx) + C_4\sin(bx)\big)$$

$$\Longrightarrow y = C_1\cos(x) + C_2\sin(x) + x\big(C_3\cos(x) + C_4\sin(x)\big),$$

where C_i, for $i = 1, 2, 3, 4$ are arbitrary constants.

Example 7. *Solve the equation*

$$y^{(7)} - 2y^{(5)} + y^{(3)} = 0.$$

Solution: In operator notation this equation becomes

$$(D^7 - 2D^5 + D^3)y = 0,$$

then the characteristic equation is

$$\alpha^7 - 2\alpha^5 + \alpha^3 = 0 \Longrightarrow \alpha^3(\alpha^4 - 2\alpha^2 + 1) = 0 \Longrightarrow \alpha^3(\alpha^2 - 1)(\alpha^2 - 1) = 0$$

$$\Longrightarrow \ \alpha^3(\alpha + 1)^2(\alpha - 1)^2 = 0 \ \Longrightarrow \ \alpha = 0, 0, 0, 1, 1, -1, -1.$$

In this case, the general form is given by

$$y = C_1 + C_2x + C_3x^2 + C_4e^x + C_5xe^x + C_6e^{-x} + C_7xe^{-x},$$

where C_i, for $i = 1, \ldots, 7$ are arbitrary constants.

Example 8. *Solve the equation*

$$y^{(5)} + 3y^{(4)} + 4y^{(3)} - 4y' - 4y = 0.$$

Solution: The characteristic equation is

$$\alpha^5 + 3\alpha^4 + 4\alpha^3 - 4\alpha - 4 = (\alpha - 1)(\alpha^2 + 2\alpha + 2)^2 = 0.$$

Clearly, the roots are $\alpha_1 = 1$, $\alpha_{2,3} = -1 + i$ and $\alpha_{4,5} = -1 - i$ $(a = -1, b = 1)$. The general solution, in this case, is

$$y = C_1e^x + e^{-x}(C_2 \cos x + C_3 \sin x) + xe^{-x}(C_4 \cos x + C_5 \sin x),$$

where C_i, for $i = 1, \ldots, 5$ are arbitrary constants.

Homework 5. *Solve the following differential equations:*

1) $y^{(6)} - y^{(5)} + 2y^{(4)} - 2y''' + y'' - y' = 0.$

2) $(D^3 + 1)y = 0.$

3) $(D^3+2D^2-5D-6)y=0.$

4) $(D^4+4D)y=0.$

5) $(D^5-5D^4+12D^3-16D^2+12D-4)y=0.$

6) $y^{(5)}-y^{(4)}+4y'-4y=0.$

Example 9. *Find a linear differential equation that has* e^{2x} *and* xe^{-3x} *among its solutions.*

Solution: Clearly, the roots are $\alpha_1=2$ and $\alpha_2=-3$. Note that $\alpha_2=-3$ is repeated root, so the characteristic polynomial is given be

$$f(D)=(D-2)(D+3)^2 \quad \Longrightarrow \quad (D-2)(D+3)^2y=0.$$

Thus, the linear differential equation is

$$y'''+4y''-3y'-18y=0.$$

Homework 6. *The equation*

$$(D^3+aD^2+bD+c)y=0,$$

where a, b and c are constants, has a solution

$$y=C_1e^{-x}+e^{-2x}(C_2\sin 4x+C_3\cos 4x).$$

Determine the values of a, b and c.

Properties of the operator D

We know that the characteristic polynomial is

$$f(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

where a_i are constants for $i = 1, \ldots, n$.

1) If b is a constant, then $f(D)\{e^{bx}\} = f(b)e^{bx}$.

Proof: Since

$$D\{e^{bx}\} = \frac{d}{dx}(e^{bx}) = be^{bx},$$

$$D^2\{e^{bx}\} = b^2 e^{bx},$$

$$\vdots$$

$$D^n\{e^{bx}\} = b^n e^{bx}.$$

So,

$$\begin{aligned} f(D)\{e^{bx}\} =&(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)\{e^{bx}\}, \\ =&a_nD^n\{e^{bx}\} + a_{n-1}D^{n-1}\{e^{bx}\} + \cdots + a_1D\{e^{bx}\} + a_0\{e^{bx}\}, \\ =&a_nb^n\, e^{bx} + a_{n-1}b^{n-1}\, e^{bx} + \cdots + a_1b\, e^{bx} + a_0\, e^{bx}, \\ =&(a_nb^n + a_{n-1}b^{n-1} + \cdots + a_1b + a_0)e^{bx}, \\ =&f(b)e^{bx}. \end{aligned}$$

Example 10. *Evaluate*

$$(D^2 + 3D + 2)e^{3x}.$$

Solution: $f(D) = D^2+3D+2$ and $b = 3$, so, $f(3) = 9+9+2 = 20$. Hence,

$$(D^2 + 3D + 2)e^{3x} = 20\, e^{3x}.$$

2) $f(D^2)\{\cos(bx)\} = f(-b^2)\cos(bx)$, where b is a constant.

Proof: We have

$$f(D) = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0,$$

then

$$f(D^2) = a_n D^{2n} + a_{n-1} D^{2(n-1)} + \cdots + a_1 D^2 + a_0.$$

Since

$$\begin{aligned}
D(\cos bx) &= \frac{d}{dx}(\cos bx) = -b \sin bx, \\
D^2(\cos bx) &= D(D(\cos bx)) = D(-b \sin bx) = -b^2 \cos bx, \\
D^3(\cos bx) &= b^3 \sin bx \\
D^4(\cos bx) &= b^4 \cos bx = (-b^2)^2 \cos bx.
\end{aligned}$$

By mathematical induction, let

$$D^{2k}(\cos bx) = (-b^2)^k \cos bx, \quad k \in \mathbb{Z}^+.$$

Now,

$$\begin{aligned}
D^{2(k+1)}(\cos bx) =& D^2 D^{2k}(\cos bx) = D^2((-b^2)^k \cos bx) \\
=& (-b^2)^k D^2(\cos bx) = (-b^2)^k(-b^2) \cos bx \\
=& (-b^2)^{k+1} \cos bx.
\end{aligned}$$

$$D^{2n}(\cos bx) = (-b^2)^n \cos bx, \quad n = 1, 2, \ldots.$$

$$
\begin{aligned}
f(D^2)\{\cos(bx)\} =& (a_n D^{2n} + \cdots + a_1 D^2 + a_0)\{\cos(bx)\} \\
=& a_n(-b^2)^n \cos bx + \cdots + a_1(-b^2)\cos bx + a_0 \cos bx \\
=& f(-b^2)\cos bx.
\end{aligned}
$$

3) $f(D^2)\{\sin(bx)\} = f(-b^2)\sin(bx)$, where b is a constant.

Proof: H.W.

Theorem 5. *If g is a function of x, then*

$$f(D)\{e^{bx}g(x)\} = e^{bx}f(D+b)\{g(x)\}.$$

Proof: Since

$$f(D) = a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0,$$

so,

$$
\begin{aligned}
f(D)\{e^{bx}g(x)\} &= (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)\{e^{bx}g(x)\} \\
&= a_nD^n\{e^{bx}g(x)\} + a_{n-1}D^{n-1}\{e^{bx}g(x)\} + \cdots \\
&+ a_1D\{e^{bx}g(x)\} + a_0\{e^{bx}g(x)\}
\end{aligned}
$$

Now,

$$
\begin{aligned}
D\{e^{bx}g(x)\} &= \frac{d}{dx}\{e^{bx}g(x)\} = e^{bx}\frac{d}{dx}(g(x)) + be^{bx}g(x) = e^{bx}D\{g(x)\} + be^{bx}g(x) \\
&= e^{bx}[D\{g(x)\} + bg(x)] = e^{bx}(D+b)g(x).
\end{aligned}
$$

and

$$
\begin{aligned}
D^2\{e^{bx}g(x)\} &= D[D\{e^{bx}g(x)\}] = \frac{d}{dx}[e^{bx}\frac{d}{dx}(g(x)) + be^{bx}g(x)] \\
&= e^{bx}\frac{d^2}{dx^2}(g(x)) + be^{bx}\frac{d}{dx}(g(x)) + be^{bx}\frac{d}{dx}(g(x)) + b^2e^{bx}g(x) \\
&= e^{bx}[D^2(g(x) + 2bD(g(x)) + b^2g(x))] \\
&= e^{bx}(D+b)^2g(x).
\end{aligned}
$$

and so on

$$
D^n\{e^{bx}g(x)\} = e^{bx}(D+b)^ng(x).
$$

Hence,

$$\begin{aligned} f(D)\{e^{bx}g(x)\} &= a_n e^{bx}(D+b)^n g(x) + a_{n-1}e^{bx}(D+b)^{n-1}g(x) + \cdots \\ &\quad + a_1 e^{bx}(D+b)g(x) + a_0 e^{bx}g(x) \\ &= e^{bx}[a_n(D+b)^n + a_{n-1}(D+b)^{n-1} + \cdots + a_1(D+b) + a_0]g(x) \\ &= e^{bx}f(D+b)\{g(x)\}. \end{aligned}$$

1.4 General solution of a non-homogeneous differential equations

Let y_p be any particular solution of the differential equation

$$b_0y^{(n)} + b_1y^{(n-1)} + \cdots + b_{n-1}y' + b_ny = R(x), \tag{1.24}$$

and let y_c be a solution of the corresponding homogeneous equation (1.24)

$$b_0y^{(n)} + b_1y^{(n-1)} + \cdots + b_{n-1}y' + b_ny = 0. \tag{1.25}$$

Then, $y = y_c + y_p$ is a general solution of (1.24). Now,

$$\begin{aligned} b_0 y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1} y' + b_n y &= (b_0 y_c^{(n)} + b_1 y_c^{(n-1)} + \cdots + b_{n-1} y_c' + b_n y_c) \\ &\quad + (b_0 y_p^{(n)} + b_1 y_p^{(n-1)} + \cdots + b_{n-1} y_p' + b_n y_p) \\ &= 0 + R(x) = R(x). \end{aligned}$$

If $y_1, y_2, \ldots, y_n$ are linearly independent solutions of (1.25), then

$$y_c = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

in which C_i's are arbitrary constants, is a general solution of (1.25) and it is called the *complementary function* (solution) for equation (1.24). The general solution of (1.24), is the sum of the complementary function and any particular solution.

The Operator $\frac{1}{f(D)}$ (Inverse of $f(D)$): To find a particular solution of

$$f(D)y = R(x),$$

it is natural to right

$$y_p = \frac{1}{f(D)}\{R(x)\}.$$

Remark 9. *Note that* $f(D) \cdot \frac{1}{f(D)}\{R(x)\} = R(x)$.

There are several cases to find the particular solution of

$$f(D)y = R(x).$$

Case 1: If $R(x) = e^{ax}$, a is a constant and we have $f(D)\{e^{ax}\} = f(a)e^{ax}$.

Case i: When $f(a) \neq 0$ and

$$f(D)\{\frac{e^{ax}}{f(a)}\} = \frac{f(a)}{f(a)}e^{ax} = e^{ax},$$

then

$$\frac{1}{f(D)}\{e^{ax}\} = \frac{1}{f(a)}e^{ax}.$$

Now,

$$f(D)y = e^{ax}, \tag{1.26}$$

then

$$y_p = \frac{1}{f(D)}\{e^{ax}\} = \frac{1}{f(a)}e^{ax},$$

which is a particular solution of the equation (1.26).

Example 11. *Solve the equation* $(D^2 + 1)y = e^{2x}$.

Solution: First we should find a complementary solution, i.e. a general solution of the corresponding homogeneous equation. Here, the roots of the characteristic equation is

$$\alpha^2 + 1 = 0 \implies \alpha = \mp i.$$

Then the complementary function is

$$y_c = C_1 \cos x + C_2 \sin x.$$

Now to find the particular solution, we have $f(D) = D^2 + 1$ and $a = 2$, then $f(a) = a^2 + 1 = 4 + 1 = 5 \neq 0$, so,

$$y_p = \frac{1}{D^2 + 1} e^{2x} = \frac{e^{2x}}{5}.$$

Thus, the general solution is

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x + \frac{e^{2x}}{5},$$

where C_1 and C_2 are arbitrary constants.

Case ii**:** When $f(a) = 0$, then $f(D)$ contains the factor $(D - a)$. Suppose that this factor occurs precisely k times in $f(D)$, that is,

$f(D) = \phi(D)(D-a)^k$; $\phi(a) \neq 0$, $k = 1, 2, \ldots, n$ (n is the order of the differential equation). Now

$$f(D)y = e^{ax} \implies y = \frac{1}{f(D)}\{e^{ax}\} = \frac{1}{(D-a)^k \phi(D)}\{e^{ax}\},$$

Then by Theorem 5, we have

$$f(D)\{e^{bx}g(x)\} = e^{bx} f(D+b)\{g(x)\}.$$

So,

$$y = e^{ax}\frac{1}{(D+a-a)^k \phi(D+a)}\{1\} = e^{ax}\frac{1}{(D)^k \phi(D+a)}\{1\} \quad \text{(Note, g(x)=1).}$$

Since,

$$\phi(D+a)\{1\} = \phi(a) \Longleftrightarrow \frac{1}{\phi(D+a)}\{1\} = \frac{1}{\phi(a)},$$

then,

$$y = e^{ax}\frac{1}{D^k \phi(a)}\{1\} = \frac{e^{ax}}{\phi(a)}\frac{1}{D^k}\{1\} = \frac{e^{ax}}{\phi(a)}\frac{x^k}{k!}$$

Case 2 If $R(x) = \sin ax$ or $R(x) = \cos ax$, where a is a constant. To find the particular solution for $f(D)y = R(x)$, there exists two types:

case i : If ai is not a root of the characteristic equation $f(D) = 0$, $i = \sqrt{-1}$, we use the following properties:

1) $f(D^2)\{\sin ax\} = f(-a^2)\{\sin ax\}$.

2) $f(D^2)\{\cos ax\} = f(-a^2)\{\cos ax\}$.

Example 12. *Solve* $(D^2 + 3D + 2)y = \cos 2x$.

Solution: The characterise equation is

$$\alpha^2 + 3\alpha + 2 = 0 \implies (\alpha + 2)(\alpha + 1) = 0 \implies \alpha_1 = -1,\ \alpha_2 = -2$$

then the complementary function (solution) is

$$y_c = C_1 e^{-x} + C_2 e^{-2x}.$$

To find the particular solution y_p, we can see that

$$y_p = \frac{1}{D^2 + 3D + 2}\{\cos 2x\} = \frac{1}{-(2)^2 + 3D + 2}\{\cos 2x\} = \frac{1}{3D - 2}\{\cos 2x\}$$

$$= \frac{3D + 2}{(3D - 2)(3D + 2)}\{\cos 2x\} = \frac{3D + 2}{9D^2 - 4}\{\cos 2x\} = \frac{3D + 2}{-36 - 4}\{\cos 2x\}$$

$$= \frac{3D+2}{-40}\{\cos 2x\} = \frac{3}{20}\{\sin 2x\} - \frac{1}{20}\{\cos 2x\}.$$

Finally, the general solution of the non-homogeneous system is

$$y = y_c + y_p = C_1 e^{-x} + C_2 e^{-2x} + \frac{3}{20}\{\sin 2x\} - \frac{1}{20}\{\cos 2x\}.$$

where C_1 and C_2 are arbitrary essential constants.

case ii : If ai is a root of the characteristic equation $f(D) = 0$, then by Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can find the particular solution from

$$f(D) = e^{iax}. \tag{1.27}$$

Since,

$$e^{iax} = \cos ax + i\sin ax,$$

then the particular solution of $f(D) = \cos ax$ is the real part of the particular solution of equation (1.27), and the particular solution of $f(D) = \sin ax$ is the imaginary part of the particular solution of equation (1.27).

Example 13. *Solve*

$$(D^2+9)y = \sin 3x. \tag{1.28}$$

Solution: In this example, the characterise equation is

$$\alpha^2+9=0 \implies \alpha = \mp 3i \implies a=0, b=3,$$

then the complementary function (solution) is

$$y_c = C_1 \cos 3x + C_2 \sin 3x.$$

To find the particular solution y_p, we can see that

$$y_p = \frac{1}{D^2+9}\{\sin 3x\} = \frac{1}{D^2+9}\{e^{3ix}\}.$$

But note that $f(3i)=0$. Therefore,

$$y_p = \frac{1}{(D-3i)(D+3i)}\{e^{3ix}\} = \frac{1}{6i}xe^{3ix} = \frac{1(-i)}{6i(-i)}xe^{3ix} = \frac{-i}{6}xe^{3ix}.$$

$$\implies y_p = \frac{1(-i)}{6i(-i)}x(\cos 3x + i\sin 3x) = \frac{x}{6}\sin 3x - \frac{x}{6}i\cos 3x.$$

Hence the particular solution pf equation (1.28) is then

$$y_p = -\frac{x}{6}\cos 3x.$$

Therefore, the general solution is

$$y = y_c + y_p = C_1 \cos 3x + C_2 \sin 3x - \frac{x}{6}\cos 3x,$$

where C_1 and C_2 are arbitrary essential constants.

Case 3: IF $R(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, $n \in \mathbb{Z}^+$, is a polynomial of degree n in x, then to find a particular solution of

$$f(D)y = R(x) \implies y_p = \frac{1}{f(D)}\{R(x)\},$$

we use the ordinary long division. Since

$$\frac{1}{1-D} = 1 + D + D^2 + \cdots + D^n + \cdots$$

then

$$\frac{1}{1-f(D)} = 1 + f(D) + (f(D))^2 + \cdots .$$

Example 14. *Find the particular solution of*

$$(D^2 - 3D + 5)y = x^2 - 1.$$

Solutions: Since

$$
\begin{aligned}
(D^2-3D+5)y = x^2-1 \implies y_p &= \frac{1}{D^2-3D+5}\{x^2-1\} \\
\implies y_p &= \frac{1}{5[1-(\frac{3}{5}D-\frac{D^2}{5})]}\{x^2-1\} \\
\implies y_p = \frac{1}{5}(1+(\frac{3}{5}D-\frac{D^2}{5})+(\frac{3}{5}D-\frac{D^2}{5})^2+\cdots)\{x^2-1\} \\
\implies y_p = \frac{1}{5}(1+\frac{3}{5}D-\frac{D^2}{5}+\frac{9}{25}D^2-\frac{6}{50}D^3+\cdots)\{x^2-1\} \\
\implies y_p = \frac{1}{5}(x^2-1+\frac{3}{5}(2x)-\frac{2}{5}+\frac{9}{25}(2)+0\cdots) = \frac{x^2}{5}+\frac{6x}{5}-\frac{7}{25}.
\end{aligned}
$$

is a particular solution.

Homework 7. *Solve the following differential equations:*

1) $(2D^2+2D+3)y = x^2+2x-1.$

2) $(D^3-2D+4)y = x^4+3x^2-5x+2.$

Remark 10. *If $R(x)$ is a polynomial of degree $n \in \mathbb{Z}^+$ in x, then to find a particular solution of $f(D)y = R(x)$, we suppose that the particular solution is a polynomial of the same degree as $R(x)$ and we must find its coefficients.*

Example 15. *Solve $y'' - 2y' - 3y = 1 - x^2$.*

Solution: Since

$$y'' - 2y' - 3y = 1 - x^2 \implies (D^2 - 2D - 3)y = 1 - x^2$$

$$\alpha^2 - 2\alpha - 3 = 0 \implies (\alpha - 3)(\alpha + 1) = 0 \implies \alpha_1 = 3 \text{ and } \alpha_2 = -1.$$

So,

$$y_c = C_1 e^{3x} + C_2 e^{-x}.$$

To find the particular solution y_p, let the $y_p = Ax^2 + Bx + C$, so $y_p' = 2Ax + B$ and $y_p'' = 2A$. Substitutes in the original differential equation, we have

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$

$$\implies 2A - 4Ax - 2B - 3Ax^2 - 3Bx - 3C = 1 - x^2$$

$$\implies (2A - 2B - 3C) - (4A + 3B)x - 3Ax^2 = 1 - x^2$$

It is easy to see that $A = \frac{1}{3}$ $B = -\frac{4}{9}$ and $C = \frac{5}{27}$. So the particular solution is

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

Case 4: If $R(x) = e^{ax}Q(x)$, where $Q(x) = \sin(bx)$, $Q(x) = \cos(bx)$ or $Q(x)$ is a polynomial in x and a, b are constants. To find y_p, we use the theorem

$$f(D)\{e^{ax}Q(x)\} = e^{ax}f(D+a)Q(x),$$

so,

$$\frac{1}{f(D)}\{e^{ax}Q(x)\} = e^{ax}\frac{1}{f(D+a)}\{Q(x)\},$$

then this case transform to the second case or third case.

Example 16. *Solve* $(D^2 - 2D)y = e^x \sin x$.

Solutions: The characteristic equation is

$$\alpha^2 - 2\alpha = 0 \implies \alpha(\alpha - 2) = 0 \implies \alpha_1 = 0 \text{ and } \alpha_2 = 2.$$

So, $y_c = C_1 + C_2 e^{2x}$.

To find the particular solution y_p,

$$y_p = \frac{1}{D^2 - 2D}\{e^x \sin x\} = e^x \frac{1}{(D+1)^2 - 2(D+1)}\{\sin x\}$$

$$\implies y_p = e^x \frac{1}{D^2 - 1}\{\sin x\} = e^x \frac{\sin x}{-(1)^2 - 1} = e^x \frac{\sin x}{-2} = -e^x \frac{\sin x}{2}.$$

Homework 8. *Solve* $(D^2 - 2D + 2)y = e^x \sin x$.

Example 17. *Solve* $(D^2 + D - 2)y = x\, e^x$.

Solutions: The characteristic equation is

$$\alpha^2+\alpha-2=0 \implies (\alpha+2)(\alpha-1)=0 \implies \alpha_1=1 \text{ and } \alpha_2=-2.$$

So, $y_c=C_1e^x+C_2e^{-2x}$.

To find the particular solution y_p,

$$y_p=\frac{1}{D^2+D-2}\{x\,e^x\}=e^x\frac{1}{(D+1)^2+(D+1)-2}\{x\}$$

$$\implies y_p=\frac{e^x}{D^2+3D}\{x\}=\frac{e^x}{3D(1-(-\frac{D}{3}))}=\frac{e^x}{3D}(1-\frac{D}{3}+\frac{D^2}{9}+\cdots)\{x\}$$

$$\implies y_p=\frac{e^x}{3D}(x-\frac{1}{3})=\frac{e^x}{3}(\frac{x^2}{2}-\frac{1}{3}x).$$

Case 5): If $R(x) = \sin ax\, p(x)$ or $R(x) = \cos ax\, p(x)$, where $p(x)$ is a polynomial in x. We use Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

To find the particular solution for

$$f(D)y = R(x),$$

first we find the particular solution for

$$f(D)y = e^{iax}p(x), \tag{1.29}$$

which is a forth case. Then the particular solution of

$$f(D)y = \sin ax\, p(x)$$

is the imaginary part of the particular solution of (1.29) and the particular solution of

$$f(D)y = \cos ax\, p(x)$$

is the real part of the particular solution of (1.29).

Example 18. *Solve* $\quad y'' - 3y = (x^2 - 1)\sin 2x.$

Solution: Clearly, the characteristic equation is

$$\alpha^2 - 3 \implies \alpha = \mp\sqrt{3},$$

so the complementary function is

$$y_c = C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x}.$$

We will now find the particular solution.

$$(D^2 - 3)y = (x^2 - 1)\sin 2x \implies (D^2 - 3)y = (x^2 - 1)e^{2ix}$$

$$\implies y_p = \frac{1}{D^2 - 3}\{(x^2 - 1)e^{2ix}\} \implies y_p = e^{2ix}\frac{1}{(D + 2i)^2 - 3}\{x^2 - 1\}$$

$$\implies y_p = e^{2ix}\frac{1}{D^2 + 4iD - 7}\{x^2 - 1\} = -\frac{e^{2ix}}{7}\frac{1}{[1 - (\frac{D^2+4iD}{7})]}\{x^2 - 1\}$$

$$\implies \quad = -\frac{e^{2ix}}{7}[1 + (\frac{D^2 + 4iD}{7}) + (\frac{D^2 + 4iD}{7})^2 + \cdots]\{x^2 - 1\}$$

$$\implies \quad = -\frac{e^{2ix}}{7}[1 + \frac{4i}{7}D + \frac{D^2}{7} - \frac{16}{49}D^2 + \cdots]\{x^2 - 1\}$$

$$\implies \quad = -\frac{e^{2ix}}{7}[x^2 - 1 + \frac{4i}{7}(2x) + \frac{2}{7} - \frac{32}{49} + 0]$$

$$\implies \quad = -\frac{e^{2ix}}{7}[x^2 + \frac{8i}{7}(x) - \frac{32 + 49 - 14}{49}]$$

$$\Longrightarrow \quad = -\frac{1}{7}(\cos 2x + i\sin 2x)(x^2 + \frac{8i}{7}x - \frac{67}{49})$$

$$\Longrightarrow \quad = -\frac{1}{7}\cos 2x(x^2 - \frac{67}{49}) + \frac{8}{49}x\,\sin 2x - \frac{i}{7}\sin 2x(x^2 - \frac{67}{49}).$$

So, y_p for the original equation is

$$-\frac{1}{7}\sin 2x(x^2 - \frac{67}{49}).$$

The general solution is

$$y = y_c + y_p = C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x} - \frac{1}{7}\sin 2x(x^2 - \frac{67}{49}),$$

where C_1 and $C - 2$ are arbitrary constants.

1.5 Variation of parameters

Suppose we have a constant coefficient second order equation

$$y'' + ay' + by = g(x), \tag{1.30}$$

where $g(x)$ is a continuous function. Let

$$y_c = C_1 y_1(x) + C_2 y_2(x),$$

where y_1 and y_2 are linearly independent solutions, denotes the complementary solution to the corresponding homogeneous equation.
We now vary the parameters C_1 and C_2 and replace them by functions $v_1(x)$ and $v_2(x)$. We propose that the particular solution is of the form

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x).$$

To solve the nonhomogeneous (1.30), we must determine the functions $v_1(x)$ and $v_2(x)$. Now,

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2',$$

and let

$$v_1' y_1 + v_2' y_2 = 0, \tag{1.31}$$

for prevent any second derivatives of v_1 and v_2 from arising. Thus,

$$y_p' = v_1 y_1' + v_2 y_2' \implies y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''.$$

Substitute the expression for y_p and its derivatives into equation (1.30), we have

$$v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' + a(v_1y_1' + v_2y_2') + b(v_1y_1 + v_2y_2) = g(x)$$

$$\Longrightarrow v_1(y_1'' + ay_1' + by_1) + v_2(y_2'' + ay_2' + by_2) + v_1'y_1' + v_2'y_2' = g(x).$$

Since y_1 and y_2 are solutions of the corresponding homogeneous equation (1.30), then we have

$$y_1'' + ay_1' + by_1 = 0 \quad \text{and} \quad y_2'' + ay_2' + by_2 = 0.$$

So,

$$v_1'y_1' + v_2'y_2' = g(x) \tag{1.32}$$

Now, from (1.31) and (1.32), we can find v_1' and v_2' by Cramer's rule:

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}.$$

Since

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0,$$

as y_1 and y_2 are linearly independent solutions, then

$$v_1' = -\frac{y_2}{W[y_1, y_2]} g(x) \implies v_1 = -\int \frac{y_2}{W[y_1, y_2]} g(x) dx$$

and

$$v_2' = \frac{y_1}{W[y_1, y_2]} g(x) \implies v_2 = \int \frac{y_1}{W[y_1, y_2]} g(x) dx$$

Example 19. *Find the general solution to the equation*

$$y'' + y = \tan x.$$

Solution: The solution of the homogeneous equation

$$y'' + y = 0,$$

is given by

$$y_c = C_1 \cos x + C_2 \sin x.$$

We now suppose that the particular solution is of the form

$$y_p = v_1(x) \cos x + v_2(x) \sin x,$$

where $v_1(x)$ and $v_2(x)$ are unknown functions of x. So, by variation of parameters method, we have

$$v_1 = -\int \frac{y_2}{W[y_1, y_2]} g(x)dx \quad \text{and} \quad v_2 = \int \frac{y_1}{W[y_1, y_2]} g(x)dx.$$

Clearly,

$$W[y_1, y_2] = W[\cos x, \sin x] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0.$$

Now,

$$\begin{aligned} v_1 &= -\int \frac{\sin x}{1} \tan dx = -\int \frac{\sin^2 x}{\cos x} dx = -\int (\sec x - \cos x)dx \\ &= -\ln|\sec x + \tan x| + \sin x, \end{aligned}$$

and

$$v_2 = \int \frac{\cos x}{1} \tan dx = \int \sin x dx = -\cos x.$$

Thus,

$$y_p = (-\ln|\sec x + \tan x| + \sin x)\cos x - \cos x \sin x,$$

and the general solution is

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x + (-\ln|\sec x + \tan x| + \sin x)\cos x - \cos x \sin x,$$

where C_1 and C_2 are arbitrary constants.

Homework 9. *Solve the following nonhonogeneous equations:*
1) $y'' - 3y' + 2y = \frac{e^{3x}}{e^x + 1}$.
2) $y'' + 2y' + y = e^{-x} \ln x$.

1.6 Reduction of orders

Consider the linear differential equation of order n with constant coefficients of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = H(x), \tag{1.33}$$

where $a_i, i = 0, \ldots, n$ are real numbers. Let $\alpha_1, \ldots, \alpha_n$ be n roots of the characteristic equation, so, equation (1.33), can be written as

$$(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n) y = H(x) \tag{1.34}$$

We now assume that

$$(D-\alpha_2)\cdots(D-\alpha_n)y = u_1, \tag{1.35}$$

therefore equation (1.34) becomes

$$(D-\alpha_1)u_1 = H(x) \quad \Longrightarrow \quad \frac{du_1}{dx} - \alpha_1 u_1 = H(x),$$

which is a first order linear differential equation. So,

$$u_1 = \frac{\int e^{-\int \alpha_1 dx}\, H(x)dx + C_1}{e^{-\int \alpha_1 dx}} = e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\, H(x)dx + C_1\Big).$$

Substitutes in equation (1.35), we have

$$(D-\alpha_2)\cdots(D-\alpha_n)y = e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\, H(x)dx + C_1\Big).$$

Let

$$(D-\alpha_3)\cdots(D-\alpha_n)y = u_2,$$

So,

$$(D-\alpha_2)u_2 = e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\, H(x)dx + C_1\Big)$$

$$\Longrightarrow \frac{du_2}{dx} - \alpha_2 u_2 = e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\, H(x)dx + C_1\Big)$$

which is a first order linear differential equation and

$$u_2 = \frac{\int e^{-\int \alpha_2 dx}\Big[e^{\alpha_1 x}\Big(\int e^{-\alpha_1 x}\ H(x)dx + C_1\Big] + C_2}{e^{-\int \alpha_2 dx}}$$

$$= e^{\alpha_2 x}\Big[e^{(\alpha_1-\alpha_2)x}\Big(\int e^{-\alpha_1 x}\ H(x)dx + C_1 + C_2\Big)\Big]$$

Continuing in this way, we get

$$y = e^{\alpha_n}\Big[e^{(\alpha_n-\alpha_{2-1})x}\cdots\Big(\int e^{-\alpha_1 x}\ H(x)dx + C_1\Big) + \cdots + C_n\Big]$$

is a general solution of (1.33).

1.7 Linear differential equations with variable coefficients

1.7.1 The Cauchy and Legendre linear equations

The linear Cauchy equation is of the form

$$p_0 x^n \frac{d^n y}{dx^n} + p_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1} x\frac{dy}{dx} + p_n y = Q(x), \qquad (1.36)$$

in which $p_0, p_1, \cdots, p_n$ are constants, and the Legendre linear equation is of the form

$$p_0(ax+b)^n \frac{d^n y}{dx^n} + p_1(ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1}(ax+b) \frac{dy}{dx} + p_n y = Q(x), \tag{1.37}$$

of which equation (1.36) is a special case of equation (1.37) ($a = 1$, $b = 0$). These equations may be reduced to a linear differential equation with constant coefficients by properly transformation of the independent variable.

1.7.2 Solving the Legendre linear equation

Let $ax + b = e^z$, then $z = \ln(ax + b)$ and $\frac{dz}{dx} = \frac{a}{ax+b}$, so,

$$Dy = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{a}{ax+b}\frac{dy}{dz}$$

and

$$(ax + b)\frac{dy}{dx} = a\frac{dy}{dz} = aD_1 y.$$

Similarly,

$$D^2 y = \frac{d^2 y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{a}{ax+b}\frac{dy}{dz}) = -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\frac{d}{dx}(\frac{dy}{dz})$$

$$= -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\frac{d}{dz}(\frac{dy}{dz})\frac{dz}{dx}$$

$$= -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a^2}{(ax+b)^2}\frac{d^2y}{dz^2}$$

and

$$(ax+b)^2D^2y = -a^2\frac{dy}{dz} + a^2\frac{d^2y}{dz^2} = a^2(D_1^2 - D_1)y = a^2D_1(D_1-1)y.$$

Continuing in this way, we get

$$(ax+b)^nD^ny = a^nD_1(D_1-1)(D_1-2)\cdots(D_1-n+1)y.$$

After making these replacements, equation (1.37) becomes

$$[p_0a^nD_1(D_1-1)(D_1-2)\cdots(D_1-n+1)+p_1D_1(D_1-1)(D_1-2)\cdots(D_1-n+2)$$

$$+\cdots+p_{n-1}aD_1+p_n]y = Q(\frac{e^z-b}{a}),$$

is a linear differential equations with constant coefficients.

Example 20. *Solve*

$$(x^3D^3+2xD-2)y = x^2\ln x + 3x. \tag{1.38}$$

Solution: The transformation $x = e^z$ reduces the equation as follows

$$xDy = D_1 y \quad \text{and} \quad x^3D^3y = D_1(D_1-1)(D_1-2)y.$$

Substitutes into equation (1.38), we have

$$[D_1(D_1-1)(D_1-2) + 2D_1 - 2]y = ze^{2z} + 3e^z$$

$$\Longrightarrow\ (D_1^3 - 3D_1^2 + 4D_1 - 2)y = ze^{2z} + 3e^z,$$

which is a third order differential equation with constant coefficients. To find the complementary solution, clearly, the characteristic equation is

$$\alpha^3 - 3\alpha^2 + 4\alpha - 2 = 0 \Longrightarrow (\alpha - 1)(\alpha^2 - 2\alpha + 2) = 0,$$

so, $\alpha_1 = 1$, $\alpha_{2,3} = 1 \mp i$ and the complementary function is

$$y_c = C_1e^z + e^z(C_2\cos z + C_3\sin z).$$

The particular solution is

$$y_p = \frac{1}{D_1^3 - 3D_1^2 + 4D_1 - 2}\{ze^{2z} + 3e^z\}$$

$$\implies y_p = e^{2z}\frac{1}{(D_1+2)^3-3(D_1+2)^2+4(D_1+2)-2}\{z\}$$

$$+\frac{3}{D_1^3-3D_1^2+4D_1-2}\{e^z\}$$

$$\implies y_p = e^{2z}\frac{1}{D_1^3+3D_1^2+4D_1+2}\{z\}+\frac{3}{(D_1-1)(D_1^2-2D_1+2)}\{e^z\}$$

$$=\frac{e^{2z}}{2[1-(-\frac{D_1^3}{2}-\frac{3}{2}D_1^2-2D_1)]}\{z\}+3e^z\frac{z}{1!}$$

$$=\frac{ze^{2z}}{2}+\frac{e^{2z}}{2}(-2)+3ze^z=\frac{ze^{2z}}{2}-e^{2z}+3ze^z.$$

Therefore, the general solution is

$$y=y_c+y_p=C_1x+x(C_2\cos\ln x+C_2\sin\ln x)+\frac{x^2\ln x}{2}-x^2+3x\ln x,$$

where C_1 and C_2 are arbitrary constants.

Homework 10. *Solve the following differential equations:*

1) $(x^2D^2-xD+4)y=\cos\ln x+x\sin\ln x.$

2) $[(3x+2)^2D^2+3(3x+2)D-36]y=3x^2+4x+1.$

Example 21. *Find the general solution of*

$$(x+2)^2\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y=3x+4.$$

Solution: Let $x+2=e^z$, so the differential equation above transform to

$$[D_1(D_1-1)-D_1+1]y=3e^z-2 \implies (D_1-1)^2y=3e^z-2,$$

so the complementary function is

$$y_c=C_1e^z+C_2ze^z.$$

In this example,

$$y_p=\frac{1}{(D_1-1)^2}\{3e^z-2\}=3e^z\frac{z^2}{2!}-2\frac{1}{(D_1-1)^2}\{e^0\}=3e^z\frac{z}{2}-2.$$

Thus, the general solution is

$$y=y_c+y_p.$$

1.8 Non-linear differential equations with variable coefficients

In this section various types of higher order differential equations with variable coefficients will be considered. There is no general procedure comparable to that for linear equations. However, for the types treated here, the procedure consists in obtaining from the given equation another of lower order. We consider the following types.

1.8.1 Dependent variable missing (absent)

If the equation is free of y, the independent variable, that is of the form

$$f(y^{(n)}, y^{(n-1)}, \ldots, y'', y', x) = 0,$$

the substitution

$$\frac{dy}{dx} = y' = p, \frac{dy^2}{dx^2} = \frac{dp}{dx}, \ldots$$

will reduces the order by one.

Example 22. *Solve*

$$y'' + (y')^2 + 1 = 0. \tag{1.39}$$

Solution: Clearly equation (1.39) is a non-linear differential equation such that the dependent variable y is absent. So, let $y' = p$ and $y'' = \frac{dp}{dx}$ and equation (1.39) becomes

$$\frac{dp}{dx} + p^2 + 1 = 0 \Longrightarrow \frac{dp}{p^2+1} = -dx \Longrightarrow \tan^{-1} p = -x + c_1 \Longrightarrow p = \tan(c_1 - x).$$

Since $p = \frac{dy}{dx}$, then

$$\frac{dy}{dx} = \tan(c_1 - x) = \frac{\sin(c_1 - x)}{\cos(c_1 - x)} \Longrightarrow y = \int \frac{\sin(c_1 - x)}{\cos(c_1 - x)} dx \Longrightarrow y = \ln |cox(c_1 - x)| +$$

is a general solution where c_1 and c_2 are arbitrary essential constants.

1.8.2 Independent variable missing (absent)

Suppose we have the equation

$$f(y^{(n)}, y^{(n-1)}, \ldots, y'', y') = 0,$$

which the independent variable x is missing. Then the substitution

$$y' = p \quad \text{and} \quad y'' = \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

will reduced the order of the differential equation bye one.

Example 23. *Solve*

$$yy'' - (y')^2 = y^2 \ \ln(y).$$

Solution: The independent variable x is absent, so, let $y' = p$ and $y'' = p\frac{dp}{dy}$, then the differential equation above takes the form

$$yp\frac{dp}{dy} - p^2 = y^2 \ln y \implies \frac{dp}{dy} - \frac{p}{y} = y\, \ln y\, p^{-1}$$

is a Bernoulli differential equation in variables p and y with $n = -1$. Let

$$z = p^{1-n} = p^2 \implies \frac{dz}{dy} = 2p\frac{dp}{dy} = 2p(y\, \ln y\, p^{-1} + \frac{p}{y})$$

$$\implies \frac{dz}{dy} = 2y\, \ln y + 2\frac{p^2}{y} = 2y\, \ln y + 2\frac{z}{y} \implies \frac{dz}{dy} - 2\frac{z}{y} = 2y\, \ln y$$

which is a first order linear differential equation. Hence

$$z = p^2 = \frac{\int e^{\int -2\frac{dy}{y}}(2y\, \ln y)dy + c_1}{e^{\int -2\frac{dy}{y}}} = \frac{\int e^{-2\ln y}(2y\, \ln y)dy + c_1}{e^{-2\ln y}}$$

$$= y^2\left(\int 2\left(\frac{\ln y}{y}\right)dy + c_1\right) = y^2\left((\ln y)^2 + c_1\right).$$

Now,

$$p = \pm\sqrt{y^2\big((\ln y)^2 + c_1\big)} = y\sqrt{(\ln y)^2 + c_1} \implies \frac{dy}{y\sqrt{(\ln y)^2 + c_1}} = dx$$

$$\implies \ln(\ln y + \sqrt{(\ln y)^2 + c_1}) = x + k \implies \ln y + \sqrt{(\ln y)^2 + c_1} = c_2 e^x$$

$$\implies \sqrt{(\ln y)^2 + c_1} = c_2 e^x - \ln y \implies c_1 = c_2^2 e^{2x} - 2c_2 e^x \ln y$$

$$\implies \ln y = Ae^x + Be^{-x},$$

where A and B are constants.

1.9 Second order linear differential equations with variable coefficients

Consider a second order equation

$$y'' + \alpha(x)y' + \beta(x)y = \gamma(x), \tag{1.40}$$

where α, β and γ are functions of x.

If $y = u(x)$ is a solution of the corresponding homogeneous equation (1.40), that is,

$$y'' + \alpha(x)y' + \beta(x)y = 0,$$

then

$$u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = 0. \tag{1.41}$$

Now, let $y(x) = u(x)v(x)$ is a solution of the differential equation (1.40), where $v(x)$ is a function of x, so

$$y' = u'v + uv' \quad \text{and} \quad y'' = u''v + 2u'v' + uv'',$$

and substitutes in equation(1.40), we have

$$u''v + 2u'v' + uv'' + \alpha(x)(u'v + uv') + \beta(x)uv = \gamma(x)$$

$$\Longrightarrow (u'' + \alpha(x)u' + \beta(x)u)v + 2u'v' + uv'' + \alpha(x)uv' = \gamma(x).$$

So, by equation (1.41), we get

$$uv'' + (2u' + \alpha(x)u)v' = \gamma(x) \tag{1.42}$$

which is a second order differential equation of variable v and x and since in equation (1.42), the dependent variable is not appeared, so, it can be solved by letting $v' = p$ and $v'' = \frac{dp}{dx} = p'$, then (1.42) becomes

$$u\frac{dp}{dx} + (2u' + \alpha(x)u)p = \gamma(x),$$

which is a first order linear differential equation in variable p and x. Then we have the following theorem.

Theorem 6. *If $y = u(x)$ is a solution of a second order homogeneous differential equation*

$$y'' + \alpha(x)y' + \beta(x)y = 0,$$

then, the substitution $y(x) = u(x)v(x)$ reduces the differential equation

$$y'' + \alpha(x)y' + \beta(x)y = \gamma(x),$$

to a linear differential equation of first order.

Example 24. *If $y = x$ is a solution of the corresponding homogeneous equation*

$$y'' + 3x^2y' - 3xy = 5x^3. \tag{1.43}$$

Solution: Let $y = vx$ be a solution of equation (1.43), then

$$y' = v'x + v \quad \text{and} \quad y'' = 2v' + xv''.$$

So, equation (1.43) becomes

$$2v' + xv'' + 3x^2(xv' + v) - 3x^2v = 5x^3 \Longrightarrow xv'' + (2 + 3x^3)v' = 5x^3.$$

Now, let $v' = p$ and $v'' = p'$, so we have

$$x\frac{dp}{dx} + (2+3x^3)p = 5x^3,$$

which is a first order linear differential equation in variables p and x.

$$\frac{dp}{dx} + \frac{2+3x^3}{x}p = 5x^2$$

$$\therefore\ p = \frac{\int e^{\int (\frac{2+3x^3}{x})dx}\, 5x^2 dx + C_2}{e^{\int (\frac{2+3x^3}{x})dx}}.$$

Since, $p = v'$, then $v = \int p\, dx$. Thus, the general solution is

$$y = C_1 x + vx,$$

where C_1 and C_2 are arbitrary constants.

1.10 How one can find the particular solution of a homogeneous differential equation with variable coefficints

Consider a second order differential equation of the form

$$y'' + P(x)y' + Q(x)y = 0. \tag{1.44}$$

To find a particular solution of equation (1.44), there are several cases:

Case 1: If $y = x$ is a particular solution of the equation (1.44), then $y' = 1$ and $y'' = 0$. Substitutes into equation (1.44), we have

$$P(x) + xQ(x) = 0.$$

Then, if $P(x)+xQ(x) = 0$, so $y = x$ is a particular solution of equation (1.44).

Example 25. *Solve*

$$(D^2 - \frac{3}{x}D + \frac{3}{x^2})y = 2x - 1. \tag{1.45}$$

Solution: Here, $P(x) + xQ(x) = -\frac{3}{x} + x\frac{3}{x^2} = 0$, so, $y = x$ is a particular solution of equation (1.45). Thus, the transformation $y = xv$ reduces the equation (1.45) to a linear first order differential equation.

Now,

$$Dy = x\frac{dv}{dx} + v \quad \text{and} \quad D^2y = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substitutes in equation (1.45), we have

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 3\frac{dv}{dx} - \frac{3}{x}v + \frac{3}{x}v = x\frac{d^2v}{dx^2} - \frac{dv}{dx} = 2x - 1$$

$$\Longrightarrow \frac{d^2v}{dx^2} - \frac{1}{x}\frac{dv}{dx} = \frac{2x-1}{x}.$$

Let, $\frac{dv}{dx} = p$ and $\frac{d^2v}{dx^2} = p'$, so

$$\frac{dp}{dx} - \frac{1}{x}p = \frac{2x-1}{x},$$

which is a linear first order differential equation.

$$\therefore p = \frac{\int e^{-\int \frac{1}{x}dx}\,(2 - \frac{1}{x})dx + C_1}{e^{-\int \frac{1}{x}dx}} = x(\int (\frac{2}{x} - \frac{1}{x^2})dx + C_1)$$

$$\Longrightarrow p = x(2\ln|x| + \frac{1}{x} + C_1).$$

But,

$$p = \frac{dv}{dx} = 2x \ln|x| + 1 + C_1 x$$

$$\Longrightarrow v = \frac{y}{x} = \int (2x \ln|x| + 1 + C_1 x)dx = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + x + \frac{C_1}{2}x^2 + C_2$$

$$\Longrightarrow y = \frac{1}{2}x^3 \ln(x) - \frac{1}{4}x^3 + x^2 + \frac{C_1}{2}x^3 + C_2 x$$

is a general solution where C_1 and C_2 are arbitrary constant.

Homework 11. *Solve*

$$x^2(x + 1\frac{d^2y}{dx^2}) - x(2 + 4x + x^2)\frac{dy}{dx} + (2 + 4x + x^2)y = -x^4 - 2x^3.$$

Case 2: Find a condition that $y = ax + b$ is a particular solution of equation (1.44).

Case 3: If $y = e^{ax}$ is a solution of the differential equation (1.44), where a is any constant. Then, $y' = ae^{ax}$ and $y'' = a^2e^{ax}$. Substitutes in equation (1.44), we have

$$a^2e^{ax} + P(x)ae^{ax} + Q(x)e^{ax} = 0 \quad \Longrightarrow \quad e^{ax}\big(a^2 + aP(x) + Q(x)\big) = 0.$$

Since, $e^{ax} \neq 0$, then $a^2+aP(x)+Q(x) = 0$. So, if $a^2 + aP(x) + Q(x) = 0$, then $y = e^{ax}$ is a particular solution of equation (1.44).

Example 26. *Solve*

$$(1+x)y'' + (4x+5)y' + (4x+6)y = e^{-2x}. \tag{1.46}$$

Solution: Clearly,

$$y'' + \frac{4x+5}{1+x}y' + \frac{4x+6}{1+x}y = \frac{e^{-2x}}{1+x},$$

So, $P(x) = \frac{4x+5}{1+x}$ and $Q(x) = \frac{4x+6}{1+x}$. Now,

1) $P(x) + xQ(x) = \frac{4x+5}{1+x} + \frac{x(4x+6)}{1+x} = \frac{4x+5+x(4x+6)}{1+x} = \frac{4x^2+10x+5}{1+x} \neq 0$. So, $y = x$ is not a particular solution of the corresponding homogeneous equation (1.46).

2) $a^2 + aP(x) + Q(x)\big) = a^2 + \frac{4ax+5a}{1+x} + \frac{4x+6}{1+x} = \frac{a^2(1+x)+4ax+5a+4x+6}{1+x} = 0$

$\Longrightarrow a^2+a^2x+4ax+5a+4x+6 = 0 \Longrightarrow (a^2+4a+4)x+a^2+5a+6 = 0$

$\Longrightarrow (a+2)^2 = 0$ and $(a+2)(a+3) = 0 \Longrightarrow a+2 = 0 \Longrightarrow a = -2$.

Thus, $y = e^{-2x}$ is a particular solution of the corresponding homogeneous equation (1.46).

Let $y = e^{-2x}v(x)$ be a solution of the differential equation (1.46), then

$$y' = -2e^{-2x}v + e^{-2x}v'$$

and

$$y'' = 4e^{-2x}v - 4e^{-2x}v' + e^{-2x}v''.$$

Substitutes in (1.46), we have

$$(1+x)e^{-2x}(4v-4v'+v'')+e^{-2x}(4x+5)(-2v+v')+e^{-2x}(4x+6)v = e^{-2x}$$

$$(1+x)v'' + (-4(1+x) + (4x+5))v' = 1 \implies (1+x)v'' + v' = 1.$$

Note that the dependent variable v is not appear, so, let $p = v'$ and $\frac{dp}{dx} = v''$, then

$$(1+x)\frac{dp}{dx} = 1 - p \implies \frac{dp}{1-p} = \frac{dx}{1+x} \implies \frac{1}{(1-p)} = C_1(1+x)$$

$\implies p = 1 - \frac{1}{C_1(1+x)}$. Since,

$$p = v' = \frac{dv}{dx} \implies v = \int [1 - \frac{1}{C_1(1+x)}]dx \implies v = x - \frac{1}{C_1}\ln(1+x) + C_2.$$

Then,

$$y = e^{-2x} + e^{-2x}(x - \frac{1}{C_1}\ln(1+x) + C_2)$$

is a general solution where C_1 and C_2 are arbitrary constants.

Case 4: Let $y = u(x)v(x)$ be a solution of

$$y'' + P(x)y' + Q(x)y = f(x). \tag{1.47}$$

So,

$$y' = uv' + u'v \quad \text{and} \quad y'' = uv'' + 2u'v' + u''v.$$

Substitutes in (1.47), we get

$$uv'' + 2u'v' + u''v + P(x)(uv' + u'v) + Q(x)uv = f(x)$$

$$\implies uv'' + (2u' + P(x)u)v' + (u'' + P(x)u' + Q(x)u)v = f(x) \tag{1.48}$$

If u is chosen so that,

$$2u' + P(x)u = 0 \implies 2\frac{du}{dx} + P(x)u = 0 \implies \frac{du}{u} + \frac{1}{2}P(x)dx = 0$$

$$\implies \ln u = -\frac{1}{2}\int P(x)dx \implies u = e^{-\frac{1}{2}\int P(x)dx}$$

$$\Longrightarrow u' = -\frac{1}{2}P(x)e^{-\frac{1}{2}\int P(x)dx} = -\frac{1}{2}P(x)u$$

$$\Longrightarrow u'' = -\frac{1}{2}P'(x)u - \frac{1}{2}P(x)u' = -\frac{1}{2}P'(x)u + \frac{1}{4}P^2(x)u$$

Substitute in

$$u'' + P(x)u' + Q(x)u = -\frac{1}{2}P'u + \frac{1}{4}P^2u - \frac{1}{2}P^2 + Qu.$$

So, equation (1.48), becomes

$$v'' + (\frac{u'' + Pu' + Qu}{u})v = \frac{f(x)}{u} \tag{1.49}$$

If $\frac{u''+Pu'+Qu}{u}$ is a constant, then,

$$\frac{-\frac{1}{2}P'u - \frac{1}{4}P^2u + Qu}{u} = -\frac{1}{2}P' - \frac{1}{4}P^2 + Q = C$$

where C is a constant.

Then, equation (1.49) becomes

$$v'' + C\,v = \frac{f(x)}{u},$$

which is a second order differential equation with constant coefficients and $u = e^{-\frac{1}{2}\int P(x)dx}$.

Now, if

$$-\frac{1}{2}P' - \frac{1}{4}P^2 + Q = \frac{C}{x^2},$$

where C is a constant, then

$$v'' + \frac{C}{x^2}v = \frac{f(x)}{u} \implies x^2v'' + Cv = \frac{x^2}{u}f(x),$$

which is a Cauchy equation.

Example 27. *Solve*

$$y'' - 4xy' + 4x^2y = xe^{x^2}. \tag{1.50}$$

Solution: Here, in this example, $P(x) = -4x$ and $Q(x) = 4x^2$.

1) Since, $P(x) + xQ(x) = -4x + 4x^3 \neq 0$, so, $y = x$, is not a particular solution of the corresponding homogeneous equation equation (1.50).

2) Since there is no number a such that $a^2 + a(-4x) + 4x^2 = 0$.

3) Note $P'(x) = -4$, $P^2 = 16x^2$, then

$$-\frac{1}{2}P' - \frac{1}{4}P^2 + Q = -\frac{1}{2}(-4) - \frac{1}{4}16x^2 + 4x^2 = 2 = constant = C.$$

So,

$$u = e^{-\frac{1}{2}\int P(x)dx} = e^{-\frac{1}{2}\int (-4x)dx} = e^{x^2}.$$

Now, let $y = e^{x^2} v$ is a solution of the differential equation (1.50), so,

$$y' = 2xe^{x^2} v + e^{x^2} v' \quad \text{and} \quad y'' = 4x^2 e^{x^2} v + 4xe^{x^2} v + e^{x^2} v'' + 2e^{x^2} v.$$

Substitute in equation (1.50), we have

$$v'' + C\, v = \frac{f(x)}{u} = \frac{xe^{x^2}}{e^{x^2}} = x$$

so, we have

$$v'' + 2v = x$$

which is a second order differential equation with constant coefficients. The characteristic equation is $\alpha^2 + 2 = 0 \implies \alpha_{1,2} = \mp i\sqrt{2}$, therefore,

$$v_c = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x.$$

To find the particular solution,

$$(D^2 + 2)v = x \implies v_p = \frac{1}{D^2 + 2}\{x\} = \frac{1}{2} = \frac{1}{[1 - (-\frac{D^2}{2})]}\{x\}$$

$$\implies v_p = \frac{1}{2}[1 - \frac{D^2}{2} + (\frac{D^2}{2})^2 + \cdots]\{x\} = \frac{1}{2}[x + 0] = \frac{1}{2}x.$$

$$\therefore v = v_c + v_p = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x + \frac{1}{2}x.$$

Hence,

$$y = e^{x^2} + uv = e^{x^2} + e^{x^2}(C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x + \frac{1}{2}x),$$

is a general solution where C_1 and C_2 are arbitrary constants.

Homework 12. *1)* $y'' - 2xy' + (x^2+2)y = e^{\frac{1}{2}(x^2+2x)}$.
2) $(1+x)^2 y'' + (x+1)(x-2)y' + (2-x)y = 0$.

1.11 Applications of Second Order Differential Equations

Chapter 2

The Laplace transformation (Laplace's transform and its application to differential equations)

2.1 Laplace Transformation

Pierre Simon de Laplace (1749-1827) was a French Mathematician who made many discoveries in mathematical physics. His last words were reported to be "*What we know is very slight; what we don't know is immense*"

Definition 3. *The Laplace transform of the function* $f(t)$, $0 \leq t \leq \infty$ *(*$t \geq 0$*) is the function* $F(s) = L\{f(t)\}$ *defined by*

$$F(s) = \int_0^{\infty} e^{-st} f(t)dt = \lim_{b \to \infty} \int_0^b e^{-st} f(t)dt. \tag{2.1}$$

Example 28. *With* $f(t) = 1, t \geq 0$*, the definition of the Laplace transform* (2.1)*, gives*

$$L\{1\} = \lim_{b\to\infty} \int_0^b e^{-st}dt = \lim_{b\to\infty} [-\frac{1}{s}e^{-st}]_0^b = \lim_{b\to\infty} [-\frac{1}{s}e^{-sb} + \frac{1}{s}] = \frac{1}{s}, \quad s > 0.$$

Remark 11. *It is good practice to specify the domain of the Laplace transform. The limit we computed in the example above, would not exists if* $s < 0$*, for then* $\frac{1}{s}e^{-bs}$ *would become unbounded as* $b \to \infty$*. Hence,* $L\{1\}$ *is defined only for* $s > 0$*.*

Homework 13. *Find the Laplace transform of* $f(t) = e^{2t}$ *and specify its domain.*

Definition 4 (Sectional or piecewise continuity)**.** *A function* $f(t)$ *is called sectional continuous or piecewise continuous in an interval* $\alpha \leq t \leq \beta$*, if the interval can be subdivided into a finite number of intervals* *in each of which the function is continuous and has finite right and left hand limits.*

Definition 5 (Functions of exponential order). *If real constants $M > 0$ and λ exists such that for all $t > N$ (N is a number)*

$$|e^{-\lambda t} f(t)| < M \quad or \quad |f(t)| < Me^{\lambda t}.$$

We say that $f(t)$ is a function of exponential order λ as $t \to \infty$ or, briefly, is of exponential order.

Theorem 7 (Sufficient condition for existence of the Laplace transform). *If $f(t)$ is sectionally continuous in every finite interval $0 \leq t \leq N$ and of exponential order λ for $t > N$, then its Laplace transform $F(s)$ exists for all $s > \lambda$.*

Theorem 8 (Linearity of the Laplace transform). *If a and b are constants, then*

$$L\{af(t) + bg(t)\} = a\, L\{f(t)\} + b\, L\{g(t)\},$$

for all s such that the Laplace transforms of the functions f and g both exists.

Proof. The proof of this theorem follows immediately from the linearity of the operation of taking limits of integration

$$\begin{aligned} L\{af(t)+bg(t)\} &= \int_0^\infty e^{-st}\big(af(t)+bg(t)\big)dt \\ &= \lim_{c\to\infty}\int_0^c e^{-st}\big(af(t)+bg(t)\big)dt \\ &= a\,\big(\lim_{c\to\infty}\int_0^c e^{-st}f(t)dt\big) + b\,\big(\lim_{c\to\infty}\int_0^c e^{-st}g(t)dt\big) \\ &= a\,L\{f(t)\} + b\,L\{g(t)\}. \end{aligned}$$

2.2 Laplace transforms of some elementary functions

Theorem 9. *(a)* $L\{k\} = \frac{k}{s}, \quad s > 0$ *for any constant* k.

(b) $L\{e^{at}\} = \frac{1}{s-a}, \quad s > a.$

(c) $L\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0,\ n = 1, 2, \ldots.$

(d) $L\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0.$

(e) $L\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0.$

(f) $L\{\sinh kt\} = \frac{k}{s^2-k^2}, \quad s > k.$

(g) $L\{\cosh kt\} = \frac{s}{s^2-k^2}, \quad s > k.$

Proof.

(a) From definition of the Laplace transform, we have

$$\begin{aligned} L\{k\} &= \int_0^\infty e^{-st}\,k\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,k\,dt = k\lim_{b\to\infty}\int_0^b e^{-st}\,dt \\ &= -\frac{k}{s}\lim_{b\to\infty}[e^{-st}]_0^b = -\frac{k}{s}(0-1) = \frac{k}{s}, \quad s>0. \end{aligned}$$

Therefore,

$$L\{k\} = \frac{k}{s}, \quad s>0.$$

If $s<0$, then the integral does not converge and the Laplace transform is not defined.

Homework 14. *Prove* $(b),\ldots,(g)$.

Homework 15. *Prove* (c) *by using Gamma function* $\Gamma(x)$ *such that*

$$\Gamma(x) = \int_0^\infty e^{-t}\,t^{x-1}\,dt \quad \textit{and} \quad \Gamma(n+1) = n! \quad \textit{for } n \textit{ is positive integer.}$$

Example 29. *1)* $L\{\sin 3t\} = \frac{3}{s^2+9}, \quad s>0.$

2) $L\{3e^{2t} + 2\sin^2 3t\} = L\{3e^{2t} + 2(\frac{1-\cos 6t}{2})\} = L\{3e^{2t}\} + L\{1\} - L\{\cos 6t\} = \frac{3}{s-2} + \frac{1}{s} - \frac{s}{s^2+36} = \frac{3s^2+144s-72}{s(s-2)(s^2+36)}, \quad for\ s>0.$

Homework 16. *Find the following Laplace transforms:*

1) $L\{\sin t\cos t\}$.

2) $L\{t^{1/2}\}$.

Theorem 10. *If a is any real number, then*

$$L\{e^{at}f(t)\} = F(s-a),$$

where $F(s) = L\{f(t)\}$.

Proof. The proof follows directly from the definition of the Laplace transform

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st}\, e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a).$$

Notation: $L\{e^{at}f(t)\} = L\{f(t)\}|_{s\to s-a}$.
This property is known as shifting property.

Example 30. *Find $L\{e^{-2t}t^3\}$.*

Solution: By Theorem above, we have

$$L\{e^{-2t}t^3\} = L\{t^3\}|_{s\to s-(-2)} = \frac{3!}{s^4}|_{s\to s+2} = \frac{3!}{(s+2)^4}.$$

Homework 17. *Find $L\{e^{2t}\cos t \sin t\}$.*

Theorem 11. *If $L\{f(t)\} = F(s)$, then $L\{f(kt) = \frac{1}{k}F(\frac{s}{k})\}$, where k is a constant*

Proof. Since we have

$$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt = F(s),$$

then

$$L\{f(kt)\} = \int_0^\infty e^{-st}\, f(kt)\, dt.$$

Now, let $u = kt$, then $t = \frac{u}{k}$ and $dt = \frac{du}{k}$, so,

$$L\{f(kt)\} = \int_0^\infty e^{-st}\, f(kt)\, dt = \int_0^\infty e^{-\frac{u}{k}s}\, f(u)\, \frac{du}{k}$$

$$= \frac{1}{k}\int_0^\infty e^{-\frac{s}{k}u}\, f(u)\, du = \frac{1}{k}L\{f(\frac{s}{k})\} = \frac{1}{k}F(\frac{s}{k}).$$

2.3 Laplace transform of a derivation

$L\{f'(t)\} = sL\{f(t)\} - f(0)$.

Proof. From the definition of the Laplace transformation, we have

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt.$$

We now use integration by parts.

$$\text{Let} \quad u = e^{-st} \quad \Longrightarrow \quad du = -se^{-st} dt$$

$$dv = f'(t)dt \quad \Longrightarrow \quad v = f(t)$$

So,

$$L\{f'(t)\} = [e^{-st}f(t)]_0^\infty + s\int_0^\infty e^{-st}f(t)dt = sL\{f(t)\} - f(0).$$

Because, the integrated term $e^{-st}f(t)$ approached to zero (when $s > 0$) as $t \to \infty$, and its value at the lower limit $t = 0$ contributes $-f(0)$. An extension of these ideas to equations of order two can easily made by letting the function $g(t) = f'(t)$, then

$$L\{f''(t)\} = L\{g'(t)\} = sL\{g(t)\} - g(0)$$

$$= sL\{f'(t) - f'(0)\}$$

$$= s[sL\{f(t)\} - f(0)] - f'(0)$$

$$= s^2L\{f(t)\} - sf(0) - f'(0).$$

A repetition of this calculation gives

$$L\{f'''(t)\} = sL\{f''(t)\} - f''(0) = s^3L\{f(t)\} - s^2f(0) - sf'(0) - f''(0).$$

After finitely many such steps, we obtain the following extensionn

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

2.4 Inverse Laplace Transform

Definition 6. *If $L\{f(t)\} = F(s)$, then $f(t)$ is the inverse Laplace transform of $F(s)$ and is written $f(t) = L^{-1}\big(F(s)\big)$.*

Example 31. *Evaluate the following inverse Laplace transforms:*
1) $L^{-1}\{\frac{1}{s^4}\}$. *2) $L^{-1}\{\frac{15}{s^2+4s+13}\}$.*

Solution:

1) $L^{-1}\{\frac{1}{s^4}\} = \frac{3!}{3!}L^{-1}\{\frac{1}{s^4}\} = \frac{1}{3!}L^{-1}\{\frac{3!}{s^4}\} = \frac{t^3}{6}$.

2) First complete the square in the denominator

$$L^{-1}\{\frac{15}{s^2+4s+13} = L^{-1}\{\frac{15}{(s+2)^2+9}\}.$$

Since, we know that $L^{-1}\{\frac{k}{s^2+k^2} = \sin kt\}$, we proceed as follows

$$L^{-1}\{\frac{15}{(s+2)^2+9}\} = L^{-1}\{\frac{5\cdot 3}{(s+2)^2+9}\} = 5L^{-1}\{\frac{3}{(s+2)^2+9}\}$$

$$= 5e^{-2t}L^{-1}\{\frac{3}{s^2+9}\} = 5e^{-2t}\sin 3t.$$

Homework 18.

Theorem 12. *If c_1 and c_2 are constants, then*

$$L^{-1}\{c_1F_1(s)+c_2F_2(s)\}=c_1L^{-1}\{F(s)\}+c_2L^{-1}\{F_2(s)\}.$$

Theorem 13. *Prove that*

$$L^{-1}\{F(s-a)\}=e^{at}L^{-1}\{F(s)\}.,$$

where a is a constant.

Proof. From

$$F(s)=\int_0^\infty e^{-st}f(t)dt=L\{f(t)\},$$

we obtain

$$F(s-a)=\int_0^\infty e^{-(s-a)t}f(t)dt=\int_0^\infty e^{-st}\big(e^{at}f(t)\big)dt=L\{e^{at}f(t)\}$$

Since, $f(t)=L^{-1}\big(F(s)\big)$, then

$$L^{-1}\{F(s-a)\}=e^{at}f(t)=e^{at}L^{-1}\{F(s)\}.$$

Example 32. *Evaluate*

$$L^{-1}\{\frac{s+1}{s^2+6s+25}\}.$$

Solution:

$$\begin{aligned} L^{-1}\{\frac{s+1}{s^2+6s+25}\} &= L^{-1}\{\frac{s+1}{(s+3)^2+16}\} = e^{-3t}L^{-1}\{\frac{s-2}{s^2+16}\} \\ &= e^{-3t}\big[L^{-1}\{\frac{s}{s^2+16}\} - L^{-1}\{\frac{2}{s^2+16}\}\big] \\ &= e^{-3t}\big[\cos 4t - \frac{1}{2}\sin 4t\big]. \end{aligned}$$

Example 33. *Evaluate*

$$L^{-1}\{\frac{3s-1}{s(s-1)}\}.$$

Solution: Using partial fraction decomposition, we have

$$\frac{3s-1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = \frac{A(s-1)+Bs}{s(s-1)} = \frac{(A+B)s-A}{s(s-1)}$$

$$\Longrightarrow A = 1 \quad \text{and} \quad A+B = 3 \Longrightarrow B = 3-1 = 2.$$

Thus,

$$L^{-1}\{\frac{3s-1}{s(s-1)}\} = L^{-1}\{\frac{1}{s} + \frac{2}{s-1}\} = L^{-1}\{\frac{1}{s}\} + L^{-1}\{\frac{2}{s-1}\} = 1 + 2e^{t}.$$

Homework 19. *Evaluate the following Laplace transforms:*

1. $L^{-1}\{\frac{1}{(s+1)(s+3)(2s-1)}\}$.

2. $L^{-1}\{\frac{s-4}{(s+1)(s^2+4)}\}$.

Example 34. *Evaluate* $L^{-1}\{\frac{5}{(s-1)^3}\}$.

Solution:

$$L^{-1}\{\frac{5}{(s-1)^3}\} = 5e^{t}L^{-1}\{\frac{1}{s^3}\} = \frac{5e^{t}}{2!}L^{-1}\{\frac{2!}{s^3}\} = \frac{5e^{t}}{2!}t^2.$$

2.5 Initial Value Problems

Let $y = y(x)$ be a solution of a differential equation satisfying

$$y(x_0) = y_0. \tag{2.2}$$

Equation (2.2) is called an initial condition of differential equation. A differential equation together with an initial condition is called initial value problem. Therefore,

$$y' - x = 1, \quad y(0) = 0,$$

is an example of initial value problem.

2.6 Transformation of Initial Value Problems

We now discuss the application of Laplace transform to solve a linear differential equation with a constant coefficients such as

$$ay''(t) + by'(t) + cy(t) = f(t),$$

with given initial condition $y(0) = y_0$ and $y'(0) = y'_0$.
We must using the following procedure:

1. Take Laplace transform to both sides of the differential equation.

2. Substitute the initial conditions.

3. We take inverse Laplace transform.

Example 35. *Solve the initial value problem*

$$y'' + y = 1; \quad y(0) = 2, \quad y'(0) = 0. \tag{2.3}$$

Solution: First, take the Laplace transform to both sides of equation (2.3) and substitute the initial conditions, we have

$$L\{y'' + y\} = L\{1\} \implies s^2 L\{y\} - sy(0) - y'(0) + L\{y\} = L\{1\}$$

$$\implies s^2 L\{y\} - 2s + L\{y\} = \frac{1}{s} \implies L\{y\}(s^2 + 1) = \frac{1}{s} + 2s = \frac{1 + 2s^2}{s}$$

$$\implies L\{y\} = \frac{1 + 2s^2}{s(s^2 + 1)} \implies y = L^{-1}\{\frac{1 + 2s^2}{s(s^2 + 1)}\}.$$

Use the partial fraction decomposition to solve this inverse Laplace transform.

$$\frac{1 + 2s^2}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} = \frac{As^2 + A + Bs^2 + Cs}{s(s^2 + 1)} = \frac{(A + B)s^2 + Cs + A}{s(s^2 + 1)}.$$

Clearly, $A = 1$, $C = 0$ and $A + B = 2 \implies B = 1$. Thus,

$$y = L^{-1}\{\frac{1}{s} + \frac{s}{s^2 + 1}\} = L^{-1}\{\frac{1}{s}\} + L^{-1}\{\frac{s}{s^2 + 1}\}$$

$$\implies y = 1 + \cos t.$$

Example 36. *Solve*

$$y'' - 5y' + 6y = 2e^{-t} \tag{2.4}$$

subject to the initial condition $y(0) = 2$ *and* $y'(0) = 1$.

Solution: Again, take the Laplace transform to both sides of equation (2.4) and substitute the initial conditions, we have

$$L\{y'' - 5y' + 6y\} = L\{2e^{-t}\}$$

$$\implies s^2L\{y\} - sy(0) - y'(0) - 5[sL\{y\} - y(0)] + 6L\{y\} = L\{2e^{-t}\} = \frac{2}{s+1}$$

$$\implies s^2L\{y\} - 1 - 5sL\{y\} + 6L\{y\} = \frac{2}{s+1} \implies L\{y\}(s^2 - 5s + 6) = \frac{2}{s+1} + 1$$

$$\implies L\{y\} = \frac{2}{(s+1)(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}$$

Now,

$$\frac{2}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}.$$

So,

$$\frac{2}{(s+1)(s-2)(s-3)} = \frac{A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)}{(s+1)(s-2)(s-3)}$$

It is easy to see that $A = \frac{1}{6}$, $B = \frac{1}{2}$ and $C = -\frac{2}{3}$. We also do the same procedure for the second fraction. Finally we get,

$$L\{y\} = \frac{\frac{1}{6}}{s+1} + \frac{\frac{1}{2}}{s-3} - \frac{\frac{2}{3}}{s-2} + \frac{1}{s-3} - \frac{1}{s-2} = \frac{\frac{1}{6}}{s+1} + \frac{\frac{3}{2}}{s-3} - \frac{\frac{5}{3}}{s-2}.$$

Therefore,

$$y = L^{-1}\{\frac{1}{6(s+1)} + \frac{3}{2}\frac{1}{s-3} - \frac{5}{3}\frac{1}{s-2}\} = \frac{1}{6}e^{-t} + \frac{3}{2}e^{3t} - \frac{5}{3}e^{2t}.$$

Homework 20. *1. Solve*

$$y'' + y = 4te^t,$$

subject to the initial condition $y(0) = -2$ *and* $y'(0) = 0$.

2. Solve the initial value problem

$$x'' - x' - 6x = 0; \quad x(0) = 2, \quad x'(0) = -1.$$

3. Solve the initial value problem

$$y'' + 4y = \sin 3t; \quad y(0) = y'(0) = 0.$$

2.7 Derivative of the Laplace transforms

Theorem 14. *If $f(t)$ is a piecewise continuous for $t \geq 0$ and of exponential order for some $c > 0$ and if $F(s) = L\{f(t)\}$, then*

$$\frac{d}{ds}F(s) = -L\{t\,f(t)\}.$$

Example 37. *Evaluate $L\{t\,e^{at}\}$.*

Solution: From theorem above (Theorem 8), we have

$$L\{t\,e^{at}\} = -\frac{d}{ds}F(s) = -\frac{d}{ds}L\{e^{at}\} = \frac{d}{ds}(\frac{1}{s-a}) = \frac{1}{(s-a)^2}.$$

Homework 21. *Evaluate the following:*

1. $L\{t^2\,e^{at}\}$.
2. $L\{t\,e^{2t}\,\cos 3t\}$.
3. $L\{\sin t + t\,\cos t\}$, *without using the linearity property.*

Chapter 3

The power series method

3.1 Power Series Method (Power Series Solutions)

Definition 7. *A function $f(x)$ is said to be analytic at x_0, if it can be represented by a Taylor series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \tag{3.1}$$

which converges for all x in some open interval containing x_0. If

$x_0 = 0$, then the sees in (3.1) is the Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n.$$

3.2 Maclaurin series expansion of some elementary functions

(1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad -\infty < x < \infty.$

(2) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad -\infty < x < \infty.$

(3) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad -\infty < x < \infty.$

(4) $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}, \qquad -\infty < x < \infty.$

(5) $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \qquad -\infty < x < \infty.$

(6) $\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \qquad -1 < x < 1 \ \big(|x| < 1\big).$

(7) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1$ (Geometric series).

(8) $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-3)}{3!}x^3 + \cdots,$ (Binomial series). If α is nonnegative integer, then the binomial series is converges for all x. Otherwise, $|x| < 1$ converges and $|x| > 1$ diverges.

Thus, if the Taylor series of the function f converges to $f(x)$ for all x in some open interval containing x_0, then we say that the function $f(x)$ is analytic at x_0. For example, every polynomial is analytic everywhere and every rational function is analytic whenever its denominator is nonzero. For instance, $\tan x = \frac{\sin x}{\cos x}$ is analytic at $x_0 = 0$.

3.3 Solutions around ordinary points

For purpose of discussion, it is useful to place the second order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{3.2}$$

in the standard form

$$y'' + P(x)y' + Q(x)y = 0, \tag{3.3}$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$, $a_2(x) \neq 0$.

Definition 8. *A point $x = x_0$ is an ordinary point of equation* (3.3)*, if both $P(x)$ and $Q(x)$ are analytic at x_0; that is, if both $P(x)$ and $Q(x)$ has a Taylor series expansion about $x = x_0$. A point that is not an ordinary point is called a singular point of the equation.*

Example 38. *The differentia equation*

$$(1-x^2)y'' - 6xy' - 4y = 0,$$

has an ordinary points at $x = 0$. The points $x = 1$ and $x = -1$ are singular points of the equation.

Example 39. *Singular points need not be real numbers. The equation*

$$(x^2+4)y'' + 2xy' - 12y = 0,$$

has singular points at $x = \mp 2i$. The point $x = 0$, is an ordinary point.

Example 40. *Solve the equation*

$$y'' + 4y = 0, \tag{3.4}$$

near the ordinary point $x = 0$.

Solution: Since $x = 0$ is an ordinary point, then the series solution is

$$y = \sum_{n=0}^{\infty} a_n x^n \implies y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1} \implies y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}.$$

We now substitute, y, y' and y'' in the equation (**??**), we have

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longrightarrow \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + 4\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\Longrightarrow \sum_{n=2}^{\infty} [n(n-1)\, a_n + 4a_{n-2}] x^{n-2} = 0$$

$$\Longrightarrow n(n-1)\, a_n + 4a_{n-2} = 0 \Longrightarrow a_n = -\frac{4}{n(n-1)} a_{n-2}, \quad n \geq 2$$

$$\begin{aligned} a_2 &= -\frac{4}{2\cdot 1} a_0, & a_3 &= -\frac{4}{3\cdot 2} a_1 \\ a_4 &= -\frac{4}{4\cdot 3} a_2, & a_5 &= -\frac{4}{5\cdot 4} a_3 \\ &\vdots & &\vdots \\ a_{2n} &= -\frac{4}{2n\cdot(2n-1)} a_{2n-2}, & a_{2n+1} &= -\frac{4}{(2n+1)\cdot 2n} a_{2n-1} \end{aligned}$$

Now,

$$a_2 \cdot a_4 \cdot \dots \cdot a_{2n} = \frac{(-1)^n 4^n}{(2n)!} a_0 \cdot a_2 \cdot \dots \cdot a_{2n-2},$$

which simplify to

$$a_{2n} = \frac{(-1)^n 4^n}{(2n)!} a_0, \quad n \geq 1.$$

Similarly,

$$a_{2n+1} = \frac{(-1)^n 4^n}{(2n+1)!} a_1, \quad n \geq 1.$$

Since,

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

so,

$$y = a_0 + \sum_{n=1}^{\infty} a_{2n} x^{2n} + a_1 x + \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}$$

$$= a_0 \Big[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n} \Big] + a_1 \Big[x + \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{(2n+1)!} x^{2n+1} \Big]$$

$$= a_0 \Big[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \Big] + \frac{1}{2} a_1 \Big[2x + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} \Big]$$

$$y = a_0 \cos 2x + \frac{1}{2} a_1 \sin 2x,$$

is the general solution where a_0 and a_1 are arbitrary constants.

Example 41. *Find the general solution in powers in x of*

$$(x^2 - 4)y'' + 3xy' + y = 0. \tag{3.5}$$

Then find the particular solution with $y(0) = 4$ and $y'(0) = 1$.

Solution: The only singular points of equation (3.5) are ∓ 2. Since, $x_0 = 0$ is an ordinary point of (3.5), then

$$y = \sum_{n=0}^{\infty} c_n x^n \implies y' = \sum_{n=1}^{\infty} n\, c_n x^{n-1} \implies y'' = \sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2}.$$

Substitutes in equation (**??**), yields

$$(x^2-4)\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} + 3x\sum_{n=1}^{\infty} n\, c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\implies \sum_{n=2}^{\infty} n(n-1)\, c_n x^n - 4\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} + 3\sum_{n=1}^{\infty} n\, c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} n(n-1)\, c_n x^n - 4\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n + 3\sum_{n=0}^{\infty} n\, c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} \Big[(n^2+2n+1)c_n - 4(n+2)(n+1)c_{n+2}\Big]x^n = 0$$

$$\implies (n+1)^2 c_n - 4(n+2)(n+1)c_{n+2} = 0$$

$$\implies c_{n+2} = \frac{n+1}{4(n+2)} c_n, \quad \text{for } n \geq 0.$$

$$\text{For } n = 0 \implies c_2 = \frac{c_0}{4 \cdot 2}$$

$$\text{For } n = 2 \implies c_4 = \frac{3\, c_2}{4 \cdot 4} = \frac{1 \cdot 3\, c_0}{4^2 \cdot 2 \cdot 4}$$

$$\text{For } n = 4 \implies c_6 = \frac{5\, c_4}{4 \cdot 6} = \frac{1 \cdot 3 \cdot 5\, c_0}{4^3 \cdot 2 \cdot 4 \cdot 6}$$

Continuing in this way, we evidently would find that

$$c_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\, c_0}{4^n \cdot 2 \cdot 4 \cdots 2n}$$

Since,

$$(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+1)!}{2^n \cdot n!}$$

and

$$2 \cdot \cdots 2n = (2n)!! = 2^n \cdot n!$$

Thus,

$$c_{2n} = \frac{(2n-1)!!}{2^{3n} n!} c_0.$$

$$\text{For } n = 1 \implies c_3 = \frac{2\, c_1}{4 \cdot 3}$$

$$\text{For } n = 3 \implies c_5 = \frac{4\, c_3}{4 \cdot 5} = \frac{2 \cdot 4\, c_1}{4^2 \cdot 3 \cdot 5}$$

$$\text{For } n = 5 \implies c_7 = \frac{6\, c_5}{4 \cdot 7} = \frac{2 \cdot 4 \cdot 6\, c_1}{4^3 \cdot 3 \cdot 5 \cdot 7}$$

So,

$$c_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{4^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n+1)} c_1 = \frac{2^n\, n!}{2^{2n}(2n+1)!!} c_1 = \frac{n!}{2^n (2n+1)!!} c_1.$$

$$y(x) = c_0 + \sum_{n=1}^{\infty} c_{2n} x^{2n} + c_1 x + \sum_{n=1}^{\infty} c_{2n+1} x^{2n+1}$$

$$= c_0\Big[1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^{3n}n!}x^{2n}\Big]+c_1\Big[1+\sum_{n=1}^{\infty}\frac{(n)!!}{2^{n}(2n+1)!!}x^{2n}\Big]$$

$$\implies y(x)=c_0(1+\frac{1}{8}x^2+\frac{3}{128}x^4+\cdots)+c_1(x+\frac{1}{6}x^3+\frac{1}{30}x^5+\cdots).$$

Since, $y(0)=4$ and $y'(0)=1$, then $c_0=4$ and $c_1=1$.

Thus,

$$y(x)=4+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{3}{32}x^4+\cdots.$$

3.4 Frobenius series solution (Solutions near regular singular points)

Definition 9. *A singular point $x = x_0$ of the equation*

$$y'' + P(x)y' + Q(x)y = 0,$$

is said to be a regular singular point, if both terms $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at x_0. Otherwise, $x = x_0$ is an irregular singular point.

Example 42. *The point $x = 0$, is a singular point of the Euler-Cauchy equation*

$$x^2y'' - xy' - 3y = 0. \tag{3.6}$$

In this example, $P(x) = -\frac{1}{x}$ and $Q(x) = -\frac{3}{x^2}$.
Since, $xP(x) = x(-\frac{1}{x}) = -1$ and $x^2Q(x) = x^2(-\frac{3}{x^2}) = -3$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point.

Example 43. *The points $x = 3$ and $x = -3$ are singular points of the equation*

$$(x^2 - 9)^2y'' + (x - 3)y' + 2y = 0. \tag{3.7}$$

Here, $P(x) = \frac{1}{(x+3)^2(x-3)}$ and $Q(x) = \frac{2}{(x+3)^2(x-3)^2}$.

Since, $(x-3)P(x) = \frac{1}{(x+3)^2}$ *and* $(x-3)^2Q(x) = \frac{2}{(x+3)^2}$ *are both analytic at* $x = 3$*, so* $x = 3$ *is a regular singular point. But,* $x = -3$ *is an irregular singular point.*

Homework 22. *Classify the singular points for the following differential equations:*

1) $(x^2+1)y'' + (x+1)y' + 5y = 0.$

2) $x^4(x^2+1)(x-1)^2y'' + 4x^3(x-1)y' + (x+1)y = 0.$

Theorem 15. *Assume that* $x = x_0$ *is a regular singular point of the differential equation*

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

Then, there is at least one series solution of the form

$$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n(x-x_0)^n = \sum_{n=0}^{\infty} c_n(x-x_0)^{n+r}, \qquad (3.8)$$

where the number r is some real constant. The series will converge on some interval $0 < |x - x_0| < R$*. The series in* (3.8) *is known as a Frobenius series. We also assume that* $c_0 \neq 0$

Example 44. *Let us solve the Euler-Cauchy equation* (3.6) *by assuming a Frobenius series solution.*

Thus,

$$y = \sum_{n=0}^{\infty} c_n \, x^{n+r} \implies y' = \sum_{n=0}^{\infty} (n+r) c_n \, x^{n+r-1}$$

$$\implies y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n \, x^{n+r-2}.$$

Substitutes in equation (**??**), gives

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n \, x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) c_n \, x^{n+r-1} - 3 \sum_{n=0}^{\infty} c_n \, x^{n+r} = 0$$

$$\implies \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n \, x^{n+r} - \sum_{n=0}^{\infty} (n+r) c_n \, x^{n+r} - 3 \sum_{n=0}^{\infty} c_n \, x^{n+r} = 0.$$

If $n = 0$, we have

$$r(r-1)c_0 - rc_0 - 3c_0 = 0.$$

Since, $c_0 \neq 0$, then

$$r(r-1) - r - 3 = 0 \tag{3.9}$$

$$\implies r^2 - 2r - 3 = 0 \implies (r-3)(r+1) = 0 \implies r = 3 \quad \text{or} \quad r = -1.$$

Now,

$$\sum_{n=0}^{\infty} \Big[(n+r)(n+r-1) - (n+r) - 3 \Big] c_n \, x^{n+r} = 0$$

$$\Longrightarrow \Big[(n+r)(n+r-1)-(n+r)-3\Big]c_n = 0$$

$$\Longrightarrow \Big[n^2+nr-n+nr+r^2-r-n-r-3\Big]c_n = 0$$

$$\Longrightarrow \Big[(n+r)^2-2(n+r)-3\Big]c_n = 0$$

$$\Longrightarrow (n+r-3)(n+r+1)c_n = 0.$$

When $r=-1$, we have $(n-4)nc_n = 0$, so $c_n = 0$ if $n \neq 0$ or $n \neq 4$, where c_0 and c_4 are arbitrary. Substituting into the Frobenius series yields

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+r} = \sum_{n=0}^{\infty} c_n\, x^{n-1} = c_0x^{-1} + c_4x^3.$$

If $r=3$, then $n(n+4)c_n = 0$, so $c_n = 0$ if $n \neq 0$. Thus,

$$y_2 = \sum_{n=0}^{\infty} c_n\, x^{n+r} = \sum_{n=0}^{\infty} c_n\, x^{n+3} = c_0x^3.$$

Therefore, the general solution is

$$y = C_1x^{-1} + C_2x^3,$$

where C_1 and C_2 are arbitrary constants.

Remark 12. *Equation* (3.9) *is called the indicate equation associate with Frobenius series solution.*

Theorem 16. *Assume that $x = 0$ is a regular singular point of the second order differential equation*

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \tag{3.10}$$

Suppose that $r_1 \geq r_2$ are two real roots to the indicated equation $p(r) = 0$.

1) If $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer, then there exists two linearly independent solutions to equation (1.4) *of the form*

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+r_1}, \quad c_0 \neq 0$$

and

$$y_2 = \sum_{n=0}^{\infty} c_n\, x^{n+r_2}, \quad c_0 \neq 0.$$

2) If $r_1 - r_2$ is a positive integer, then there exists two linearly independent solutions to equation (1.4) *of the form*

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+r_1}, \quad c_0 \neq 0$$

and

$$y_2 = Cy_1(x)\, \ln(x) + \sum_{n=1}^{\infty} b_n\, x^{n+r_2}, \quad b_0 \neq 0,$$

where C is a constant that could be zero.

3) If $r_1 = r_2$, then there exists two linearly independent solutions to equation (1.4) of the form

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+r_1}, \quad c_0 \neq 0$$

and

$$y_2 = y_1(x)\, \ln(x) + \sum_{n=1}^{\infty} b_n\, x^{n+r_1}, \quad b_0 \neq 0.$$