Dynamical Systems I

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Existence and Uniqueness Theorems

Agenda

- Lipschitz condition.

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- Existence Theorem.

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- The method of successive approximations.

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Lipschitz condition

In this lecture we are concerned with the first order vector differential equation

$$\mathbf{x}' = f(t, \mathbf{x}). \tag{1}$$

We assume throughout this chapter that $f : D \to \mathbb{R}^n$ is continuous, where D is an open subset of $\mathbb{R} \times \mathbb{R}^n$.
We recall some basic definition in order to this chapter be self-contained.

Definition

We say that $\mathbf{x}$ *is a solution of* (1) *on an interval* I *provided* $\mathbf{x} : I \to \mathbb{R}^n$ *is differentiable,* $(t, \mathbf{x}(t)) \in D$, *for* $t \in I$ *and* $\mathbf{x}' = f(t, \mathbf{x})$ *for* $t \in I$.

Lipschitz condition

Definition

Let $(t_0, \mathbf{x}_0) \in D$*. We say that* $\mathbf{x}$ *is a solution of the initial value problem*

$$\mathbf{x}' = f(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

on an interval I *provided* $t_0 \in I$*,* $\mathbf{x}$ *is a solution of* (1) *on* I *and* $\mathbf{x}(t_0) = \mathbf{x}_0$*.*

Closely related to the initial value problem (2) is the integral equation

$$\mathbf{x} = \mathbf{x}_0 + \int_{t_0}^{t} f(s, \mathbf{x}(s))ds. \tag{3}$$

Lipschitz condition

Definition

We say that $\mathbf{x} : \mathbf{I} \to \mathbb{R}^{\mathbf{n}}$ *is a solution of the vector integral equation* (3) *on an interval* I *provided* $t_0 \in I$, $\mathbf{x}$ *is continuous on* I, $(t, \mathbf{x}(t)) \in D$, *for* $t \in I$, *and* (3) *is satisfied for* $t \in I$.

The relation between the initial value problem (2) and the integral equation (3) is given by the following lemma. So, of this result, we say that the initial value problem (2) and the integral equation (3) are equivalent.

Lipschitz condition

Lemma

Assume D is an open subset of $\mathbb{R} \times \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is continuous, and $(t_0, \mathbf{x}_0) \in D$; then $\mathbf{x}$ is a solution of the initial value problem (2) *on an interval I iff $\mathbf{x}$ is a solution of the integral equation* (3) *on an interval I.*

Proof. Assume that $\mathbf{x}$ is a solution of the initial value problem (2) on an interval I . Then $t_0 \in I$, $\mathbf{x}$ is differentiable on I (hence is continuous on I), $(t, \mathbf{x}(t)) \in D$, for $t \in I$, $\mathbf{x}(t_0) = \mathbf{x}_0$, and

$$\mathbf{x}'(t) = f(t, \mathbf{x}(t)),$$

for $t \in I$. Integrating this last equation and using $\mathbf{x}(t_0) = \mathbf{x}_0$, we get

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} f(s, \mathbf{x}(s))ds,$$

Lipschitz condition

for $t \in I$. Thus we have shown that $\mathbf{x}$ is a solution of the integral equation (3) on the interval I.

Conversely, assume $\mathbf{x}$ is a solution of the integral equation (3) on an interval I. Differentiating (3), using the fundamental theorem of integral calculus, we see that

$$\mathbf{x}'(t) = f(t, \mathbf{x}(t)),$$

for all $t \in I$. Moreover form (3) it is clear that $\mathbf{x}(t_0) = \mathbf{x}_0$ and thus $\mathbf{x}$ is a solution of the initial value problem (2).

Lipschitz condition

Here we introduce the notion of *Lipschitz* condition within the context of differential equation. Lipschitz conditions have important applications to the existence, uniqueness and approximation of solutions to equations, including ordinary differential equations.

Definition

A vector-valued function $F(t, \mathbf{x})$ *is said to satisfy a Lipschitz condition in a region* $\mathcal{R}$ *in* $(t, \mathbf{x})$*-space if, for some constant* L *(called the Lipschitz constant), we have*

$$\|F(t, \mathbf{x}) - F(t, \mathbf{y})\| \leq L \|\mathbf{x} - \mathbf{y}\|, \tag{4}$$

whenever $(t, \mathbf{x}) \in \mathcal{R}$ *and* $(t, \mathbf{y}) \in \mathcal{R}$.

Lipschitz condition

Lemma

If $F(t, \mathbf{x})$ has continuous partial derivatives on a bounded closed convex domain D, then it satisfies a Lipschitz condition in D.

This Lemma means that there exists some constant K such that

$$\|\frac{\partial F(t, \mathbf{x})}{\partial \mathbf{x}}\| \leq K.$$

Proof. By the fundamental theorem of calculus, for all $(t, \mathbf{x}), (t, \mathbf{y}) \in D$, we have

$$F(t, \mathbf{x}) - F(t, \mathbf{y}) = \int_{\mathbf{y}}^{\mathbf{x}} \frac{\partial F(t, \mathbf{s})}{\partial \mathbf{s}} d\mathbf{s}$$

and so

Lipschitz condition

$$
\begin{aligned}
\|F(t,\mathbf{x}) - F(t,\mathbf{y})\| &= \| \int_{\mathbf{y}}^{\mathbf{x}} \frac{\partial F(t,\mathbf{s})}{\partial \mathbf{s}} d\mathbf{s} \| \\
&\le \| \int_{\mathbf{y}}^{\mathbf{x}} \| \frac{\partial F(t,\mathbf{s})}{\partial \mathbf{s}} \| d\mathbf{s} \| \\
&\le \| \int_{\mathbf{y}}^{\mathbf{x}} K d\mathbf{s} \| \\
&= K \|\mathbf{x} - \mathbf{y}\|.
\end{aligned}
$$

Remark

1) The constant L is independent of $\mathbf{x}$, $\mathbf{y}$ and t. However, it depends on $\mathcal{R}$. In other words, for a given function $F(t, \mathbf{x})$, its Lipschitz constant may change if the domain $\mathcal{R}$ is different. In fact, for the same function $F(t, \mathbf{x})$, it can be a Lipschitz function in some regions, but not a Lipschitz function in some other regions.

2) That $F(t, \mathbf{x}) = \big(f_1(t, x_1, \ldots, x_n), \ldots, f_n(t, x_1, \ldots, x_n)\big)$ has continuous partial derivatives means that $\frac{\partial f_i}{\partial \mathbf{x}_j}$ and $\frac{\partial f_i}{\partial t}$ are continuous for all i, j. Sometimes we use $F(t, \mathbf{x}) \in \mathcal{C}^1$ for this.

An example of a function satisfying a Lipschitz condition is

$$f(t,x) = t^2 + 2x$$

where $D = \mathbb{R}^2$. For each $(t,x),(t,y) \in D$, consider

$$|f(t,x) - f(t,y)| = |(t^2 + 2x) - (t^2 + 2y)| = 2|x - y|$$

and so our f does satisfy a Lipschitz condition on $D = \mathbb{R}^2$ with $L = 2$.

Another example is

$$f(x,y) = xy^2$$

on

$$\mathcal{R}: \quad |x| \leq 1, \quad |y| \leq 1.$$

Here

$$\left|\frac{\partial f(x,y)}{\partial y}\right| = |2xy| \leq 2,$$

for $(x,y) \in \mathcal{R}$. This function does not satisfy a Lipschitz condition on the

$$\mathcal{S}: \quad |x| \leq 1, \quad |y| < \infty.$$

since

$$\left|\frac{f(x, y_1) - f(x, 0)}{y_1 - 0}\right| = |x||y_1|,$$

which tends to infinity as $|y_1| \to \infty$, if $|x| \neq 0$.
An example of a continuous function not satisfying a Lipschitz condition is

$$f(x, y) = y^{2/3}$$

on

$$\mathcal{R}: \quad |x| \leq 1, \quad |y| \leq 1.$$

Indeed, if $y_1 > 0$,

$$\frac{|f(x, y_1) - f(x, 0)|}{|y_1 - 0|} = \frac{y_1^{2/3}}{y_1} = \frac{1}{y_1^{1/3}},$$

which is unbounded as $y_1 \to 0$.

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Example

Determine whether the function

$$F(x,t) = \frac{x^2+1}{x} \cdot t$$

satisfies a Lipschitz condition in the domains:

1. $\mathcal{R}_1 = [1,2] \times [0,1]$.
2. $\mathcal{R}_2 = (1,2) \times [0,1]$.
3. $\mathcal{R}_3 = [1,2] \times [0,\infty)$.
4. $\mathcal{R}_4 = (0,1) \times [0,1]$.

Solution:

1) Since the function $F(x,t)$ is continuously differentiable in the bounded closed convex domain $\mathcal{R}_1$, by Lemma 2, we know that the function satisfy a Lipschitz in the given region.

2) Since the function $F(x,t)$ satisfies a Lipschitz condition in $\mathcal{R}_1$, and $\mathcal{R}_2 \subset \mathcal{R}_1$, we know that the function satisfies the same Lipschitz inequality in $\mathcal{R}_2$. Hence the function $F(x,t)$ satisfis a Lipschitz condition in $\mathcal{R}_2$.
3) Since $\mathcal{R}_3 = [1,2] \times [0,\infty)$ is not a bounded region, we cannot apply Lemma 2 in this case. Since

$$\frac{|F(x,t) - F(y,t)|}{|x-y|} = |\frac{xy-1}{xy}| \cdot |t| > |t| \to \infty$$

as $t \to \infty$, there exists no constant L, independent of x, y and t, such that

$$|F(x,t) - F(y,t)| = L|x-y|$$

Hence, the function F is not a Lipschitz function in $\mathcal{R}_3$.
4) Homework.