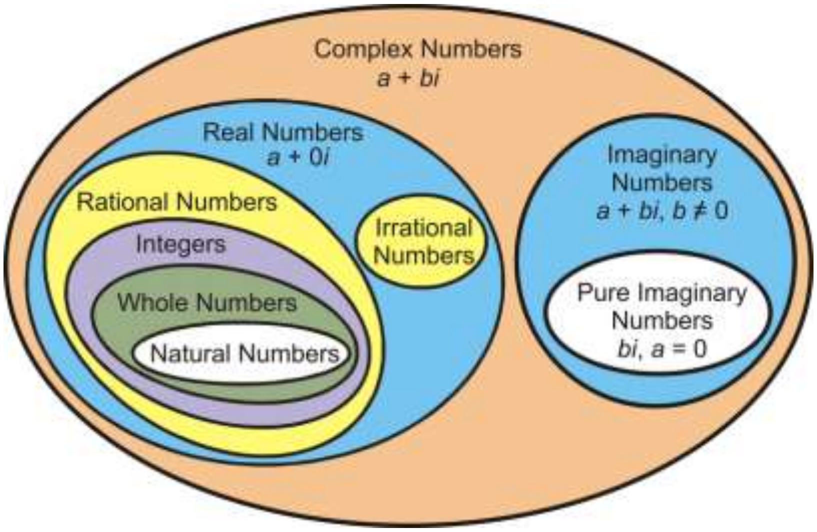
Complex numbers



Complex number :Is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i is a solution of the equation $x^2 = -1$. Because no real number satisfies this equation, i is called an imaginary number. For the complex number $a + bi$, a is called the **real part**, and b is called the **imaginary part**

Complex numbers : Is an order pair written as

$$z = (x, y) \text{ ; where } x, y \in \mathbb{R}$$

or

$$z = x + iy \,;\, x, y \in \mathbb{R} \,;\, i = \sqrt{-1}$$

The real numbers x and y are, moreover, known as the real and imaginary parts of z, respectively; and we write $x = Rez, y = Imz$

Complex set : $\mathbb{C} = \{z: z = x + iy: x, y \in \mathbb{R}\}$

The sum and product of two complex numbers:

Let $z1 = (x1, y1)$ and $z2 = (x2, y2)$ then

1. $z_1 \pm z_2 = (x_1, y_1) \pm (x_2, y_2) = (x_1 \pm x_2, y_1 \pm y_2)$,
2. $z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)$.

Note that :

1. *Any complex number* $z = (x, y)$ *can be written* $z = (x, 0) + (0, y)$
2. $(0,1)(y, 0) = (0, y)$. *Hence* $z = (x, 0) + (0,1)(y, 0)$
3. $x = (x, 0)$ *and* i=$(0,1)$, $i^2 = (0,1)(0,1) = (-1,0)$, *or* $i^2 = -1$

Basic algebra properties

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

I- *The commutative laws:*

1- $z_1 + z_2 = z_2 + z_1$

2- $z_1 z_2 = z_2 z_1$ *hw*

II- *The associative laws:*

1- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$,

2- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

III- *The distributive law:*

$$z(z_1 + z_2) = zz_1 + zz_2$$

VI- *The identity*

The additive identity 0 = (0,0) and the multiplicative identity 1 = (1,0)

That is,

$z + 0 = z$ *and* $z \cdot 1 = z$

for every complex number z . Furthermore, 0 and 1 are the only complex numbers with such properties

V- *Inverse*

1- *For each complex number* $z = (x, y)$ *a unique additive inverse* $-z = (-x, -y)$

satisfying the equation $z + (-z) = 0.$

2- *For* **any nonzero** *complex number* $z = (x, y)$, *there is a number* z^{-1} *such that* $zz^{-1} = 1$ *where* $z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$*to here*

Note that:

a. $\frac{z_1}{z_2} = z_1 z_2^{-1}$; $z_2 \neq 0$

b. *If* $z_1 . z_2 = 0$ *either* $z_1 = 0$ *or* $z_2 = 0$

Example *: write the following as complex number*

1- $\frac{1}{2-3i}\frac{1}{1+i}$ *Ans:*$\frac{5}{26} + \frac{1}{26}i$

2- $\frac{4+i}{2-3i}$

Solution:

$$\frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{(2-3i)(2+3i)} = \frac{5+14i}{13} = \frac{5}{13} + \frac{14}{13}i.$$

Note that :

The binomial formula involving real numbers remains valid with complex numbers. That is, if z_1 *and* z_2 *are any two nonzero complex numbers, then*

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \qquad (n = 1, 2, \ldots)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n)$$

and where it is agreed that $0! = 1$. The proof is left as an exercise.

Example:

1. Reduce each of these quantities to a real number:

(a) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$; (b) $\dfrac{5i}{(1-i)(2-i)(3-i)}$; (c) $(1-i)^4$.

Ans. (a) $-2/5$; (b) $-1/2$; (c) -4.

2. Show that

$$\frac{1}{1/z} = z \qquad (z \neq 0).$$

The modului (the length)(absolute value)

of complex number $z = x + iy$ is defined to be non negative real number $\sqrt{x^2 + y^2}$ which is length of vector interpretation of z and is denoted by $|z|$ it means $|z| = \sqrt{x^2 + y^2}$

Theorem:

let z , z_1 and z_2 be two complex number ,then we have the following:

1. $|z| \geq 0$
2. $|z| = 0 \ iff \ z = 0$
3. $|(x, 0)| = \sqrt{x^2} = |x|$
4. $|z_1 - z_2| = |z_2 - z_1|$
5. $|z_1 z_2| = |z_2| \quad |z_1|$
6. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$; *where* $z_2 \neq 0$.
7. $|z_1 + z_2| \leq |z_1| + |z_2|$
8. $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

Proof:

Note that :

1. $Re\, z \leq |Re\, z| \leq |z|$
2. $Im\, z \leq |Im\, z| \leq |z|$

Proof:

The distance between z_1 and z_2 *:Let z_1 and z_2 be two complex numbers then the distance between z_1 and z_2 as follows:*

$$|z_1 - z_2| = \sqrt{(x_1 - x_2) + (y_1 - y_2)}.$$

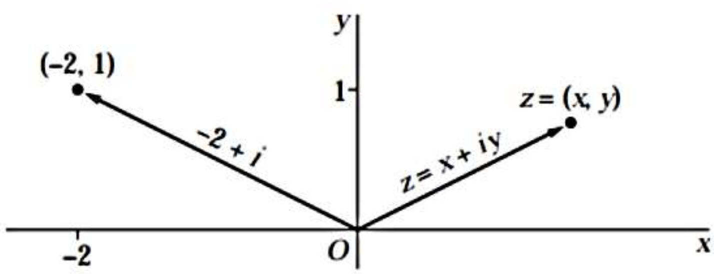
$|z_1 - z_2| = r$ *is represent the equation of circle with center* z_2 *and radius r because*

$$\left(\sqrt{(x_1 - x_2) + (y_1 - y_2)}\right)^2 = r^2$$

H.W*:* *Show that* $|z|^2 = (Re\, z)^2 + (Im\, z)^2$

Geometric representation of complex number

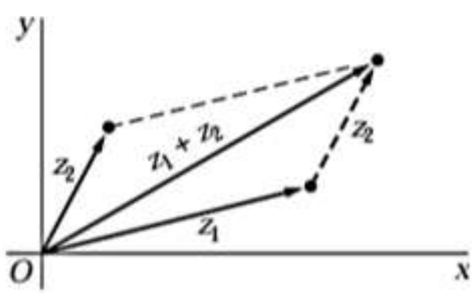
It is natural to associate any nonzero complex number $z = x + iy$ *with the directed vector, from the origin to the point (x, y) that represents z in thecomplex plane. In fact, we often refer to z as the point z or the vector z.the numbers* $z = x + iy$ *and* $-2 + i$ *are displayed graphically as both points and radius vectors*



When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as



Although the product of two complex numbers z_1 *and* z_2 *is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for* z_1 *and* z_2.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The modulus, or absolute value, of a complex number $z = x + iy$ *is defined as the nonnegative real number* $\sqrt{x^2 + y^2}$ *and is denoted by* $|z|$*;thatis,*

$$|z| = \sqrt{x^2 + y^2}.$$

Geometrically, *the number* $|z|$ *is the distance between the point* (x, y) *and the origin, or the length of the radius vector representing* z*. It reduces to the usual absolute value in the real number system when* $y = 0$.

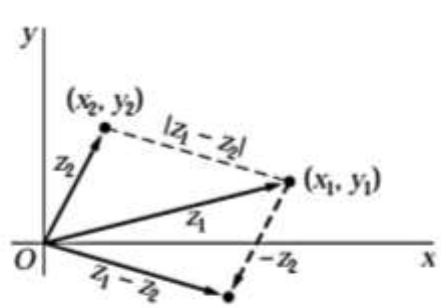
Note that :

1. *It is meaningless the inequality* $z_1 < z_2$ *unless both* z_1 *and* z_2 *are real.*
2. $|z1| < |z2|$ *means that the point* z_1 *is closer to the origin than the point* z_2 .

Example: *Since* $|-3 + 2i| = \sqrt{13}$ *and* $|1 + 4i| = \sqrt{17}$*, we know that the point* $-3 + 2i$ *is closer to the origin than* $1 + 4i$.

The distance between two points (x_1, y_1) *and* (x_2, y_2) *is* $|z1 - z2|$ *and* $|z_2 - z_1|$ *is the length of the vector representing the number*

$$z_1 - z_2 = z_1 + (-z_2)$$



EXERCISES

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

(a) $z_1 = 2i, \quad z_2 = \frac{2}{3} - i$; (b) $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;

(c) $z_1 = (-3, 1), \quad z_2 = (1, 4)$; (d) $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$.

2. Verify inequalities (3), Sec. 4, involving $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$.

3. Use established properties of moduli to show that when $|z_3| \neq |z_4|$.

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

4. Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

5. In each case, sketch the set of points determined by the given condition:

(a) $|z - 1 + i| = 1$; (b) $|z + i| \leq 3$; (c) $|z - 4i| \geq 4$.

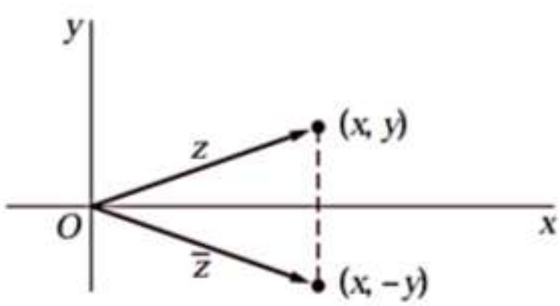
6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that

(a) $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;

(b) $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

Complex conjugate

The conjugate of complex number $z = x + iy$ is defined as complex number $x - iy$ and denoted by $\bar{z}$. That is if $z = x + iy$ then $\bar{z} = x - iy$.



Theorem:

let z , z_1 and z_2 be two complex numbers, then we have the following:

1- $z = 0 \; iff \; \bar{z} = 0$

2- $\bar{\bar{z}} = z$

3- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

4- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

5- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad ; z_2 \neq 0$

6- $z\bar{z} = |z|^2 = x^2 + y^2 \quad it \; means \quad z = \sqrt{z\bar{z}}$

7- $|z| = |\bar{z}|$

8- $z^{-1} = \frac{\bar{z}}{|z|^2}$

Example:

Sketch the following

1- $Im\,(\bar{z} - i) = 2$

2- $Re\,(\bar{z} - i) = 2$

3- $|z - i| = |z + i|$

Solution:

Example:

Show that

$$|\mathrm{Re}(2 + \bar{z} + z^3)| \leq 4 \qquad \text{when } |z| \leq 1.$$

H.W:

1. Use properties of conjugates and moduli established in Sec. 5 to show that

(a) $\overline{\bar{z}+3i} = z - 3i$; (b) $\overline{iz} = -i\bar{z}$;

(c) $\overline{(2+i)^2} = 3 - 4i$; (d) $|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}\,|2z+5|$.

2. Sketch the set of points determined by the condition

(a) $\operatorname{Re}(\bar{z}-i) = 2$; (b) $|2\bar{z}+i| = 4$.

3. Show that

I. $Re\ z = \frac{z+\bar{z}}{2}$

II. $Im\ z = \frac{z-\bar{z}}{2i}$

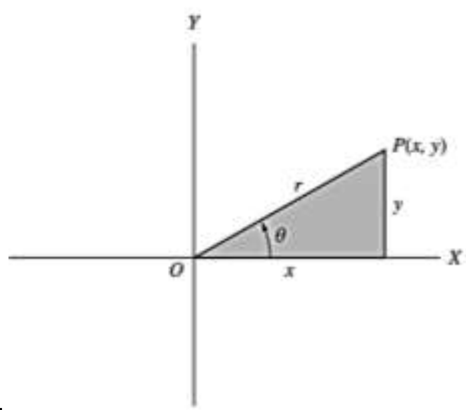
III. $\bar{z} = z\ iff\ z$ *is real*

IV. $\bar{z} = z\ iff\ z$ *iff z is purely imaginary*

4-	Prove that (a) z is real if and only if $\bar{z} = z$; (b) z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.
5-	Use mathematical induction to show that when $n = 2, 3, \ldots$, (a) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$; (b) $\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}$.

Exponential form (polar form) of complex numbers

Let r and θ be polar coordinates of the point (x, y) that corresponds to a nonzero



complex number $z = x + iy$.

Since

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

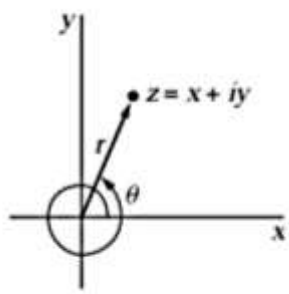
the number z can be written in polar form as

$$z = r(\cos\theta + i\sin\theta).$$

$$z = (r\cos\theta, r\sin\theta).$$

If $z = 0$, the coordinate θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinates are used.

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z ; that is, $r = |z|$. The real number θ represents the angle, measured in radians ($\tan\theta = \frac{y}{x}$), that z makes with the positive real axis is called argument of z and denoted by $\arg z = \theta$.



Note that:

1. for a $z \in \mathbb{C}$, $\arg(z)$is not unique because can adding any integer multiple of 2π.

2. For any complex number $z \neq 0$ there corresponds only one value of θ in $-\pi < \theta \leq \pi$. However, any other interval of length 2π, is called principal argument of z and denoted by $Arg\ z$

   It means $\arg z = Arg\ z + 2n\pi, \quad n = 0, \mp 1, \mp 2, \ldots$

3. When z is negative real numbers $Arg\ z = \pi$ not $-\pi$.

Example:

Write $z = 1 - i$ in polar form

Solution

1- Find r

2- Find $Arg\ z$

3- $1 - i = \sqrt{2}(\cos\frac{7\pi}{4} + \sin\frac{7\pi}{4})$.

Type equation here.

Example:

. Express each of the following complex numbers in polar form.

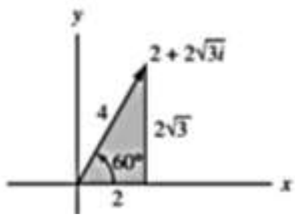
(a) $2 + 2\sqrt{3}i$, (b) $-5 + 5i$, (c) $-\sqrt{6} - \sqrt{2}i$, (d) $-3i$

Solution

(a) $2 + 2\sqrt{3}i$

Modulus or absolute value, $r = |2 + 2\sqrt{3}i| = \sqrt{4 + 12} = 4$.

$$2 + 2\sqrt{3}i = r(\cos\theta + i\sin\theta) = 4(\cos 60^\circ + i\sin 60^\circ) = 4(\cos\pi/3 + i\sin\pi/3)$$



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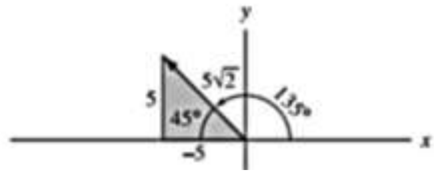


(b) $-5+5i$

$$r = |-5+5i| = \sqrt{25+25} = 5\sqrt{2}$$
$$\theta = 180° - 45° = 135° = 3\pi/4 \text{ (radians)}$$

Then

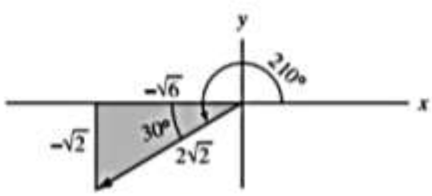
$$-5+5i = 5\sqrt{2}(\cos 135° + i \sin 135°) = 5\sqrt{2} \text{ cis } 3\pi/4 = 5\sqrt{2}e^{3\pi i/4}$$

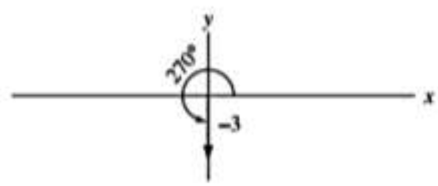
(c) $-\sqrt{6}-\sqrt{2}i$

$$r = |-\sqrt{6}-\sqrt{2}i| = \sqrt{6+2} = 2\sqrt{2}$$
$$\theta = 180° + 30° = 210° = 7\pi/6 \text{ (radians)}$$

Then

$$-\sqrt{6}-\sqrt{2}i = 2\sqrt{2}(\cos 210° + i \sin 210°) = 2\sqrt{2} \text{ cis } 7\pi/6 = 2\sqrt{2}e^{7\pi i/6}$$





(d) $-3i$

$$r = |-3i| = |0-3i| = \sqrt{0+9} = 3$$
$$\theta = 270° = 3\pi/2 \text{ (radians)}$$

Then

$$-3i = 3(\cos 3\pi/2 + i \sin 3\pi/2) = 3 \text{ cis } 3\pi/2 = 3e^{3\pi i/2}$$

H.W

Write the followings in polar form and exponential form

1. $z = -1 - i$
2. $z = -1 + i$
3. $z = 1 + i$

De Moivre's Theorem:

Let $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we can show that

$$z_1z_2 = r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

A generalization leads to

$$z_1z_2\cdots z_n = r_1r_2\cdots r_n\{\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)\}$$

and if $z_1 = z_2 = \cdots = z_n = z$ this becomes

$$z^n = \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$$

which is often called *De Moivre's theorem.*

Note : Euler's formula $e^{i\theta} = \cos\theta + i\ sin\ \theta$

Then $z = re^{i\theta}$

Then Demover's theorem reduce to $\left(e^{i\theta}\right)^n = e^{in\theta}$

That is

$\frac{z_1}{z_2} =$

$z_1z_2 =$

$\arg(z_1z_2) = argz_1 + argz_2 = \theta_1 + \theta_2$

Note that : $Arg\ z_1z_2 \neq Arg\ z_1 + Argz_2$ for example $z_1 = -1$ and $z_2 = i$

Example: Find $Arg z; z = -\frac{2}{1+\sqrt{3}i}$ $Ans: \frac{2\pi}{3}$

Example : Find

1. $(\sqrt{3}+i)^7$ Ans :$-64(\sqrt{3}+i)$
2. $(1+i)^8$ Ans :16

$Solution$:

Roots of Complex Numbers

A number w is called an nth *root* of a complex number z if $w^n = z$, and we write $w = z^{1/n}$. From De Moivre's theorem we can show that if n is a positive integer,

$$
\begin{aligned}
z^{1/n} &= \{r(\cos\theta + i\sin\theta)\}^{1/n} \\
&= r^{1/n}\left\{\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right\} \quad k = 0, 1, 2, \ldots, n-1
\end{aligned}
$$

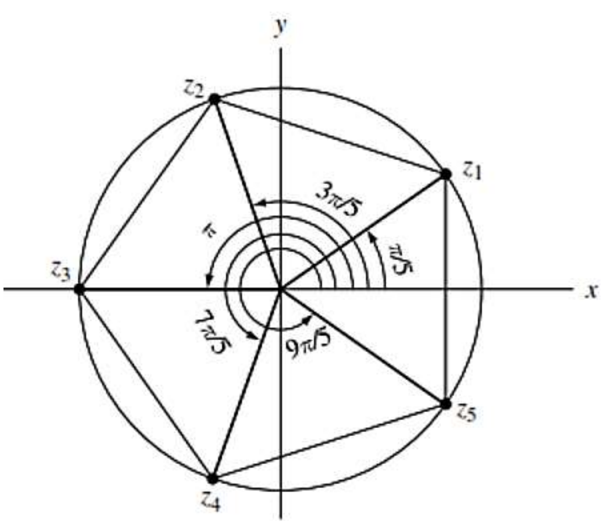
from which it follows that there are n different values for $z^{1/n}$, i.e., n different nth roots of z, provided $z \neq 0$.

The nth Roots of Unity

The solutions of the equation $z^n = 1$ where n is a positive integer are called the nth *roots of unity* and are given by

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{2k\pi i/n} \quad k = 0, 1, 2, \ldots, n-1 \qquad ($$

If we let $\omega = \cos 2\pi/n + i\sin 2\pi/n = e^{2\pi i/n}$, the n roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Geometrically, they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with center at the origin. This circle has the equation $|z| = 1$ and is often called the *unit circle*.



Example:

Find each of the indicated roots and locate them graphically.

(a) $(-1+i)^{1/3}$, (b) $(-2\sqrt{3}-2i)^{1/4}$

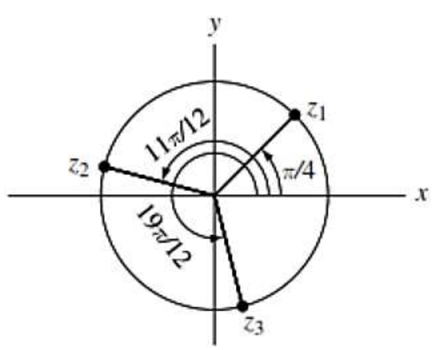
Solution

(a) $(-1+i)^{1/3}$

$$-1+i=\sqrt{2}\{\cos(3\pi/4+2k\pi)+i\sin(3\pi/4+2k\pi)\}$$

$$(-1+i)^{1/3}=2^{1/6}\left\{\cos\left(\frac{3\pi/4+2k\pi}{3}\right)+i\sin\left(\frac{3\pi/4+2k\pi}{3}\right)\right\}$$

If $k=0$, $z_1=2^{1/6}(\cos\pi/4+i\sin\pi/4)$.
If $k=1$, $z_2=2^{1/6}(\cos 11\pi/12+i\sin 11\pi/12)$.
If $k=2$, $z_3=2^{1/6}(\cos 19\pi/12+i\sin 19\pi/12)$.

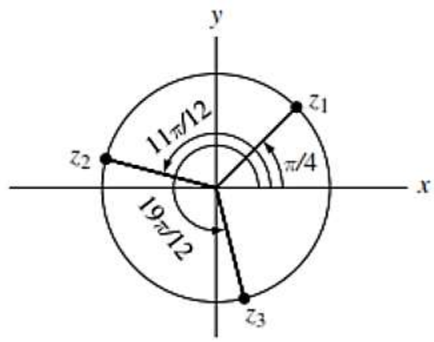


(b) $(-2\sqrt{3}-2i)^{1/4}$

$$-2\sqrt{3}-2i=4\{\cos(7\pi/6+2k\pi)+i\sin(7\pi/6+2k\pi)\}$$

$$(-2\sqrt{3}-2i)^{1/4}=4^{1/4}\left\{\cos\left(\frac{7\pi/6+2k\pi}{4}\right)+i\sin\left(\frac{7\pi/6+2k\pi}{4}\right)\right\}$$

If $k=0$, $z_1=\sqrt{2}(\cos 7\pi/24+i\sin 7\pi/24)$.
If $k=1$, $z_2=\sqrt{2}(\cos 19\pi/24+i\sin 19\pi/24)$.
If $k=2$, $z_3=\sqrt{2}(\cos 31\pi/24+i\sin 31\pi/24)$.
If $k=3$, $z_4=\sqrt{2}(\cos 43\pi/24+i\sin 43\pi/24)$.



H.W

Find the followings

1. **Cube root of $1 - i$**
2. **Square roots of $\frac{-1-\sqrt{3}i}{-1+i}$**
3. $(-8i)^{\frac{1}{3}}$
4. $z = \sqrt{\sqrt{6} - \sqrt{2}i}$