Ordinary Differential Equations

Lecture notes

Mathematics

Chiman M. Qadir

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Contents

List of References

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Chapter One

Basic definitions and elimination of essential constants

1.1 Introduction:

The general laws of science, engineering, medicine, social sciences, population dynamics and the like where the rate of change of quantities is involved are usually modelled as differential equations. Formation of differential equations, solution and interpretation of the results are of practical interest, especially for engineers and physicists, as they deal with many engineering and physical problems. So, we study in this chapter, methods of solution of ordinary differential equations of the first order and first degree, their applications to Newton’s law of cooling the law of natural growth and decay, and orthogonal trajectories. Further, methods of solution of linear differential equations of second and higher order with constant coefficients are also considered in this chapter

Definition 1. An equation expressing a relation between functions, their derivatives and the variables is called a differential equation.

Differential equations are classified into (1) ordinary and (2) partial differential equations.

Definition 2. A differential equation containing derivatives of a function (or functions) of a single variable is called an ordinary differential equation.

Definition 3. A differential equation containing partial derivatives of a function (or functions) of more than one variable, with each derivative referring to one of the variables is called a partial differential equation.

The following are examples of differential equations

$$y' + y = t \tag{1}$$

$$ay'' + by' + cy = g(t) \tag{2}$$

$$\sin(y)\frac{d^2 y}{dx^2} = (1-y)\frac{dy}{dx} + y^2 e^{-5y} \tag{3}$$

$$y^{(4)} + 10y''' - 4y' + 2y = \cos(t) \tag{4}$$

$$ydy - x\, e^y dx = 0 \tag{5}$$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial t} \tag{6}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 0 \tag{7}$$

$$y^2 \frac{\partial u}{\partial x} + xy \frac{\partial u}{\partial y} = nxu \tag{8}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \tag{9}$$

Definition 4. The order of a differential equation is the order of the highest derivative appearing in it.

The general form of an nth order ordinary differential equation in variables x and y is

$$F\left(x, y, y', y'', y''', \ldots, y^{(n)}\right) = 0$$

where $y^{(n)} = \frac{d^n y}{dx^n}$.

Definition . The degree of a differential equation is the degree or power of the highest derivative, when the equation is freed from radicals and fractions in respect of the derivatives, i.e., when the expression $F\left(x, y, y', y'', y''', \ldots, y^{(n)}\right)$ is written as a rational integral algebraic expression in $y^{(n)}$.

If F cannot be expressed in this manner then the degree of the differential equation is not defined.

The differential equations

1) $\frac{d^3 y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$ is of order 3 and degree 1.
2) $\sqrt[4]{(y'')^5} = \sqrt{7 + 3(y')^2}$

   Solution: The highest derivative is 2 then the order is 2.

   To obtain degree of differential equation we have make differential equation free from radicals

$$\left(\sqrt[4]{(y'')^5}\right)^4 = \left(\sqrt{7 + 3(y')^2}\right)^4$$

$$(y'')^2 = (7 + 3(y')^2)^2$$

   And the power of highest of derivative is 5

Which shows that the order of differential equation is 2 and degree 5.

Also the order and degree of the above examples

1) 1:1, 2) 2:1, 3) 2:1, 4) 4:1, 5) 1:1, 6) 2:1, 7) 1:2, 8,9) 1:1

Homework 1. Give an example of a differential equation for which a degree is not defined.

1.2 Solution of a differential equation

A relation $\phi(x, y) = 0$ defining a function $y = f(x)$ in some interval I, which has derivatives $f', f'', \dots, f^n$ such that $F(x, f, f', f'', \dots, f^n) = 0$, i.e., satisfying the differential equation, is called a solution of the differential equation:

$$F(x, y, f, f', f'', \dots, f^n) = 0.$$

Definition. A relation $\phi(x, y, c_1, c_2, \dots, c_n) = 0$ containing n independent arbitrary constants c_i which is a solution of the differential equation

$$F(x, y, f, f', f'', \dots, f^n) = 0.$$

Is called the general (complete) solution of the differential equation.

Definition. Any solution obtained from the general solution of a differential equation, by giving particular values to the arbitrary constants in it, is called a particular solution of the differential equation.

Definition. A solution $\phi(x, y) = 0$ of the differential equation

$$F(x, y, f, f', f'', \dots, f^n) = 0$$

Which is neither a general solution nor a particular solution of it, is called a singular solution of the differential equation. Only some equations have singular solutions.

In the context of differential equations, solution and 'integral' have the same meaning; and the general solution is sometimes called primitive.

To find a singular solution of a differential equation of first order and of any degree, we have process the following steps:

1) Let $f(x, y, c) = 0$ is a general solution
2) Find $\frac{\partial f}{\partial c}$
3) We take $\frac{\partial f}{\partial c} = 0$, then we find the value of c, and substitute in the general solution to get the singular solution.

Example. Find the singular solution to the differential equation

$$(y')^2 - xy' + y = 0$$

if it has the general solution

$$y = cx - c^2$$

$$f(x, y, c) = y - cx + c^2$$
$$\frac{\partial f}{\partial c} = -x + 2c$$
$$\frac{\partial f}{\partial c} = 0 \Rightarrow \quad -x + 2c = 0 \Rightarrow \quad c = \frac{1}{2}x$$
$$y = cx - c^2 = (\frac{1}{2}x)x - (\frac{1}{2}x)^2 = \frac{1}{2}x^2 - \frac{1}{4}x^2 = \frac{1}{4}x^2$$

$y = \frac{1}{4}x^2$ is a singular solution.

Example. $y = (x + a)^2$ is the general solution;

$y = x^2$ is the a particular solution; and

$y = 0$ is a singular solution of the first order and second degree differential equation.

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0.$$

Note that the singular solution $y = 0$ cannot be obtained from the general solution for any value of the arbitrary constant a.

1.3 Formation of a Differential Equation

We now find the differential equation if its general solution is known. We start with a relation involving essential arbitrary constants, and, by elimination of these constants, come to a differential equation. Since each differentiation yields a new relation, the number of derivatives that need be used is the same as the number of essential constants to be eliminated.

We shall in each case determine the differential equation that is

Of order equal to the number of essential constants in the given relation.

Free from essential constants.

Example. Form the differential equation of the $y = Acos5x + Bsin5x$, where A and B are arbitrary constants.

Solution. Given $y = Acos5x + Bsin5x$

Differentiate twice the above equation with respect to x

$\frac{dy}{dx} = -5A\sin 5x + 5B\cos 5x$

$\frac{d^2y}{dx^2} = -25A\cos 5x - 25B\sin 5x = -25(Acos5x + Bsin5x) = -25y$

$$\frac{d^2y}{dx^2} = -25y$$

Example. Form the differential equation of the $y = ae^{3x} + be^{x}$, where a and b are arbitrary constants.

Solution.

$$y = ae^{3x} + be^{x} \qquad (10)$$

Differentiate twice the equation (10) with respect to x

$$\frac{dy}{dx} = 3ae^{3x} + be^{x} \qquad (11)$$

$$\frac{d^2y}{dx^2} = 9ae^{3x} + be^{x} \qquad (12)$$

Subtracting (10) from (11) $\quad \Rightarrow \frac{dy}{dx} - y = 2ae^{3x} \qquad (13)$

Subtracting (11) from (12)

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = 6ae^{3x} = 3(\frac{dy}{dx} - y) \qquad \text{using (13)}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

Example. Find the differential equation of the $y = acos(mx + b)$, where a and b are being arbitrary constants.

Solution. $\quad y = a\ cos(mx + b), \qquad (14)$

Differentiate twice the equation (14) with respect to x

$$\frac{dy}{dx} = -ma\ sin(mx + b)$$

$$\frac{d^2y}{dx^2} = -m^2a\ cos(mx + b) \qquad (15)$$

By multiplying equation (14) then adding new equation with equation

(15) we get the differential equation as follows

$$\frac{d^2y}{dx^2} + m^2y = 0$$

Example. Find the differential equation by eliminating the arbitrary constants a and b from$y = a\tan x + b\sec x$

Solution. $$y = a\tan x + b\sec x \tag{16}$$

Multiplying both sides of (16) by $\cos x$, we get

$$y\cos x = a\sin x + b \tag{17}$$

Differentiate (17) with respect to x, we get

$$y(-\sin x) + \cos x\frac{dy}{dx} = \text{ a } \cos x \tag{18}$$

Divide both sides of (18) by $\cos x$

$$-\mathrm{y}\tan x + \frac{dy}{dx} = a \tag{19}$$

Differentiate (19) with respect to x, the differential equation is obtained as follows

$$\frac{d^2y}{dx^2} - \frac{dy}{dx}\tan x - y\sec^2 x = 0.$$

Example.Obtain the differential equation of the family of circles with centers on the x-axis and passing through the origin.

Solution:The equation of the family of circles passing through the origin and with centres on the x-axis is

$$x^2 + y^2 + 2ax = 0 \tag{20}$$

Where a is an arbitrary constant

Differentiating (20) with respect to x

$$2x + 2y\frac{dy}{dx} + 2a = 0$$

$$\Longrightarrow a = -x - y\frac{dy}{dx} \tag{21}$$

Putting the value of a in (20)

$$x^2 + y^2 + 2x(-x - y\frac{dy}{dx}) = 0$$

$$\Longrightarrow y^2 - x^2 - 2xy\frac{dy}{dx} = 0$$

Which is the required differential equation.

Exercise

Form the differential equation in each of the following cases by eliminating the parameters mentioned against each.

1. $y = ax + bx^2$ $\quad (a, b)$

   Ans: $2y + x^2 y'' = 2xy'$

2. $x = A\cos(pt + B)$ $\quad (A, B)$

   Ans: $x'' + p^2 x = 0$ ($'$ denotes differentiation with respect to 't')

3. $y = mx + \dfrac{a}{m}$ $\quad (m)$

   Ans: $y = xy' + \dfrac{a}{y'}$

4. $y = ax^2 + bx + c$ $\quad (a, b, c)$

   Ans: $y''' = 0$

5. $(x-h)^2+(y-k)^2=r^2 \qquad (h,k)$

Ans: $(1+y_1^2)^3=r^2y_2^2, \left(y_1=\frac{dy}{dx}, y_2=\frac{d^2y}{dx^2}\right)$

6. $y=e^x(A\cos x+B\sin x) \qquad (A,B)$

Ans: $y_2-2y_1+2y=0$

7. Find the differential equation for the family of circles with their centres on the x-axis.
(Hint: $x^2+y^2+2gx+c=0$ g, c parameters)

Ans: $1+y_1^2+yy_2=0$

8. Form the differential equation for the family of circles, touching the x-axis at (0, 0).
(Hint: $x^2+y^2-2fy=0, f$ parameter)

Ans: $(y^2-x^2)y'+2xy=0$

9. Form the differential equation of all parabolas each having its latus-return = $4a$ and its axis parallel to the x-axis.
(Hint: $(y-k)^2=4a(x-h)$; h,k parameters)

Ans: $2ay_2+y_1^3=0$

10. Find the differential equation by eliminating c from $y=cx+x^3$.

Ans: $x\frac{dy}{dx}-y-2x^3=0$

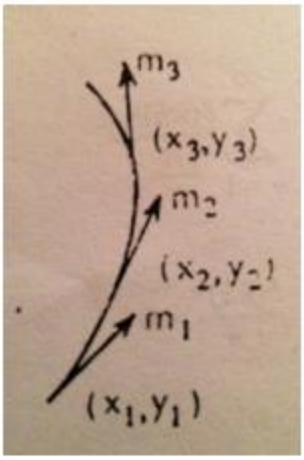
1.4 Geometrical interpretation of differential equations

Consider a differential equation

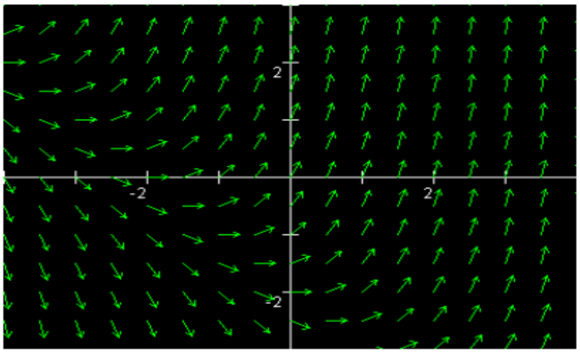
$$\frac{dy}{dx}=f(x,y), \tag{22}$$

which is a first order and first degree. Since, from calculus, the derivative is the slope of the tangent line, we interpret this equation geometrically to mean that at any point (x,y) in the plane, the tangent line must have slope $f(x,y)$.

Take any point (x_1, y_1) in xy-plane, equation (22) will determine corresponding value of $\frac{dy}{dx}$, say m_1. A point that moves, subject to the restriction imposed by (22), on passing through (x_1, y_1) must go in the direction m1. Let it moves infinitesimal distance to a point (x_2, y_2) and m_2 be the value of $\frac{dy}{dx}$ corresponding to (x_2, y_2) as determined by (22). Thence under the same condition to (x_3, y_3) and so on through successive points. In the proceeding thus the point will describe the coordinate of every point of which and the direction of the tangent thereat will satisfy the differential equation (22).



For instance, the slope field for $\frac{dy}{dx} = x + y + 2$ is illustrated as follows:



The solution to a differential equation is a curve that is tangent to the arrows of the slope field. Since differential equations are solved by integrating, we call such a curve an integral curve. This picture illustrates some of the integral curves for $\frac{dy}{dx} = x + y + 2$. You can see there are a lot of possible integral curves, infinitely many in fact. This corresponds to the fact that there are infinitely many solutions to a typical differential equation. To specify a particular integral curve, you must specify a point on the curve. Once you specify one specific point, the rest of the curve is determined by following the arrows. This corresponds to finding a particular solution by specifying an initial value.