زانكۆی سەلاحەدین - هەولێر
Salahaddin University-Erbil

Matrix Eigenvalue problem

Research Project

Submitted to the department of (mathematics) in partial fulfillment of the requirements for the degree of BSc. in (mathematics)

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April – 2023

Certification of the supervisor

I certify that this work was prepared under my supervision at the Department of Mathematics/ College of Education /Salaheddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics

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Date: 6 /4 /2023

In view of the available recommendations, I forward this work for debate by the examining committee.

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Acknowledgement

First of all, praises and thanks to the Almighty Allah for his blessing throughout my research project to complete the research successfully.

I would like to express my sincere gratitude to my research supervisor, **Dr. Adnan Ali Jalal** for helping me through giving advice, checking my research and making useful notes which enhanced the research and teaching me to present the project at it is finest and for being so helpful.

Finally, I extremely grateful to my parents and friends for their love and support.

Abstract

First, Talk about some basic consents of matrix.

Also, we used some methods to find inverse of matrix exactly and numerically such as (Gauss-Jordan Method, Factorization Method, Partition Method and Iterative Method). Finally, we used power method to find the eigenvalue problem.

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Introduction

There are two main numerical exercises which arise in connection with the matrices. One of these is the problem of finding the inverse of a matrix. The other problem is that of finding the eigenvalues and the corresponding eigenvectors of a matrix. When a student first encounters an eigenvalue problem, it appears to him somewhat artificial and theoretical only. In fact the computation of eigenvalues is required in many engineering and scientific problems. For instance, the frequencies of the vibrations of beams are the eigenvalues of a matrix. Eigenvalues are also required while finding the frequencies associated with

(i) the vibrations of a system of masses and springs,

(ii) the symmetric vibrations of an annular membrane,

(iii) the oscillations of a triple pendulum,

(iv) the torsional oscillations of a uniform cantilever,

(v) the torsional oscillations of a multi-cylinder engine etc.

Once the physical formulation in any of the above situations is completed, all these Exercises have the same mathematical approach: that of finding an eigenvalue for a numerical matrix.

Chapter One:

Definition 1.1 (Burden & Faires, 2010)

An $n \times m$ (n by m) **matrix** is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array.

The notation for an $n \times m$ matrix will be a capital letter such as A for the matrix and lowercase letters with double subscripts, such as a_{ij}, to refer to the entry at the intersection of the i th row and j th column; that is,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Definition 1.2 (B.S.GREWAL, 2018)

Non-Singular matrix is a square matrix whose determinant is a non-zero value.

Definition 1.3 (B.S.GREWAL, 2018)

If *A* is a non-singular square matrix, there is a matrix *B* of same order, which is called the **inverse matrix** of A such that *AB* = *BA*= *I*, where *I* is the unit matrix.

Definition 1.4 (Poole, 2010)

Let A be a square matrix. A factorization of A as $A = LU$, where L is unit lower triangular and U is upper triangular, is called an **LU factorization** of A.

Definition 1.5 (Burden & Faires, 2010)

If A is a square matrix, the **characteristic polynomial** of A is defined by

$$p(\lambda) = \det(A - \lambda)$$

It is not difficult to show that p is an nth-degree polynomial and, consequently, has at most n distinct zeros, some of which might be complex. If λ is a zero of p, then, since det $(A - \lambda I) = 0$, the linear system defined by $(A - \lambda l)\mathbf{x} = 0$ has a solution with $\mathbf{x} \neq 0$. We wish to study the zeros of p and the nonzero solutions corresponding to these systems.

Definition 1.5 (Burden & Faires, 2010)

If p is the characteristic polynomial of the matrix A, the zeros of p are **eigenvalues**, or characteristic values, of the matrix A. If λ is an eigenvalue of A and $x \neq 0$ satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is an **eigenvector**, or characteristic vector, of A corresponding to the eigenvalue λ.

To determine the eigenvalues of a matrix, we can use the fact that

- λ is an eigenvalue of A if and only if $\det(A - \lambda l) = 0$.
  Once an eigenvalue λ has been found a corresponding eigenvector $x \neq 0$ is determined by solving the system
- $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Chapter two

2.1 Methods to find inverse matrix:

2.1.1 Gauss-Jordan Method: (B.S.GREWAL, 2018)

This is similar to the Gauss elimination method except that instead of first converting A into upper triangular form; it is directly converted into the unit matrix.

In practice, the two matrices A and I are written side by side and the same row transformations are performed on both. As soon as A is reduced to I, the other matrix represents A^{-1}.

Example: Find the inverse of the matrix A using the Gauss-Jordan method.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Solution:

$$[\mathrm{A} \mid \mathrm{I}] = \begin{bmatrix} 3 & 1 & 2 & : & 1 & 0 & 0 \\ 2 & -3 & -1 & : & 0 & 1 & 0 \\ 1 & -2 & 1 & : & 0 & 0 & 1 \end{bmatrix} \qquad \left(\tfrac{1}{3}\mathrm{R}_1\right)$$

$$= \begin{bmatrix} 1 & \tfrac{1}{3} & \tfrac{2}{3} & : & \tfrac{1}{3} & 0 & 0 \\ 2 & -3 & -1 & : & 0 & 1 & 0 \\ 1 & -2 & 1 & : & 0 & 0 & 1 \end{bmatrix} \qquad (\mathbf{R}_2 - 2\mathbf{R_1}, \mathbf{R_3} - \mathbf{R_1})$$

$$= \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & : & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{7}{3} & : & -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{7}{3} & \frac{1}{3} & : & -\frac{1}{3} & 0 & 1 \end{bmatrix} \qquad \left(-\frac{3}{11}\mathrm{R}_2\right)$$

$$=\begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & : & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{7}{11} & : & \frac{2}{11} & -\frac{3}{11} & 0 \\ 0 & -\frac{7}{3} & \frac{1}{3} & : & -\frac{1}{3} & 0 & 1 \end{bmatrix} \qquad (\mathbf{R}_1-\frac{1}{3}\mathbf{R}_2, \mathbf{R}_3+\frac{7}{3}\mathbf{R}_2)$$

$$=\begin{bmatrix} 1 & 0 & \frac{5}{11} & : & \frac{3}{11} & \frac{1}{11} & 0 \\ 0 & 1 & \frac{7}{11} & : & \frac{2}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & \frac{20}{11} & : & \frac{1}{11} & -\frac{7}{11} & 1 \end{bmatrix} \qquad \left(\frac{11}{20}\mathbf{R}_3\right)$$

$$=\begin{bmatrix} 1 & 0 & \frac{5}{11} & : & \frac{3}{11} & \frac{1}{11} & 0 \\ 0 & 1 & \frac{7}{11} & : & \frac{2}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 1 & : & \frac{1}{20} & -\frac{7}{20} & \frac{11}{20} \end{bmatrix} \qquad \left(\mathbf{R}_1-\frac{5}{11}\mathbf{R}_3, \mathbf{R}_2-\frac{7}{11}\mathbf{R}_3\right)$$

$$=\begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & : & \frac{3}{20} & -\frac{1}{20} & -\frac{7}{20} \\ 0 & 0 & 1 & : & \frac{1}{20} & -\frac{7}{20} & \frac{11}{20} \end{bmatrix}$$

Thus we obtain:

$$A^{-1}=\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{20} & -\frac{1}{20} & -\frac{7}{20} \\ \frac{1}{20} & -\frac{7}{20} & \frac{11}{20} \end{bmatrix}$$

2.1.2 Factorization Method (B.S.GREWAL, 2018)

In this method, we factorize the given matrix as A = LU (1)

where L is a lower triangular matrix with unit diagonal elements and U is an upper triangular matrix

i.e. $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Now (1) gives $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$ (2)

To find L^{-1}, let $L^{-1} = X$, where X is a lower triangular matrix. Then $LX = I$

i.e. $$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the matrices on the L.H.S. and equating the corresponding elements, we have

$$x_{11} \doteq 1, x_{22} = 1, x_{33} = 1 \quad (3)$$

$$l_{21}x_{11} + x_{21} = 0\,,\ l_{31}x_{11} + l_{32}x_{21} + x_{31} = 0 \text{ and } l_{32}x_{22} + x_{32} = 0 \quad (4)$$

(3) gives $x_{11} = x_{22} = x_{33} = 1$

(4) $x_{21} = -l_{21}x_{11}\,,\ x_{31} = -(l_{31} + l_{32}x_{21})$ and $x_{32} = -l_{32}$

Thus $L^{-1} = X$ is completely determined.

To find U^{-1}, let $U^{-1} = Y$, where Y is an upper triangular matrix.

Then YU = I

i.e., $$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the matrices on the L.H.S. and then equating the corresponding elements, we have

$$y_{11}u_{11} = 1, y_{22}u_{22} = 1, y_{32}u_{33} = 1$$

$$y_{11}u_{12} + y_{12}u_{22} = 0, y_{11}u_{13} + y_{12}u_{23} + y_{13}u_{33} = 0 \\ y_{22}u_{23} + y_{23}u_{33} = 0$$

From (5), $y_{11} = 1/u_{11}, y_{22} = 1/u_{22}, y_{33} = 1/u_{33}$

From (6), $y_{12} = -y_{11}u_{12}/u_{22}, y13 = -(y_{11}u_{13} + y_{12}u_{23})/u_{33}; y_{23} = -y_{22}u_{23}/u_{33}$.

∴ We get $U^{-1} = Y$, completely.

Hence, by (2), we obtain A^{-1}.

Example:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

Solution:

taking $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

A=LU $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

∴ $1 = u_{11},\ 0 = u_{12},\ 2 = u_{13};$

$l_{21}u_{11} = 2,\ l_{21}u_{12} + u_{22} = -1,\ l_{21}u_{13} + u_{23} = 3\,;$

$l_{31}u_{11} = 4, \quad l_{31}u_{12} + l_{32}u_{22} = 1, \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 8\,;$

0r

$$1 = u_{11}, 0 = u_{12}, 2 = u_{13}, l_{21} = 2, l_{31} = 4, l_{32} = -1, u_{22} = -1, u_{23} = -1, u_{33} = -1$$

Thus $U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \quad and \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$

$$A = LU\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 4 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & -1 & -1\\ 0 & 0 & -1\end{bmatrix}$$

To find L^{-1}, let $L^{-1} = X$. Then LX = I

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 4 & -1 & 1\end{bmatrix}\begin{bmatrix}x_{11} & 0 & 0\\ x_{21} & x_{22} & 0\\ x_{31} & x_{32} & x_{33}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$x_{11} = 1$, $2x_{11}+x_{21} = 0$, $x_{22} = 1$, $4x_{11} - x_{21} + x_{31} = 0$, $-x_{22} + x_{32} = 0$,

$x_{33} = 1$

Then $x_{11} = x_{22} = x_{33} = x_{32} = 1$, $x_{21} = -2$, $x_{31} = -6$

Thus $L^{-1} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -6 & 1 & 1\end{bmatrix}$

To find U^{-1}, let $U^{-1} = Y$. Then YU = I

$$\begin{bmatrix}y_{11} & y_{12} & y_{13}\\ 0 & y_{22} & y_{23}\\ 0 & 0 & y_{33}\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & -1 & -1\\ 0 & 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$y_{11} = 1$, $y_{12} = 0$, $2y_{11} - y_{12} - y_{13} = 0$, $y_{22} = -1$, $-y_{22} - y_{23} = 0$, $y_{33} = -1$

Then $y_{11} = y_{23} = 1$, $y_{22} = y_{33} = -1$, $y_{12} = 0$, $y_{13} = 2$

Thus $U^{-1} = \begin{bmatrix}1 & 0 & 2\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}$

Hence, $A^{-1} = U^{-1}L^{-1} = \begin{bmatrix}1 & 0 & 2\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -6 & 1 & 1\end{bmatrix}$

$$= \begin{bmatrix}-11 & 2 & 2\\ -4 & 0 & 1\\ 6 & -1 & -1\end{bmatrix}$$

An Easy Way to Find *LU* Factorizations (Poole, 2010)

We computed the matrix L as a product of elementary matrices. Fortunately, L can be computed directly from the row reduction process without our needing to compute elementary matrices at all. Remember that we are assuming that A can be reduced to row echelon form without using any row interchanges. If this is the case, then the entire row reduction process can be done using only elementary row operations of the form $R_r - kR_f$ (Why do we not need to use the remaining elementary row operation, multiplying a row by a nonzero scalar?) In the operation $R_1 - kR_p$ we will refer to the scalar k as the multiplier.

In Example, the elementary row operations that were used were, in order,

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow[R_1+R_1]{R_1-2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow{R_1+2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$\begin{array}{ll} R_2 - 2R_1 & (\text{ multiplier } = 2) \\ R_3 + R_1 = R_3 - (-1)R_1 & (\text{ multiplier } = -1) \\ R_3 + 2R_2 = R_3 - (-2)R_2 & (\text{ multiplier } = -2) \end{array}$$

The multipliers are precisely the entries of L that are below its diagonal! Indeed,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

and $L_{21} = 2, L_{31} = -1$, and $L_{32} = -2$. Notice that the efementary row operation $R_t - kR_j$ has its multiplier k placed in the (i, j) entry of L.

Example: find inverse matrix by factorization method

$$\text{A=} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

Solution:

taking $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

A=LU $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \quad (\mathbf{R}_2 - 2\mathbf{R_1}, \mathbf{R_3} - 4\mathbf{R_1}) \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\mathbf{R}_3 + \mathbf{R_2}) \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \quad and \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$

$A = LU \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

To find L^{-1}, let $L^{-1} = X$. Then LX = I

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_{11} = 1$, $2x_{11}+x_{21} = 0$, $x_{22} = 1$, $4x_{11} - x_{21} + x_{31} = 0$, $-x_{22} + x_{32} = 0$,
$x_{33} = 1$

Then $x_{11} = x_{22} = x_{33} = x_{32} = 1$, $x_{21} = -2$, $x_{31} = -6$

Thus $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -6 & 1 & 1 \end{bmatrix}$

To find U^{-1}, let $U^{-1} = Y$. Then YU = I

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$y_{11} = 1$, $y_{12} = 0$, $2y_{11} - y_{12} - y_{13} = 0$, $y_{22} = -1$, $-y_{22} - y_{23} = 0$, $y_{33} = -1$

Then $y_{11} = y_{23} = 1,\ y_{22} = y_{33} = -1,\ y_{12} = 0,\ y_{13} = 2$

Thus $U^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Hence, $A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -6 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

2.1.3 Partition Method (B.S.GREWAL, 2018)

According to this method, if the inverse of a matrix A_n of order n is known, then the inverse of a matrix A_{n+1} of order $(n+1)$ ean be determined by adding $(n+1)$ the row and $(n+1)$ th column to A_n.

Suppose $A = \begin{bmatrix} A_1 & : & A_2 \\ \dots & \dots & \dots \\ A_3' & : & \alpha \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \dots & \dots & \dots \\ X_3' & : & x \end{bmatrix}$

where A_2, X_2 are column vectors and A_3', X_3' are row vectors (i.e., transposes of column vectors A_3, X_3) and α, x are ordinary numbers.

Also we assume that A_1^{-1} is known. Actually A_3 and X_3 are column vectors since their transposes are row vectors.

Now $AA^{-1} = I_{n+1}$ gives

$$A_1X_1 + A_1X_3' = I_n$$
$$A_1X_2 + A_2x = 0$$
$$A_3'X_1 + \alpha X_3' = 0$$
$$A_3'X_3' + \alpha x = 1$$

From (2), $X_2 = -A_1^{-1}A_2x$ and using this, (4) gives $(\alpha - A_3'A_1^{-1}A_2)x = 1$.

Hence x and then X_2 an be fourd.

Also from (1), $X_1 = A_1^{-1}(I_n - A_2X_3')$

and using this, (3) gives $(\alpha - A_3' A_1^{-1} A_2) X_3' = -A_3' A_1^{-1}$ whence X_3' and then X_1 are determined.

Thus, having found X_1, X_2, X_3' and x, A^{-1} is completely known.

Or $x = (\alpha - A_3' A_1^{-1} A_2)^{-1}$

$$X_2 = -A_1^{-1} A_2 x$$

$$X_3' = -A_3' A_1^{-1} x$$

$$X_1 = A_1^{-1}(I_n - A_2 X_3')$$

NOTE: The partition method is also known as the "Escalator method".

Example: Find inverse matrix by partition method

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} A_1 & \vdots & A_2 \\ \cdots & \cdots & \cdots \\ A_3' & \vdots & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 2 & \vdots & 3 \\ 0 & 1 & & 4 \\ \cdots & \cdots & & \cdots \\ 5 & 6 & \vdots & 1 \end{bmatrix}$$

So that $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $A_3' = [5 \quad 6]$ $x = 1$

We find $A_1^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Let $A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \cdots & \cdots & \cdots \\ X_3' & : & x \end{bmatrix}$. Then $AA^{-1} = I$

Hence

$$x = (\alpha - A_3' A_1^{-1} A_2)^{-1} = \left(1 - [5 \quad 6] \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)^{-1} = \frac{1}{2}$$

$$X_2 = -A_1^{-1} A_2 x = -\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \frac{1}{2} = \begin{bmatrix} 2.5 \\ -2 \end{bmatrix}$$

$$X_3' = -A_3' A_1^{-1} x = -[5 \quad 6] \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \frac{1}{2} = [-2.5 \quad 2]$$

$$X_1 = A_1^{-1}(I_n - A_2X_3') = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}3\\4\end{bmatrix}[-2.5 \quad 2]\right) = \begin{bmatrix}-11.5 & 8\\10 & -7\end{bmatrix}$$

Then

$$A^{-1} = \begin{bmatrix}X_1 & : & X_2\\ \cdots & \cdots & \cdots\\ X_3' & : & x\end{bmatrix} = \left[\begin{array}{cc:c}-11.5 & 8 & 2.5\\ 10 & -7 & -2\\ \hdashline -2.5 & 2 & 0.5\end{array}\right] = \begin{bmatrix}-11.5 & 8 & 2.5\\10 & -7 & -2\\-2.5 & 2 & 0.5\end{bmatrix}$$

2.2 Numerical Methods for finding the inverse matrix

2.2.1 Iterative Method (B.S.GREWAL, 2018)

Suppose we wish to compute A^{-1} and we know that B is an approximate inverse of A. Then the error matrix is given by $E = AB - I$ (by definition of error) or

$$\begin{aligned} AB &= I + E\\ \therefore \quad (AB)^{-1} &= (I+E)^{-1} \text{ i.e. } B^{-1}A^{-1} = (I+E)^{-1}\\ A^{-1} &= B(I+E)^{-1}\\ &= B(I - E + E^2 - \cdots) \quad (\text{ by Taylor series })\end{aligned}$$

provided the series converges.

Thus we can find further approximations of A^{-1}, by using $A^{-1} = B(1 - E + E^2 - \cdots)$

Example: Using iterative method, find the inverse of the matrix

$$A = \begin{bmatrix}5 & 2\\3 & -1\end{bmatrix} \text{ taking } B = \begin{bmatrix}0.1 & 0.2\\0.3 & -0.4\end{bmatrix}$$

Solution:

$$E = AB - I = \begin{bmatrix}5 & 2\\3 & -1\end{bmatrix}\begin{bmatrix}0.1 & 0.2\\0.3 & -0.4\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1.1 & 0.2\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0.1 & 0.2\\0 & 0\end{bmatrix}$$

then $\quad E^2 = \begin{bmatrix} 0.01 & 0.02 \\ 0 & 0 \end{bmatrix}$

$\therefore \quad E^3 = \begin{bmatrix} 0.001 & 0.002 \\ 0 & 0 \end{bmatrix}$

To the second approximation, we have

$$A^{-1} = B(I - E + E^2 - E^3)$$

$$= B - BE + BE^2 - BE^3$$

$$= \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{bmatrix}\begin{bmatrix} 0.1 & 0.2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{bmatrix}\begin{bmatrix} 0.01 & 0.02 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{bmatrix}\begin{bmatrix} 0.001 & 0.002 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{bmatrix} - \begin{bmatrix} 0.01 & 0.02 \\ 0.03 & 0.06 \end{bmatrix} + \begin{bmatrix} 0.001 & 0.002 \\ 0.003 & 0.006 \end{bmatrix} - \begin{bmatrix} 0.0001 & 0.0002 \\ 0.0003 & 0.0006 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0909 & 0.1818 \\ 0.2727 & -0.4546 \end{bmatrix}$$

2.3 Numerical eigenvalues:

2.3.1 The Power Method: (Poole, 2010)

The power method applies to an $n \times n$ matrix that has a dominant eigenvalue $\lambda_1 -$ that is, an eigenvalue that is larger in absolute value than all of the other eigenvalues. For example, if a matrix has eigenvalues $-4, -3, 1$, and 3, then -4 is the dominant eigenvalue, since $4 = |-4| > |-3| \geq |3| \geq |1|$. On the other hand, a matrix with eigenvalues $-4, -3, 3$, and 4 has no dominant eigenvalue.

The power method proceeds iteratively to produce a sequence of scalars that converges to λ_1 and a sequence of vectors that converges to the corresponding eigenvector $\mathbf{v}_1$, the dominant eigenvector. For simplicity, we will assume that the matrix A is diagonalizable. The following theorem is the basis for the power method.

Theorem (2.3.1): Let A be an $n \times n$ diagonalizable matrix with dominant eigenvalue λ_1. Then there exists a nonzero vector $\mathbf{x}_0$ such that the sequence of vectors $\mathbf{x}_k$ defined by

$$\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1, \mathbf{x}_3 = A\mathbf{x}_2, \dots, \mathbf{x}_k = A\mathbf{x}_{\mathrm{k}-1}, \dots$$

approaches a dominant eigenvector of A.

Proof: We may assume that the eigenvalues of A have been labeled so that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be the corresponding eigenvectors. Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent (why?), they form a basis for $\mathbb{R}^n$. Consequently, we can write $\mathbf{x}_0$ as a linear combination of these eigenvectors - say,

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Now $\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0, \mathbf{x}_3 = A\mathbf{x}_2 = A(A^2\mathbf{x}_0) = A^3\mathbf{x}_0$, and, generally,

$$\mathbf{x}_k = A^k\mathbf{x}_0 \text{ for } k \geq 1$$

$$\begin{aligned} A^k\mathbf{x}_0 \quad &= c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n \\ &= \lambda_1^k\left(c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{v}_n\right) \end{aligned}$$

where we have used the fact that $\lambda_1 \neq 0$.

The fact that λ_1 is the dominant eigenvalue means that each of the fractions $\lambda_2/\lambda_1, \lambda_3/\lambda_1, \dots, \lambda_n/\lambda_1$, is less than 1 in absolute value. Thus,

$$\left(\frac{\lambda_2}{\lambda_1}\right)^k, \left(\frac{\lambda_3}{\lambda_1}\right)^k, \dots, \left(\frac{\lambda_n}{\lambda_1}\right)^k$$

all go to zero as $k \to \infty$. It follows that

$$\mathbf{x}_k = A^k \mathbf{x}_0 \to \lambda_1^k c_1 \mathbf{v}_1 \ \text{ as } k \to \infty$$

Now, since $\lambda_1 \neq 0$ and $\mathbf{v}_1 \neq 0, \mathbf{x_k}$ is approaching a nonzero multiple of $\mathbf{v}_1$ (that is, an eigenvector corresponding to λ_1) provided $c_1 \neq 0$. (This is the required condition on the initial vector $\mathbf{x}_0$: It must have a nonzero component c_1 in the direction of the dominant eigenvector $\mathbf{v}_1$.

- There is an alternative way to estimate the dominant eigenvalue λ_1 of a matrix A in conjunction with the power method. First, observe that if $Ax = \lambda_1 \mathrm{x}$, then

$$\frac{(A\mathrm{x}) \cdot \mathrm{x}}{\mathrm{x} \cdot \mathrm{x}} = \frac{(\lambda_1 \mathrm{x}) \cdot \mathrm{x}}{\mathrm{x} \cdot \mathrm{x}} = \frac{\lambda_1 (\mathrm{x} \cdot \mathrm{x})}{\mathrm{x} \cdot \mathrm{x}} = \lambda_1$$

The expression $R(\mathrm{x}) = ((A\mathrm{x}) \cdot \mathrm{x})/(\mathrm{x} \cdot \mathrm{x})$ is called a **Rayleigh quotient**.

Example: Complete four iterations of the power method to approximate a dominant eigenvector of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Use theorem (2.2.2) to get the dominant eigenvalue.

Solution: We will take $\mathrm{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the initial vector. Then

Approximate

$$X_1 = AX_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} \to -4 \begin{bmatrix} 2.50 \\ 1.00 \end{bmatrix}$$

$$X_2 = AX_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix} \to 10 \begin{bmatrix} 2.8 \\ 1.00 \end{bmatrix}$$

$$X_3 = AX_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 28 \\ 10 \end{bmatrix} = \begin{bmatrix} -64 \\ -22 \end{bmatrix} \rightarrow -22 \begin{bmatrix} 2 \cdot 91 \\ 1.00 \end{bmatrix}$$

$$X_4 = AX_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -64 \\ -22 \end{bmatrix} = \begin{bmatrix} 136 \\ 46 \end{bmatrix} \rightarrow 46 \begin{bmatrix} 2 \cdot 96 \\ 1.00 \end{bmatrix}$$

So after the fourth iteration, the approximate a dominant eigenvector is $v = \begin{pmatrix} 2.96 \\ 1.00 \end{pmatrix}$

Rayleigh quotient, $\lambda = \frac{Ax \cdot x}{x \cdot x}$

$$Ax = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 2.96 \\ 1.00 \end{pmatrix} = \begin{pmatrix} -6.08 \\ -2.04 \end{pmatrix}$$

$$\begin{aligned} Ax \cdot x &= \begin{pmatrix} -6.08 \\ -2.04 \end{pmatrix} \cdot \begin{pmatrix} 2.96 \\ 1.00 \end{pmatrix} \\ &= -20.0368 \\ x \cdot x &= \begin{pmatrix} 2.96 \\ 1.00 \end{pmatrix} \cdot \begin{pmatrix} 2.96 \\ 1.00 \end{pmatrix} = 9.7616 \end{aligned}$$

$$\lambda = \frac{-20.0368}{9.7616} = -2.05$$

Note that exact solution is
dominant eigenvalue = -2
dominant eigenvector = $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

References

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