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Numerical solution of Ordinary Differential Equations

Research Project

Submitted to the department of (Mathematic) in partial fulfillment of the requirements for the degree of BSc. in (Mathematic)

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Certification of the supervisor

I certify that this work was prepared under my supervision at the Department of Mathematics/ College of Education /Salaheddin University-Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics

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Abstract

This report provides an overview of numerical methods for solving ordinary differential equations (ODEs), including the Picard method, Taylor series methods, Euler's method, modified Euler's method, and Runge-Kutta methods. Our results show that Runge-Kutta methods generally provide the most accurate and efficient solutions, while the Picard method and Taylor series methods are useful for understanding the behavior of solutions near specific points. Finally, some examples are shown for declaring these methods.

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Introduction

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realize that computing machines are now readily available which reduce numerical work considerably.

Solution of a differential equation. The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the closed or finite form of solution. In the absence of such a solution, we have recourse to numerical methods of solution.

Let us consider the first order differential equation

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0 \qquad (1)$$

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solutions. In these methods, y in (1) is approximated by a truncated series, each term of which is a function of x. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as **single-step methods**.

The methods of Euler, Runge-Kutta, etc. belong to the latter class of solutions. In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations until sufficient accuracy is achieved. As such, these methods are called **step-by-step methods**.

Chapter One

Definitions and Background

Definition 1.1 (Stewart 2015)

A **series**, in mathematics, refers to the sum of the terms of a sequence. It is typically represented using the sigma notation (Σ), where the terms of the sequence are written below the sigma symbol, and the index indicating the starting value of the sequence is written above the sigma symbol.

Example: $\sum_{x=1}^{4} 3x$.

Definition 1.2 (Stewart 2015)

A **differential equation** is an equation that involves an unknown function and its derivatives. In other words, it is an equation that relates a function to its rate of change.

Definition 1.3 (Zill 2013)

A **second-order** differential equation is an equation of the form

$$y''(x) = f(x, y(x), y'(x))$$

where $y(x)$ is an unknown function, $y'(x)$ and $y''(x)$ are its first and second derivatives with respect to x, and $f(x, y(x), y'(x))$ is a given function.

Definition 1.4 (Stewart 2015)

A **power series** is a series of the form $\sum_{n=0}^{\infty} an(x-c)^{\mathrm{n}}$, where an are constants, x is a variable, and c is a fixed point called the center of the series. A power series may converge for some values of x and diverge for others, depending on the values of the coefficients an and the value of $(x-c)$.

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

Definition 1.5 (Apostol 1973)

A **Taylor series** is a type of power series that approximates a function near a fixed point (usually denoted by "a") by a polynomial. The coefficients of the polynomial are given by the function's derivatives evaluated at the point a.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f_0 + (x-x_0)\acute{f}_0 + \frac{(x-x_0)^2}{2!} f_0'' + \cdots$$

Chapter Two

Numerical methods for solving ordinary differential equation

2.1 PICARD'S METHO (Grewal 2018)

consider the first order equation $\frac{d_y}{d_x} = f(x, y)$ ------(1)

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get.

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y) dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y) dx \quad (2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign. As a first approximation $y_1$to the solution, we put $y = y_0$ in f (x, y) and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0) dx$$

For a second approximation y_2, we put $y = y_1$ In $f\ (x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1) dx$$

Similarly, a third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2) dx$$

Continuing this process, we obtain $y_4, y_5, \ldots, y_n$ where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}) dx$$

Hence this method gives a sequence of approximations $y_1, y_2, y_3, \ldots$

Each giving a better result than the preceding one.

Example 2.1.1: Solve $\frac{dy}{dx} = x + y^2$ up to third approximation by Picard's method.
Given, $y = 0$ when $x = 0$.
Solution: Given, $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{x} + \mathrm{y}^2$ … (1)

$y = 0;$ i. e. $y = 0; x = 0$; i. e. $\mathrm{x}_0 = 0$

"Integrating equ. " (1)" w.r.t. 'x' between the limits 0 and x.

$$\therefore \int_0^y dy = \int_0^x (x + y^2) dx$$

$$[y]_0^y = \int_0^x (x + y^2) dx$$

$$\therefore \; y = \int_0^x (x + y^2) dx \; \ldots (2)$$

1st Approximation
Put $y = y_0 = 0$ in $(x + y^2)$ of equ. (2), we get

$$y_1 = \int_0^x (x + 0^2) dx = \left[\frac{x^2}{2}\right]_0^x$$

$$\therefore y_1 = \frac{x^2}{2}$$

2nd Approximation

Put $y = y_1 = \frac{x^2}{2}$ in $(x + y^2)$ of equ. (2), we get

$$y_2 = \int_0^x \left(x + \left(\frac{x^2}{2}\right)^2\right) dx = \int_0^x \left(x + \frac{x^4}{4}\right) dx = \left[\frac{x^2}{2} + \frac{1}{4}\frac{x^5}{5}\right]_0^x$$

$$\therefore\ y_2 = \frac{x^2}{2} + \frac{x^5}{20}$$

3rd Approximation

Put $y = y_2$ in $(x + y^2)$ of equ $^n(2)$, we get

$$y_3 = \int_0^x \left[x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2\right] dx$$

$$= \int_0^x \left[x + \frac{x^4}{4} + \frac{2x^2}{2} \cdot \frac{x^5}{20} + \frac{x^{10}}{400}\right] dx = \left[\frac{x^2}{2} + \frac{1}{4}\frac{x^5}{5} + \frac{1}{20}\frac{x^8}{8} + \frac{1}{400}\frac{x^{11}}{11}\right]_0^x$$

2.2 Taylor's Series Methods (Sastry 2012)

Consider the first order operation $\frac{dy}{dx} = f_{x,y}$ (1)

With the initial condition y_{x_0} =y_0

Successive differentiation of equation (1), gives:

$$\frac{d^2y}{dx^2} = \frac{df}{dx} + \frac{df}{dy} * \frac{dy}{dx}$$

$$\frac{d^3y}{dx^3} = \frac{d^2f}{dx^2} + 2 * \frac{d^2f}{dxdy} * \frac{df}{dy} + \frac{d^2f}{dy^2} * \left(\frac{df}{dy}\right)^2 + \frac{df}{dy}\left(\frac{df}{dy}\right)^2$$

$$\frac{d^4f}{dx^4} = \frac{d^3f}{dx^3} + 3 * \left(\frac{d^3f}{dx^2dy} + \frac{d^3f}{dxdy^2} * \frac{df}{dy}\right) * \frac{df}{dy} + \frac{d^3f}{dy^3} * \left(\frac{df}{dy}\right)^3 + 3 * \left(\frac{d^2f}{dxdy} + \frac{d^2f}{dy^2} * \frac{df}{dy}\right) * \left(\frac{df}{dy}\right)^2 + \frac{df}{dy}\left(\frac{df}{dy}\right)^3$$

The values $y_0^i, y_0^{ii}, y_0^{iii}, y_0^{iv}, \ldots$ can be obtained Taylor's series

Y= $y_0 + (x - x_0)y^i + \frac{(x-x_0)^2}{2!}y_0^{ii} + \frac{(x-x_0)^3}{3!}y^{iii} + \frac{(x-x_0)^4}{4!} + \cdots$

Example 2.2.1: Find he Taylor's series method, the values of y at x=0.2 to five places of decimals from $\frac{dy}{dx} = x^2y - 1, y(0) = 1.$

Solution: Differentiating successively, we get

$y' = x^2y - 1$ $\quad\quad (y')_0 = (0)^2(y) - 1 = -1$

$y'' = 2xy + x^2y'$ $\quad\quad (y'')_0 = 0$

$y''' = 2y + 4xy' + x2y''$ $\quad\quad (y''')_0 = 2$

$(y'''') = 6y' + 6xy'' + x2y'''$ $\quad\quad (y'''')_0 = -6$

Putting these values in the Taylor's series, we have

$$y = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) = 1 + x + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hence y (0. 2) =0.90033 and y (0. 21) =0.80227

2.3 Euler's Method (Boyer n.d.)**,** (Grewal 2018), (Sastry 2012)

Leonhard Euler, born April 15, 1707, Basel, Switzerland—died September 18, 1783, St. Petersburg, Russia), Swiss mathematician and physicist, one of the founders of pure mathematics. He not only made decisive and formative contributions to the subjects of geometry, calculus

Figure 1 Leonhard Euler

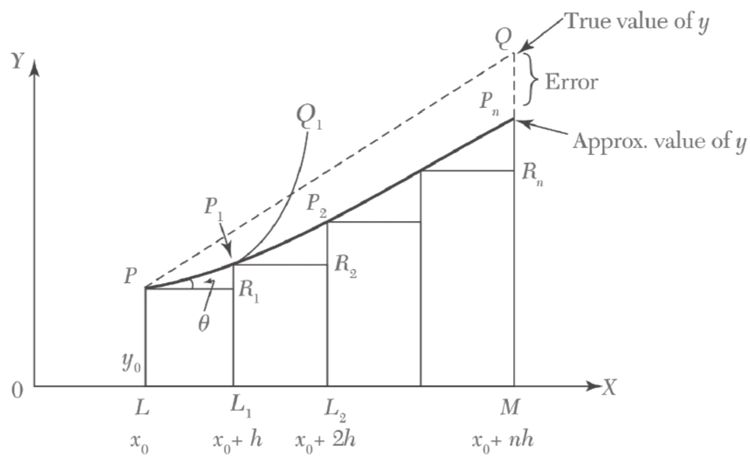

Figure 2 Euler method

Let us divide LM into n subintervals each of width h at $L_1, L_2, \ldots$ so that h is quite small in the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$ Then

$$y_1 = L_1P_2 = LP + R_1P_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_p = y_0 + hf(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in P_2 $(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

$$y_n = y_{n-1} + hf(x_0 + (n-1)h, y_{n-1})$$

This is Euler's method of finding an approximate solution of (1).

Example 2.3.1: Using Euler's method, find an approximate value of x corresponding to $x = 1$,given $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$

Solution:

We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{x + y = \left(\frac{dx}{dy}\right)}$	$\boldsymbol{Old\ y + 0.1\left(\frac{dy}{dx}\right) = new\ y}$
0.0	1.00	$(0)\ +\ (1.00)$	$New\ y\ = 1.00 + 0.1(1.00)$ $= 1.10$
0.1	1.10	$(0.10)\ +\ (1.10)$	$1.10 + 0.1(1.20)\ = 1.22$
0.2	1.22	1.42	$1.22 + 0.1(1.42)\ = 1.36$
0.3	1.36	1.66	$1.36 + 0.1(1.66)\ = 1.53$
0.4	1.53	1.93	$1.53 + 0.1(1.93)\ = 1.73$
0.5	1.73	2.22	$1.73 + 0.1(2.22)\ = 1.94$
0.6	1.94	2.54	$1.94 + 0.1(2.53)\ = 2.19$
0.7	2.19	2.89	$2.19 + 0.1(2.89)\ = 2.48$
0.8	2.48	3.29	$2.48 + 0.1(3.29)\ = 2.81$
0.9	2.81	3.71	$2.81 + 0.1(3.71)\ = 3.18$
$\mathbf{1.0}$	$\mathbf{3.18}$		

2.4 Modified Euler's Method (Sastry 2012)

Instead of approximating $f(x, y)$ by $f(x_0, y_0)$ in

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx. \quad (1)$$

we now approximate the integral given in Eq. (1) by means of trapezoidal rule to obtain

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \quad (2)$$

We thus obtain the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right],\ n = 0,1,2, \dots \ (3)$$

where $y_1^{(n)}$ is the nth approximation to y_1. The iteration formula (3) can be started by choosing $y_1^{(0)}$ from Euler's formula:

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

Example 2.4.1: Determine the value of y when $x = 0.1$ given that

$$y(0) = 1 \text{ and } y' = x^2 + y$$

We take $h = 0.05$. With $x_0 = 0$ and $y_0 = 1.0$, we have $f(x_0, y_0) = 1.0$. Hence Euler's formula gives

$$y_1^{(0)} = 1 + 0.05(1) = 1.05$$

Further, $x_1 = 0.05$ and $f\left(x_1, y_1^{(0)}\right) = 1.0525$. The average of $f(x_0, y_0)$ and $f\left(x_1, y_1^{(0)}\right)$ is 1.0262. The value of $y_1^{(1)}$ can therefore be computed by using Eq. (8.14) and we obtain

$$y_1^{(1)} = 1.0513.$$

Repeating the procedure, we obtain $y_1^{(2)} = 1.0513$. Hence, we take $y_1 = 1.0513$, which is correct to four decimal places.

Next, with $x_1 = 0.05, y_1 = 1.0513$ and $h = 0.05$, we continue the procedure to obtain y_2, i.e., the value of y when $x = 0.1$. The results are

$$y_2^{(0)} = 1.1040,\ y_2^{(1)} = 1.1055,\ y_2^{(2)} = 1.1055.$$

Hence, we conclude that the value of y when $x = 0.1$ is 1.1055.

2.5 RUNGE-KUTTA METHODS (Sastry 2012)

As already mentioned, Euler's methods is less efficient in practical problems since it requires h to be small for obtaining reasonable accuracy. The Runge-kutta methods are designed to give greater accuracy and possess the advantage of requiring only the function values at some selected points on the subinterval.

If we substitute

$$y_1 = y_0 + hf(x_0, y_0)$$

on the right side of the Eq. $y_1 = y_0 + \frac{h}{2}[f\left(x_{0,}y_0\right) + f(x_1, y_1)$ we obtain

$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h, y_0 + hf_0)]$,Where $f_0 = f(x_0, y_0)$.

if we now set $k_1 = hf_0 \quad and \quad k_2 = hf(x_0 + h, y_o + k_1)$

Then the above equation becomes $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$,

Which the *second-order Runge-kutta* formula the error in the formula can be showed to be of order h^3 by expanding both sides by Taylor's series.

Thus, the left side gives $y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{3}y_0''' + \cdots$

And on the right side

$k_2 = hf\ (x_0 + h, y_o + hf_0) = h\left[f_0 + h\frac{\partial f}{\partial x_0} + hf_0\frac{\partial f}{\partial y_0} + O(h^2)\right]$.

Since

$$\frac{df(x, y)}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y},$$

We obtain

$k_2 = h[f_0 + hf_0' + O(h^2)] = hf_0 + h^2\ f_0'\ + O(h^3) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + O(h^3)$.

Example 2.5.1: Apply the Runge-Kutta method to find the approximate value of y for $x = 0.2$, in step of 0.1, if $\frac{dy}{dx} = x + y^2$, $y = 1$ where $x = 0$

Solution:

Given $f(x, y) = x + y^2$ here we take $h = 0.1$ and carry out the calculations in two Steps General form second order Runge-Kutta $y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$ where

$k_1 = hf(x_i, y_i), k_2 = hf(x_i + h, y_i + k_1 h)$

First step:

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

$k_1 = hf_0 = hf(x_0, y_0) = (0.1)(0 + (0.1)^2) = 0.1$

$k_2 = hf_0 = hf(x_0 + h, y_0 + hk_1) = (0.1)\left(0 + 0.1 + \left(1 + (0.1)(0.1)\right)^2\right) = 0.11201$

So $y_1 = 1 + \frac{1}{2}(0.1 + 0.11201\,) = 1.106005$

Second step:

$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

$k_1 = hf_1 = hf(x_1, y_1) = (0.1)(0.1 + (1.106005)^2) = 0.132325$

$k_2 = hf_1 = hf(x_1 + h, y_1 + hk_1) = (0.1)\left(0.1 + 0.1 + \left(1.106005 + (0.132325)(0.1)\right)^2\right) = 0.145269$

So $y_2 = 1.106005 + \frac{1}{2}(0.132325 + 0.145269) = 1.244802$

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پوختە

ئەم ڕاپۆرتە تێڕوانینێکی گشتی لە شێوازە ژمارەییەکان بۆ چارەسەرکردنی هاوکێشە جیاوازە ئاساییەکان (ODEs)دەخاتە ڕوو، لەوانە شێوازی پیکارد، شێوازەکانی زنجیرەی تایلۆر، شێوازی ئۆیلەر، شێوازی ئۆیلەری دەستکاریکراو وشێوازەکانی ڕونگ-کوتا .ئەنجامەکانمان دەریدەخەن کە شێوازەکانی ڕونگ-کوتا بەگشتی وردترین وکاراترین چارەسەرەکان دابین دەکەن، لەکاتێکدا شێوازی پیکارد وشێوازەکانی زنجیرەی تایلۆر بەسوودن بۆ تێگەیشتن لە هەڵسوکەوتی چارەسەرەکانی نزیک خاڵە تایبەتەکان .
لە کۆتاییدا هەندێک نموونە بۆ ڕاگەیاندنی ئەم شێوازانە نیشان دەدرێن.