Lectures on Partial Differential Equations

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Chapter 1

1.1 INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. In the present part of the book, we propose to study various methods to solve partial differential equations.

1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E.)

Definition. An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation.

For examples of partial differential equations, we list the following:

$$\partial z/\partial x + \partial z/\partial y = z + xy \qquad \ldots(1)$$

$$(\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x\left(\frac{\partial z}{\partial x}\right) \qquad \ldots(2)$$

$$z(\partial z/\partial x) + \partial z/\partial y = x \qquad \ldots(3)$$

$$\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz \qquad \ldots(4)$$

$$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2} \qquad \ldots(5)$$

$$y\{(\partial z/\partial x)^2 + (\hat{\partial} z/\partial y)^2\} = z\left(\frac{\partial z}{\partial y}\right) \qquad \ldots(6)$$

Definition. The **order of a partial differential equation** is defined as the order of the highest partial derivative occuring in the partial differential equation.

Equations $(1), (3), (4)$ and (6) are of the first order, (5) is of the second order and (2) is of the third order.

Definition. The **degree of a partial differential equation** is the degree of the highest order derivative which occurs in it after the equation has been rationalised, i.e., made free from radicals and fractions so far as derivatives are concerned.

Equations $(1), (2), (3)$ and (4) are of first degree while equations (5) and (6) are of second degree.

1.3 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be **linear** if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a **non-linear** partial differential equation.

equations (1) and (4) are linear while equations $(2), (3), (5)$ and (6) are nonlinear.

1.4 NOTATIONS

When we consider the case of two independent variables, we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$$p = \partial z/\partial x,\ q = \partial z/\partial y,\ r = \partial^2 z/\partial x^2,\ s = \partial^2 z/\partial x\,\partial y \text{ and } t = \partial^2 z/\partial y^2$$

1.5 CLASSIFICATION OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Linear equation. A first order equation $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$$

For examples, $yx^2p + xy^2q = xyz + x^2y^3$ and $p + q = z + xy$ are both first order linear partial differential equations.

Semi-linear equation. A first order partial differential equation $f(x, y, z, p, q) = 0$ is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

For **examples**, $xyp + x^2yq = x^2y^2z^2$ and $yp + xq = (x^2z^2/y^2)$ are both first order semi-linear partial differential equations.

Quasi-linear equation. A first order partial differential equation $f(x, y, z, p, q) = 0$ is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

For **examples**, $x^2zp + y^2zp = xy$ and $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ are first order quasi-linear partial differential equations.

Non-linear equation. A first order partial differential equation $f(x, y, z, p, q) = 0$ which does not come under the above three types, in known as a non-liner equation. For **examples**, $p^2 + q^2 = 1$, $pq = z$ and $x^2 p^2 + y^2q^2 = z^2$ are all non-linear partial differential equations.

1.6 Origin of partial differential equations. We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

Rule I. **Derivation of a partial differential equation by the elimination of arbitrary constants.**

Consider an equation $F(x, y, z, a, b) = 0 \quad \dots (1)$
where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$$\partial F/\partial x + p(\partial F/\partial z) = 0 \quad \text{and} \quad \partial F/\partial y + q(\partial F/\partial z) = 0 \quad \dots (2)$$

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0, \dots (3)$$

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems: For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$$F(x, y, z, a, b) = 0,\ \ \partial F/\partial x = 0 \text{ and } \partial F/\partial y = 0$$

The equation free from a and b will be the required partial differential equation. Three situations may arise:

<u>Situation I.</u> When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.

For **example (Raisin)**, consider $z = ax + y \quad \ldots (1)$

where a is the only arbitrary constant and x, y are two independent variables.
Differentiating (1) partially w.r.t. ' x ', we get $\partial z/\partial x = a \quad \ldots (2)$
Differentiating (1) partially w.r.t. ' y ', we get $\partial z/\partial y = 1 \quad \ldots (3)$
Eliminating a between (1) and (2) yields

$$z = x(\partial z/\partial x) + y \quad \ldots (4)$$

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Example:

Situation II. When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.

Example (Raisin): Eliminate a and b from $az + b = a^2 x + y \quad ...(1)$

Differentiating (1) partially w.r.t ' x ' and ' y ', we have

$a(\partial z/\partial x) = a^2 \quad ...(2)$ and $a(\partial z/\partial \mathrm{y}) = 1 \quad ...(3)$

Eliminating a from (2) and (3), we have $(\partial z/\partial x)(\partial z/\partial y) = 1$

which is the unique partial differential equation of order one.

Example:

<u>Situation III.</u> When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.

Example (Raisin): Eliminate a, b and c from $z = ax + by + cxy \quad \dots (1)$
Differentiating (1) partially w.r.t., ' x ' and ' y, we have

$\frac{\partial z}{\partial x} = a + cy \; \dots (2)$ and $\frac{\partial z}{\partial y} = b + cx \; \dots (3)$
From (2) and (3), $\frac{\partial^2 z}{\partial x^2} = 0, \; \frac{\partial^2 z}{\partial x^2} = 0 \quad \dots (4)$ and $\frac{\partial^2 z}{\partial x \, \partial y} = c \; \dots (5)$
Now, (2) and (3) $\Rightarrow$ $x(\partial z / \partial x) = ax + cxy$ and $y(\partial z / \partial y) = by + cxy$

$\therefore \; x(\partial z / \partial x) + y(\partial z / \partial y) = ax + by + cxy + cxy$

then "using (1) and (5) " $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = z + xy\left(\frac{\partial^2 z}{\partial x \, \partial y}\right) \; \dots (6)$

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

Exercise: IAN

1- Ex $x^2 + y^2 + (z - c)^2 = a^2$
2- Ex $x^2 + y^2 = (z - c)^2 \tan^2 a$
3- $(x - a)^2 + (y - b)^2 + z^2 = 1$
4- $z = (x + a)(y + b)$
5- $2z = (ax + y)^2 + b$
6- $ax^2 + by^2 + z^2 = 1$

Quiz one:

Rule II. Derivation of partial differential equation by the elimination of arbitrary function $\boldsymbol{\phi}$ from the equation $\boldsymbol{\phi(u,v)=0}$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are functions of $\boldsymbol{x},\boldsymbol{y}$ and $\mathbf{z}$.

Proof. Given $\phi(u,v)=0.\ \ldots(1)$
We treat z as dependent variable and x and y as independent variables so that
$\partial z/\partial x=p,\ \partial z/\partial y=q,\ \partial y/\partial x=0$ and $\partial x/\partial y=0$
Differentiating (1) partially with respect to x, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

or

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

$\frac{\partial\phi}{\partial u}/\frac{\partial\phi}{\partial v}=-\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)/\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)$ ...(2)

Similarly, differentiating (1) partially w.r.t. 'y', we get

$\frac{\partial\phi}{\partial u}/\frac{\partial\phi}{\partial v}=-\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)/\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)$ ...(3)

Eliminating ϕ with the help of (2) and (3), we get

$$\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)/\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)=\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)/\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)$$
$$\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)$$
$$Pp+Qq=R$$

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y},\ Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z},\ R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Thus we obtain a linear partial differential equation of first order and of first degree in p and q.

Note. If the given equation between x,y,z contains two arbitrary functions, then in general, their elimination gives rise to equations of higher order.

Example: Form a partial differential equation by eliminating the arbitrary functions

$$z = f(x^2 + y^2)$$

Exercise: IAN

1- $z = xy + f(x^2 + y^2)$

2- $z = x + y + f(xy)$

3- $z = f(\frac{xy}{z})$

4- $z = f(x - y)$

5- $z = f(x^2 + y^2 + z^2, z^2 - 2xy)$

The Origin of Second order Equations:

Suppose that the function z is given by an expression of the type $z = f(u) + g(v) + w$ ...(1) where f and g are arbitrary functions of u and v, respectively, and u, v and w are prescribed functions of x and y.

We find, on differentiating both sides of (1) w.r.t. x and y, respectively, that

$$p = f'(u) \cdot u_x + g'(v) \cdot v_x + w_x \to P - w_x = f' \cdot u_x + g' \cdot v_x$$

$$q = f'(u)u_y + g'(v) \cdot v_y + w_y \to q - w_y = f' \cdot u_y + g' \cdot v_y$$

and hence that

$$\begin{aligned}
&r = f'(u) \cdot u_{xx} + f''(u) \cdot u_x^2 + g'(v) \cdot v_{xx} + g''(v) \cdot v_x^2 + w_{xx} \\
&\longrightarrow r - w_{xx} = f'(u) \cdot u_x + g'(v) \cdot v_{xx} + f''(u) \cdot u_x^2 + g'(v) \cdot v_x^2 \\
&S = f'(u) \cdot u_{xy} + f''(u) \cdot u_x u_y + g'(v) \cdot v_{xy} + g''(v) \cdot v_x \cdot v_y + w_{xy} \\
&\longrightarrow S - w_{xy} = f'(u)u_{xy} + g(v)v_{xy} + f''(u) \cdot u_x u_y + g''(v) \cdot v_x v_y. \\
&t = f'(u) \cdot u_{yy} + f''(u) \cdot u_y^2 + g'(v) \cdot v_{yy} + g''(v) \cdot v_y^2 + w_{yy} \\
&\longrightarrow t - w_{yy} = f'(u) \cdot u_{yy} + g(v) \cdot v_{yy} + f''(u) \cdot u_y^2 + g''(v) \cdot v_y^2.
\end{aligned}$$

We now have five equations involving the four arbitrary quantities f', f'', g', g''.

If we eliminate these four quantities from the five equations, we obtain the relation:

$$\begin{vmatrix}
p - w_x & u_x & v_x & 0 & 0 \\
q - w_y & u_y & v_y & 0 & 0 \\
r - w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\
s - w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\
s - t_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2
\end{vmatrix} = 0 \quad ...(2)$$

which involves only the derivatives p, q, r, s, t and unknown functions of x and y.

It is therefore a P.D.E. of the second order.

Example : Form a partial differential equation by eliminating the arbitrary functions f and g from the relation $z = f(x + y) + g(x - y)$ where a is a constant.

Ans. $P = f' + g' \longrightarrow r = f'' + g''$

$q = f' - g' \to t = f'' + g''$

$\longrightarrow t = r$ is the P.D.E. of second order.

Example IAN: Form a partial differential equation by eliminating the arbitrary functions

1- $z = f(x + ay) + g(x - ay)$

Exercise: (IAN)

1- Verify that the partial differential equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x^2}$ is satisfied by $z = \frac{1}{x}\phi(y - x) + \phi'(y - x)$ where ϕ is an arbitrary function.

2- If $u = f(x + iy) + g(x - iy)$, where the functions f and g are arbitrary, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

3- Show that if f and g are arbitrary functions of a single variable, then

$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ provided that $\alpha^2 = 1 - v^2/c^2$.

4- If $z = f(x^2 - y) + g(x^2 + y)$ where the functions f, g are arbitrary, prove that $\frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 4x^2\frac{\partial^2 z}{\partial y^2}$

5- A variable z is defined in terms of variables x, y as the result of eliminating t from the equations

$$z = \mathrm{t}x + yf(t) + g(t)$$
$$0 = x + yf'(t) + g'(t)$$

Prove that, whatever the functions f and g may be, the equation $rt - s^2 = 0$ is satisfied.

Quiz two:

Chapter 2

Partial differential equations of order one

2.1 LAGRANGE'S EQUATION

A quasi–linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z. Such a partial differential equation is known as *Lagrange equation.*

For example $xyp + yzq = zx$ is a Lagrange equation.

Theorem. The general solution of Lagrange equation $Pp + Qq = R$ ... (1) is $\phi(u, v) = 0$ ... (2) where ϕ is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \ldots (3)$$

Here, c_1 and c_2 are arbitrary constants and at least one of u, v must contain z.

Proof. Differentiating (2) partially w.r.t. ' x ' and ' y ', we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \quad \ldots (4)$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \quad \ldots (5)$$

Eliminating $\partial\phi/\partial u$ and $\partial\phi/\partial v$ between (5) and (6), we have

$$\begin{vmatrix} \partial u/\partial x + p(\partial u/\partial z) & \partial v/\partial x + p(\partial v/\partial z) \\ \partial u/\partial y + q(\partial u/\partial z) & \partial v/\partial y + q(\partial v/\partial z) \end{vmatrix} = 0$$

or $\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$

or $\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0$

$$\therefore \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \quad \ldots (6)$$

Hence (2) is a solution of the equation (6)

Taking the differentials of $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$, we get

$$\left(\frac{\partial u}{\partial x}\right)dx+\left(\frac{\partial u}{\partial y}\right)dy+\left(\frac{\partial u}{\partial z}\right)dz=0 \quad \dots(7) \text{ or } u_x dx+u_y dy+u_z dz=0$$

and $$\left(\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial v}{\partial y}\right)dy+\left(\frac{\partial v}{\partial z}\right)dz=0 \quad \dots(8) \text{ or } v_x dx+v_y dy+v_z dz=0$$

Since u and v are independent functions, solving (7) and (8) for the ratios $dx{:}\,dy{:}\,dz$, gives

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \quad \dots(9)$$

Comparing (3) and (9), we obtain

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P}=\frac{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q}=\frac{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}{R}=k, \text{ say}$$

$$\Rightarrow \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}=kP,\ \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=kQ \text{ and } \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=kR$$

Substituting these values in (6), we get $k(Pp+Qq)=kR$ or $Pp+Qq=R$, which is the given equation (1).

Therefore, if $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ are two independent solutions of the system of differential equations $(dx)/P=(dy)/Q=(dz)/R$, then $\phi(u,v)=0$ is a solution of $Pp+Qq=R$, ϕ being an arbitrary function. This is what we wished to prove.

2.2 Methods for solving Lagrange's auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ... (1)

2.2.1 Type 1

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of given equations (1).

Example:

1-

2.2.2 Type 2

Suppose that one integral of (1) is known by using rule I explained in section 2.2.1 and suppose also that another integral cannot be obtained by using rule I of section. 2.5. Then one integral known to us is used to find another integral. Note that in the second integral, the constant of integration of first integral should be removed later on.

Example:

2.2.3 Type 3

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well-known principle of algebra, each fraction in (1) will be equal to

$$\frac{(P_1dx + Q_1dy + R_1dz)}{(P_1P + Q_1Q + R_1R)} \quad \dots (2)$$

If $P_1P + Q_1Q + R_1R = 0$, then we know that the numerator of (2) is also zero. This gives $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible by use of multipliers. In such cases second integral should be obtained by using rule I of section. $2.2.1$ or rule II of section. $2.2.2$ as the case may be.

Example:

1-

2.2.4 Type 4

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well-known principle of algebra, each fraction of (1) will be equal to

$$\frac{(P_1 dx + Q_1 dy + R_1 dz)}{(P_1 P + Q_1 Q + R_1 R)} \quad \dots (2)$$

Suppose the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$\frac{(P_2 dx + Q_2 dy + R_2 dz)}{(P_2 P + Q_2 Q + R_2 R)} \quad \dots (3)$$

is such that its numerator is exact differential of denominator. Fractions (2) and (3) are then combined to given an integral. This method may be repeated in some problems to get another integral. Sometimes only one integral is possible by using the above rule IV. In such cases second integral should be obtained by using rule 1 of section. 2.2.1 or rule 2 of section. 2.2.2 or rule 3 of section. 2.2.3.

Example:

Exercise: (P55)

1- $z(xp - yq) = y^2 - x^2$ Ans. $dx = dy\ ;\ x, y, z\ F(xy, x^2 + y^2 + z^2) = 0$

2- $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

3- $px(x + y) = qy(x + y) - (x - y)(2x + 2y + z)$

$dx = dy: dx + dy = dx + dy + dz$ Ans. $F(xy, (x + y)(x + y + z)) = 0$

4- $y^2p - xyq = x(z - 2y)$ Ans. $F(x^2 + y^2, yz - y^2) = 0$

5- $(y + xz)p - (x + yz)q = x^2 - y^2$ Ans. $y, x, 1: x, y, -z\ F(xy + z, x^2 + y^2 - z^2) = 0$

6- $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$

Quiz three:

2.3 Integral surfaces passing through a given curve.

We shall now present two methods of using such a general solution for getting the integral surface which passes through a given curve.

2.3.1 Method I.

Let $Pp + Qq = R$ ...(1) be the given equation. Let its auxiliary equations give the following two independent solutions $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. ...(2)
Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by $x = x(t), y = (t), z = z(t)$ ...(3)
where t is a parameter.
Then (2) may be expressed as:
$u[x(t), y(t), z(t)] = c_1$ and $v[x(t), y(t), z(t)] = c_2$. ...(4)
We eliminate single parameter t from the equations of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

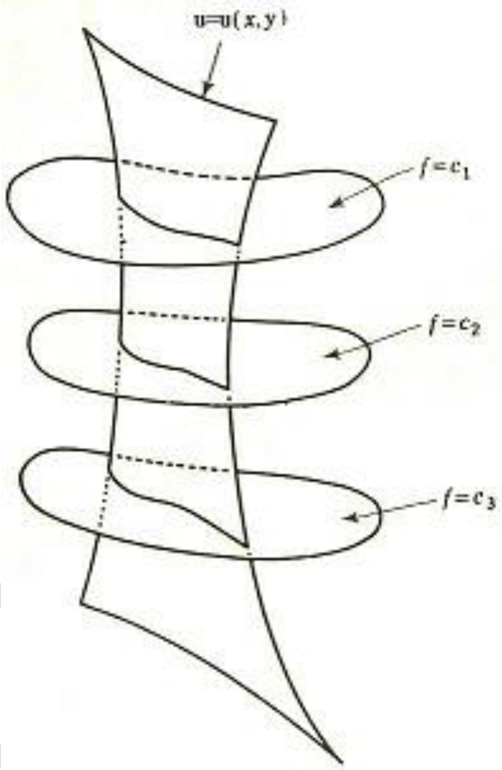
2.3.2 Method II.

Let $Pp + Qq = R$ ...(1) be the given equation. Let is Lagrange's auxiliary equations give the following two independent integrals $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. ...(2)
Suppose we wish to obtain the integral surface passing though the curve which is determined by the following two equations $G(x, y, z) = 0$ and $H(x, y, z) = 0$. ...(3)
We eliminate x, y, z from four equations of (2) and (3) and obtain a relation between c_1 and c_2. Finally, replace c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and obtain the desired integral surface.

Example:

1. (IAN ex) Find the integral surface of the linear PDE $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ *which contains the straight line* $x + y = 0, z = 1$.
   Ans. $xyz = c_1$; $x^2 + y^2 - 2z = c_2$, $x = t$, $2xyz + x^2 + y^2 - 2z + 2 = 0$.

1.4 SURFACES ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES

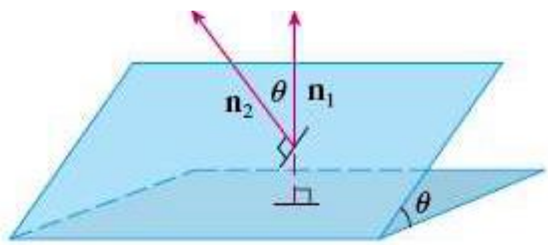


Let $f(x, y, z) = C$ ...(1) represents a system of surfaces where C is parameter.

We want to find a collection of surfaces which cut each of these given surfaces (1) at right angles.

Let the surface $F(x, y, z) = z(x, y) - z = 0$ ...(2) cuts each surface of (1) at right angles.

At a point of intersection (x, y, z) , observe that $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is the normal to the surface (1). Similarly, $\nabla F = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ is the normal to the surface (2).



Since both the surfaces intersect orthogonally, at point of intersection (x, y, z) their respective normals are perpendicular.

$$\nabla f . \nabla F = f_x z_x + f_y z_y + f_z(-1) = 0$$

$$f_x p + f_y q = f_z \quad ...(3)$$

Therefore, integral surface of quasi-linear P.D.E. (3) is orthogonal to the given surface (1).

Conversely, we easily verify that any solution of (3) is orthogonal to every surface of (1).

Example:

1- (IAN ex) Find the surface which intersects the surfaces of the system $z(x + y) = c(3z + 1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$.

Exercise: (IAN57,P59)

1- Find the equation of the integral surface of the differential equation $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z = 0, x^2 + y^2 = 2x$.
Ans. $1,2y,-2; dx = dz: x + y^2 - 2z = c_1; x^2 - 3x - z^2 + 6z = c_2, x = t \rightarrow c_1 + c_2 = 0.$
2- Find the general integral of the P.D.E. $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ and also the particular integral which passes through the line $x = 1, y = 0$.
3- Find the integral surface of the equation $(x-y)y^2p + (y-x)x^2q = z(x^2+y^2)$ through the curve $xz = a^3, y = 0$.
Ans. $dx = dy; \frac{1}{x-y}, \frac{-1}{x-y}, \frac{-1}{z}: x^3 + y^3 = c_1; \frac{x-y}{z} = c_2, x = t \rightarrow (x^3+y^3)^2 = a^9 \frac{(x-y)^3}{z^3}.$

4- Find the general solution of the equation $2x(y+z^2)p + y(2y+z^2)q = z^3$ and deduce that $yz(z^2+yz-2y) = x^2$ is a solution.
Ans. $\frac{1}{x}dx - \frac{1}{y}dy = dz; dy = dz: y^2 + z^2 = f(x^2 - y^2).$

5- Find the general integral of the equation $(x-y)p + (y-x-z)q = z$ and the particular solution through the circle $z = 1, x^2 + y^2 = 1$.
6- Find the general solution of the differential equation

$$x(z+2a)p + (xz + 2yz + 2ay)q = z(z-a)$$

Find also the integral surfaces which pass through the curves:
(a) $y = 0,\ z^2 = 4ax$
(b) $y = 0,\ z^3 + x(z+a)^2 = 0$

7- Find the surface which is orthogonal to the one-parameter system $z = cxy\,(x^2+y^2)$ and which passes through the hyperbola $x^2 - y^2 = a^2, z = 0$.

Ans. $xdx + ydy = -zdz$

8- Find the equation of the system of surfaces which cut orthogonally the cones of the system $x^2 + y^2 + z^2 = cxy$.

Ans. $x, y, z; xdx - ydy = dz$

9- Find the general equation of surfaces orthogonal to the family given by:

a) $x(x^2+y^2+z^2) = c_1y^2$ Ans. $4xdx + 2ydy = dz$
showing that one such orthogonal set consists of the family of spheres given by
b) $x^2 + y^2 + z^2 = c_2z$
If a family exists, orthogonal to both (a) and (b), show that it must satisfy

$$2x(x^2-z^2)dx + y(3x^2+y^2-z^2)dy + 2z(2x^2+y^2)dz = 0$$

Show that such a family in fact exists, and find its equation.

10- Show that the integral surface of $(x^2+y^2-a^2)(xp+yq) = z(x^2+y^2)$ are generated by conics, and find the integral surface through the curve $x = 2z, x^2 + y^2 = 4a^2$.
Ans. $\frac{x^2+y^2-a^2}{z^2} = c_1; \frac{y}{x} = c_2; z = t \rightarrow 3z^2(x^2+y^2) = x^2(x^2+y^2-a^2)$

Quiz four:

Chapter 3

Nonlinear Partial Differential Equations of the First Order

3.1 Charpit's method

Let the given partial differential equation of first order and non-linear in p and q be

$$f(x, y, z, p, q) = 0 \;\; \dots (1).$$

We know that $dz = pdx + qdy \dots (2)$.

The next step consists in finding another relation $F(x, y, z, p, q) = 0 \dots (3)$ such that when the values of p and q obtained by solving (1) and (3), are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (2), differentiate partially (1) and (3) with respect to x and y and get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \;\; \dots (4)$$
$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \;\; \dots (5)$$
$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \;\; \dots (6)$$
$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0 \;\; \dots (7)$$

Eliminating $\partial p / \partial x$ from (4) and (5), we get

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p} = 0$$

or $$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \;\; \dots (8)$$

Similarly, eliminating $\partial q / \partial y$ from (6) and (7), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right) q + \left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \;\; \dots (9)$$

Since $\partial q / \partial x = \partial^2 z / \partial x \, \partial y = \partial p / \partial y$, the last term in (8) is the same as that in (9), except for a minus sign and hence they cancel on adding (8) and (9).

Therefore, adding (8) and (9) and rearranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0 \quad \dots(10)$$

This is a linear equation of the first order to obtain the desired function F. As in chapter 2, integral of (10) is obtained by solving the auxiliary equations

$$\frac{dp}{(\partial f/\partial x)+p(\partial f/\partial z)}=\frac{dq}{(\partial f/\partial y)+q(\partial f/\partial z)}=\frac{dz}{-p(\partial f/\partial p)-q(\partial f/\partial q)}=\frac{dx}{-\partial f/\partial p}=\frac{dy}{-\partial f/\partial q}=\frac{dF}{0} \quad \dots(11).$$

Since any of the integrals of (11) will satisfy (10), an integral of (11) which involves p or q (or both) will serve along with the given equation to find p and q. In practice, however, we shall select the simplest integral.

Note. In what follows we shall use the following standard notations:

$$\partial f/\partial x=f_x,\ \partial f/\partial y=f_y,\ \partial f/\partial z=f_z,\ \partial f/\partial p=f_p,\ \partial f/\partial q=f_q.$$

Therefore, Charpit's auxiliary equations (11) may be re-written as

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dF}{0} \quad \dots(11')$$

WORKING RULE WHILE USING CHARPIT'S METHOD

Step 1. Transfer all terms of the given equation to L.H.S. and denote the entire expression by f.

Step 2. Write down the Charpit's auxiliary equations (11) or (11)'.

Step 3. Using the value of f in step 1 write down the values of $\partial f/\partial x, \partial f l \partial y \dots$, i.e., $f_x, f_y, \dots$ etc. occuring in step 2 and put these in Charpit's equations (11) or (11)'.

Step 4. After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5. The simplest relation of step 4 is solved along with the given equation to determine p and q. Put these values of p and q in $dz=pdx+qdy$ which on integration gives the complete integral of the given equation.

The Singular and General integrals may be obtained in the usual manner.

Remark. Sometimes Charpit's equations give rise to $p = a$ and $q = b$, where a and b are constants. In such cases, putting $p = a$ and $q = b$ in the given equation will give the required complete integral.

Special methods of solutions applicable to certain standard forms:

We have already discussed the general method (*i.e.*, Charpit's method). We now discuss four standard forms to which many equations can be reduced, and for which a complete integral can be obtained by inspection or by other shorter methods.

Standard Form I. Only p and q present (Equations involving only p and q):

Under this standard form, we consider equations of the form $f(p, q) = 0 \ldots (1)$.
Charpit's auxiliary equations are $\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$
giving $\frac{dp}{0} = \frac{dq}{0}$, by (1)
Taking the first ratio, $dp = 0$ so that $p =$ constant $= a$, say ...(2)
Taking the second ratio, $dq = 0$ so that q =constant $=$ b, say ...(3)
substituting (2) and (3) in (1), we get $f(a, q) = 0 \;\ldots (4)$
Then, $dz = pdx + qdy = adx + bdy$, using (2) and (3).
Integrating, $z = ax + by + c \;\ldots (5)$
where c is an arbitrary constant. (5) together with (4) give the required solution.
Now solving (4) for b, suppose we obtain $b = F(a)$.
Putting this value of b in (5), the complete integral of (1) is

$$z = ax + yF(a) + c$$

Examples: Find a complete integral of

1- (IAN ex71)$pq = 1$.

Standard form II. Clairaut equation:

A first order partial differential equation is said to be of Clairaut form if it can be written in the form $z = px + qy + f(p, q)$. ...(1).

Let $F(x, y, z, p, q) \equiv px + qy + f(p, q) - z$ ...(2)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or $\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px-qy-p(\partial f/\partial p)-q(\partial f/\partial q)} = \frac{dx}{-x-(\partial f/\partial p)} = \frac{dy}{-y-(\partial f/\partial q)}$, by (1)

Then, first and second fractions $\Rightarrow dp = 0$ and $dq = 0 \Rightarrow p = a$ and $q = b$.

Substituting these values in (1), the complete integral is $z = ax + by + f(a, b)$

Remark 1. Observe that the complete integral of (1) is obtained by merely replacing p and q by a and b respectively. Singular and general integrals can be obtained by usual methods.

Remark 2. Sometimes change of variables can be employed to transform a given equation to standard form II.

Examples: Find a complete integral of

1- (IAN ex73)$(z - px - qy)(p + q) = 1$.
2- (IAN61)$z = px + qy + p + q - pq$

Standard form III. Only p, q and z present (Not involving the independent variable):

Under this standard form we consider differential equation of the form

$$f(p,q,z)=0 \quad \dots(1)$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $\frac{dp}{p(\partial f/\partial z)}=\frac{dq}{q(\partial f/\partial z)}=\frac{dz}{-p(\partial f/\partial p)-q(\partial f/\partial q)}=\frac{dx}{-\partial f/\partial p}=\frac{dy}{-\partial f/\partial q}$, using (1)

Taking the first two ratios,

$(1/p)dp=(1/q)dq$

Integrating, $q=ap, a$ being an arbitrary constant. $\dots(2)$

Now, $dz=pdx+qdy=pdx+apdy$, using (2)

or $dz=p(dx+ady)=pd(x+ay)=pdu, \quad \dots(3)$

Where $u=x+ay \quad \dots(4)$

Now, (3) → $p=dz/du$ and so by (2) $q=ap=a(\frac{dz}{du})$

Substituting these values of p and q in (1), we get $f\left(\frac{dz}{du},a\frac{dz}{du},z\right)=0 \quad \dots(5)$

which is an ordinary differential equation of first order. Solving (5), we get z as a function of u. Complete integral is then obtained by replacing u by $(x+ay)$.

Examples: Find a complete integral of

1- (IAN ex72)$p^2z^2+q^2=1$.
2- (IAN60)$z^2(p^2+q^2+1)=1$.
3- (IAN61)$z=\frac{1}{p}+\frac{1}{q}$.
4- (IAN66)$z=pq$.
5- (IAN66)$(1+q^2)z=px$.
6- (IAN66)$z=p^2-q^2$.
7- (IAN66)$p^2+q^2=4z$.

Standard form IV. Equation of the form $\boldsymbol{f_1(x,p) = f_2(y,q)}$ (Separable Equ.):

i.e., a form in which z does not appear and the terms containing x and p are on one side and those containing y and q on the other side.

Let $F(x,y,z,p,q) = f_1(x,p) - f_2(y,q) = 0$ ... (1)
Then Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

Or by (1) $\frac{dp}{\partial f_1/\partial x} = \frac{dq}{-\partial f_2/\partial y} = \frac{dz}{-p(\partial f_1/\partial p)+q(\partial f_2/\partial q)} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_2/\partial q}$

Taking the first and the fourth ratios, we have
$(\partial f_1/\partial p)dp + (\partial f_1/\partial x)dx = 0$ or $df_1 = 0$
Integrating, $f_1 = a, a$ being an arbitrary constant.

$\therefore$ (1) $\Rightarrow$ $f_1(x,p) = f_2(y,q) = a$ ... (2)

Now, (2) $\Rightarrow$ $f_1(x,p) = a$ and $f_2(y,q) = a$ ... (3)
From (3), on solving for p and q respectively, we get
$p = F_1(x,a)$, and $q = F_2(y,a)$ ... (4)

Substituting these values in $dz = pdx + qdy$, we get $dz = F_1(x,a)dx + F_2(y,a)dy$
Integrating,

$$z = \int F_1(x,a)\,dx + \int F_2(y,a)\;dy + b$$

which is a complete integral containing two arbitrary constants a and b.

Examples: Find a complete integral of

1- (IAN ex72)$p^2y(1+x^2) = qx^2$.

SOLVING EXAMPLES USING CHARPIT'S GENERAL FORMULA:

Examples: Find a complete integral of

1- (IAN ex70)$z = p^2x + q^2y$.
2- (IAN ex65)$z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$.

Exercise: (IAN70)

1- $(p^2 + q^2)y = qz$.
2- $p = (z + qy)^2$.
3- $z^2 = pqxy$.
4- $xp + 3yq = 2(z - x^2q^2)$.
5- $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$.
6- $2(y + zq) = q(xp + yq)$.
7- $2(z + xp + yq) = yp^2$.

Exercise: (IAN73)

1- $p + q = pq$.
2- $z = p^2 - q^2$.
3- $zpq = p + q$.
4- $p^2q(x^2 + y^2) = p^2 + q$.
5- $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$.
6- $pqz = p^2(xq + p^2) + q^2(yp + q^2)$