AX=λX

AX-λX=0

| A-λI |X=0

Eigen Values & Eigen Vectors

Let A be an $n \times n$ matrix, a real or complex number λ is called an Eigen value of A if $\det(A - \lambda I) = 0$ and with $[A - \lambda I]X = 0$ then X is called an Eigen vector with respect to λ where I is the identity matrix.

Example

Find the Eigen values and Eigen vectors of A if

(a) $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, (c) $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Solution

(a) To find Eigen values, we have $\det(A - \lambda I) = 0$ then

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda) + 2 = 0 \quad \Rightarrow \quad -2 - \lambda + 2\lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow \quad \lambda^2 + \lambda = 0 \quad \Rightarrow \quad \lambda(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda_1 = 0,\ \lambda_2 = -1$$

To find the Eigen vectors, we have $[A-\lambda I]X=0$

For $\lambda_1=0 \;\Rightarrow\; \begin{bmatrix}1 & 2\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$

$x_1+2x_2=0 \;\Rightarrow\; x_1=-2x_2$

Let $x_2=1 \;\Rightarrow\; x_1=-2 \Rightarrow$ The Eigen vector for $\lambda_1=0$ is $\begin{bmatrix}-2\\ 1\end{bmatrix}$

For $\lambda_2=-1 \;\Rightarrow\; \begin{bmatrix}2 & 2\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$

$2x_1+2x_2=0 \;\Rightarrow\; 2x_1=-2x_2 \;\Rightarrow\; x_1=-x_2$

Let $x_2=1 \;\Rightarrow\; x_1=-1 \Rightarrow$ The Eigen vector for $\lambda_2=-1$ is $\begin{bmatrix}-1\\ 1\end{bmatrix}$

(b) To find Eigen values, we have $\det(A-\lambda I)=0$ then

$$\begin{vmatrix}1-\lambda & 3\\ 2 & 1-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)^2-6=0 \;\Rightarrow\; 1-2\lambda+\lambda^2-6=0$$

$$\Rightarrow \lambda^2-2\lambda-5=0$$

$$\Rightarrow \lambda_1=1+\sqrt{6},\ \lambda_2=1-\sqrt{6}$$

To find the Eigen vectors, we have $[A-\lambda I]X=0$

For $\lambda_1=1+\sqrt{6} \;\Rightarrow\; \begin{bmatrix}-\sqrt{6} & 3\\ 2 & -\sqrt{6}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$

$-\sqrt{6}x_1+3x_2=0 \;\Rightarrow\; -\sqrt{6}x_1=-3x_2 \;\Rightarrow\; x_1=\dfrac{3}{\sqrt{6}}x_2$

Let $x_2 = 1 \Rightarrow x_1 = \frac{3}{\sqrt{6}} \Rightarrow$ The Eigen vector for $\lambda_1 = 1 + \sqrt{6}$ is $\begin{bmatrix} \frac{3}{\sqrt{6}} \\ 1 \end{bmatrix}$

For $\lambda_2 = 1 - \sqrt{6} \Rightarrow \begin{bmatrix} \sqrt{6} & 3 \\ 2 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\sqrt{6}x_1 + 3x_2 = 0 \Rightarrow \sqrt{6}x_1 = -3x_2 \Rightarrow x_1 = \frac{-3}{\sqrt{6}} x_2$$

Let $x_2 = 1 \Rightarrow x_1 = \frac{-3}{\sqrt{6}} \Rightarrow$ The Eigen vector for $\lambda_2 = 1 - \sqrt{6}$ is $\begin{bmatrix} \frac{-3}{\sqrt{6}} \\ 1 \end{bmatrix}$

(c) To find Eigen values, we have $\det(A - \lambda I) = 0$ then

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(1-\lambda)^2(1+\lambda) = 0$$

$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = 1, \text{ and } \lambda_3 = -1$$

To find the Eigen vectors, we have $[A - \lambda I]X = 0$

For $\lambda_{1,2} = 1 \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$-x_2 = 0,\ x_3 = 0,\ -2x_3 = 0$$

The Eigen vector for $\lambda_{1,2} = 1$ is $\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$

Each choice of α gives us an Eigen vector associated with $\lambda = 1$

For $\lambda_3 = -1 \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$2x_1 - x_2 = 0 \Rightarrow 2x_1 = x_2$$

$$2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3$$

Let $x_1 = 1 \Rightarrow x_2 = 2 \Rightarrow x_3 = -4$

The Eigen vector for $\lambda_3 = -1$ is $\begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

Exercises

Find the Eigen values and Eigen vectors of the following matrices

1) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ **Ans.** $1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}; -3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **Ans.** 0***, any non-zero vector***

3) $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ **Ans.** $5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}; -5, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ ***Ans.*** $1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}; -1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

5) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ ***Ans.*** $4, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; 8, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; 6, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

6) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ***Ans.*** $0, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; -2, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

7) $\begin{bmatrix} -2 & 0 \\ 0 & 0.4 \end{bmatrix}$ ***Ans.*** $-2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}; 0.4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

8) $\begin{bmatrix} 4 & 0 \\ 2 & -4 \end{bmatrix}$ ***Ans.*** $4, \begin{bmatrix} 4 \\ 1 \end{bmatrix}; -4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

9) $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ ***Ans.*** $-4, \begin{bmatrix} 2 \\ 9 \end{bmatrix}; 3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

10) $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ ***Ans.*** $0.8 + j0.6, \begin{bmatrix} 1 \\ -j \end{bmatrix}; 0.8 - j0.6, \begin{bmatrix} 1 \\ j \end{bmatrix}$

11) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ***Ans.*** $5, \begin{bmatrix} 1 \\ 2 \end{bmatrix}; 0, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

12) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ ***Ans.*** $4, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; 0, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; -1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

13) $\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$ ***Ans.*** $3, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}; 6, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}; 9, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

14) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 6 & 4 & 2 \end{bmatrix}$ ***Ans.*** $1, \begin{bmatrix} -3 \\ 2 \\ 10 \end{bmatrix}; 4, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}; 2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

15) $\begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}$ ***Ans.*** $-3, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}; 5, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

16) $\begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$ ***Ans.*** $9, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

17) $\begin{bmatrix} 0 & -2 & 2 & 0 \\ -4 & 2 & -2 & 4 \\ 0 & 2 & 2 & -4 \\ 0 & 2 & -6 & 4 \end{bmatrix}$ ***Ans.*** $4, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}; -4, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}; 0, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; 8, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -3 \end{bmatrix}$

18) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 4 & 2 & -6 \end{bmatrix}$ ***Ans.*** $2, \begin{bmatrix} 8 \\ 8 \\ -16 \\ 1 \end{bmatrix}; 1, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 4 \end{bmatrix}; 3, \begin{bmatrix} 0 \\ 0 \\ 9 \\ 2 \end{bmatrix}; -6, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

19) $\begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$ ***Ans.*** $-1, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; -5, \begin{bmatrix} -3 \\ -3 \\ 1 \\ 1 \end{bmatrix}; 3, \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \end{bmatrix}$

Gauss Elimination Method

The system of linear equations is denoted by

$$AX = B$$

where A is a matrix

X and B are vectors

This system is solved by using the Gauss elimination method as shown bellow.

Example

The system

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ 4x_1 + 6x_2 &= 5 \end{aligned}$$

can be written as

$$\begin{bmatrix} 1 & -2 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

So, by using the Gauss elimination method as follows:

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 4 & 6 & 5 \end{array}\right]$$

$$4R_1 - R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & -14 & 7 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \Rightarrow \begin{aligned} x_1 - 2x_2 &= 3 \\ -14x_2 &= 7 \end{aligned}$$

$$\Rightarrow \ -14x_2 = 7 \ \Rightarrow \ x_2 = -\frac{1}{2} \ \Rightarrow \ x_1 = 2\left(-\frac{1}{2}\right) + 3 = 2$$

The solution is $\begin{bmatrix} 2 \\ -1/2 \end{bmatrix}$

Example

Solve the system

$$x + y - 3z = 2$$
$$2x - y + z = 1$$
$$x + 2y - z = 0$$

Solution

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \rightarrow \\ -R_1 + R_3 \rightarrow \end{array}\left[\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 0 & -3 & 7 & -3 \\ 0 & 1 & 2 & -2 \end{array}\right] \Rightarrow \begin{array}{r} \\ \\ R_2 + 3R_3 \rightarrow \end{array}\left[\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 0 & -3 & 7 & -3 \\ 0 & 0 & 13 & -9 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -3 & 7 \\ 0 & 0 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -9 \end{bmatrix} \Rightarrow \begin{array}{r} x + y - 3z = 2 \\ -3y + 7z = -3 \\ 13z = -9 \end{array}$$

$$13z = -9 \Rightarrow z = -\frac{9}{13}$$

$$3y = 3 + 7\left(-\frac{9}{13}\right) \Rightarrow y = \frac{1}{3}\left(3 - \frac{63}{13}\right) = \frac{13 - 21}{13} = \frac{-8}{13}$$

$$x = 2 + 3z - y = 2 + 3\left(-\frac{9}{13}\right) + \frac{8}{13} = \frac{26 - 27 + 8}{13} = \frac{7}{13}$$

Matrix Inverse

We can use the Gauss elimination method to find the inverse of a matrix. We know that a matrix A and its inverse A^{-1} must satisfy the equation $A \times A^{-1} = I$ (identity matrix). The following example illustrates the procedure.

Example

Find A^{-1} if

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \Rightarrow \frac{1}{2}R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

$$R_2 - R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 7/2 & -1/2 & 1 \end{array}\right] \Rightarrow \frac{2}{7}R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1/7 & 2/7 \end{array}\right]$$

$$R_1 - \frac{1}{2}R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 4/7 & -1/7 \\ 0 & 1 & -1/7 & 2/7 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 4/7 & -1/7 \\ -1/7 & 2/7 \end{bmatrix}$$

Note:

The matrix A has an inverse if $|A| \neq 0$

Example

Find A^{-1} if

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \quad \frac{1}{2}R_1 \to \left[\begin{array}{ccc|ccc} 1 & -1/2 & 3/2 & 1/2 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_2 - R_1 \to \\ R_3 - 4R_1 \to \end{array} \left[\begin{array}{ccc|ccc} 1 & -1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & -7/2 & -1/2 & 1 & 0 \\ 0 & 2 & -4 & -2 & 0 & 1 \end{array}\right]$$

$$2R_2 \to \left[\begin{array}{ccc|ccc} 1 & -1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 2 & -4 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_1 + \frac{1}{2}R_2 \to \\ \\ R_3 - 2R_2 \to \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 0 & 10 & 0 & -4 & 1 \end{array}\right]$$

$$\frac{1}{10}R_3 \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -2/5 & 1/10 \end{array}\right]$$

$$\begin{array}{l} R_1 + 2R_3 \to \\ R_2 + 7R_3 \to \\ \ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/5 & 1/5 \\ 0 & 1 & 0 & -1 & -4/5 & 7/10 \\ 0 & 0 & 1 & 0 & -2/5 & 1/10 \end{array}\right]$$

$$\Rightarrow \ A^{-1} = \begin{bmatrix} 0 & 1/5 & 1/5 \\ -1 & -4/5 & 7/10 \\ 0 & -2/5 & 1/10 \end{bmatrix}$$

Exercises

Solve the Following Systems of Linear Equations using Gauss Elimination

1) $3x + y = -5$
$2x + 3y = 6$

2) $x - 2y = -8$
$5x + 3y = -1$

3) $5x - 2y = 1$
$6x + 8y = 22$

4) $2x + 3y = 4$
$3x + 2y = -4$

5) $3x + 2y = -17$
$10x + y = 0$

6) $-x + 2y = 4$
$3x + 4y = 38$

7) $7x - y - 2z = 0$
$9x - y - 3z = 0$
$2x + 4y - 7z = 0$

8) $3x - y + z = -2$
$x + 5y + 2z = 6$
$2x + 3y + z = 0$

9) $x + 2y - 8z = 0$
$2x - 3y + 5z = 0$
$3x + 2y - 12z = 0$

10) $5x + 3y - 3z = -1$
$3x + 2y - 2z = -1$
$2x - y + 2z = 8$

11) $9x + 4y + 3z = -1$
$5x + y + 2z = 1$
$7x + 3y + 4z = 1$

12) $2y - z = -1$
$x + 3z = 11$
$2x - 4y + 2z = 6$

13) $4y - 2z = 2$
$6x - 2y + z = 29$
$4x + 8y - 4z = 24$

14) $14x - 2y - 4z = 0$
$18x - 2y - 6z = 0$
$4x + 8y - 14z = 0$

15) $y + z = -2$
$4y + 6z = -12$
$x + y + z = 2$

16) $2x + y - 3z = 8$
$5x + 2z = 3$
$8x - y + 7z = 0$

17) $4y + 4z = 24$
$3x - 11y - 2z = -6$
$6x - 17y + z = 18$

18) $-x + 3y - 2z = 7$
$3x + 3z = -3$
$2x + y + 2z$

19) $4x - y = 1$
$-2x + 5y = 1$

20) $3x - 2y = 9$
$-x + 6y = -3$

21) $2x + 5y + 3z = 1$
$-x + 2y + z = 2$
$x + y + z = 0$

22) $3x + y - z = 3$
$2x + 2y - 3z = 1$
$-x + y - 2z = -2$

Rank

The rank r of a matrix A is the highest order of the matrix with $|A| \neq 0$

Example

Find the rank of the following matrices

(a) $X = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$, (b) $Y = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$, (c) $Z = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}$

Solution

(a) $|X| = 1\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = (6-8) - 2(4-6) = -2 + 4 = 2 \Rightarrow r = 3$.

(b) $|Y| = 2\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = 2(2-4) - 3(6+2) + 4(6+1) = 0$

Since $\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7 \neq 0$, so the rank $r = 2$.

(c) $|Z| = 1\begin{vmatrix} 4 & 6 \\ -6 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 6 \\ -3 & -9 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ -3 & -6 \end{vmatrix} = 0$

Since all the minor matrices of degree 2 have determinants of zero, then the rank $r = 1$.

Exercises

Determine the Rank of the Following Matrices

1) $\begin{bmatrix} 4 & 7 \\ 5 & 0 \end{bmatrix}$ ***Ans.*** $r = 2$

2) $\begin{bmatrix} -4 & 0 & 3 \\ 1 & 7 & 9 \end{bmatrix}$ ***Ans.*** $r = 2$

3) $\begin{bmatrix} 4 & 1 & -3 \\ 2 & 0 & 10 \end{bmatrix}$ ***Ans.*** $r = 2$

4) $\begin{bmatrix} 9 & 1 \\ 0 & 4 \\ 2 & 6 \end{bmatrix}$ ***Ans.*** $r = 2$

5) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ ***Ans.*** $r = 3$

6) $\begin{bmatrix} -1 & 3 \\ 3 & 9 \end{bmatrix}$ ***Ans.*** $r = 2$

7) $\begin{bmatrix} 0 & 8 & 2 & 6 \\ 1 & 12 & 3 & 9 \end{bmatrix}$ ***Ans.*** $r = 2$

8) $\begin{bmatrix} 3 & -3 & 0 \\ 1 & 4 & 5 \\ 4 & 4 & 8 \end{bmatrix}$ ***Ans.*** $r = 2$

9) $\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ -3 & 6 \end{bmatrix}$ ***Ans.*** $r=1$

10) $\begin{bmatrix} 0 & -2 & 1 & 3 \\ 1 & 4 & 0 & 7 \\ 5 & 5 & 5 & 5 \end{bmatrix}$ ***Ans.*** $r=3$

11) $\begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix}$ ***Ans.*** $r=2$

12) $\begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}$ ***Ans.*** $r=2$

13) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 5 & 8 & -37 \\ 3 & 8 & 7 & 0 \\ 0 & -37 & 0 & 37 \end{bmatrix}$ ***Ans.*** $r=3$

14) $\begin{bmatrix} 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \\ 2 & 16 & 8 & 4 \end{bmatrix}$ ***Ans.*** $r=4$