Chapter Four

Partial Derivatives

4.1 Recall: ordinary derivatives

If y is a function of x then $\frac{dy}{dx}$ is the derivative meaning the gradient (slope of the graph) or the rate of change with respect to x.

4.2 Functions of 2 or more variables

Functions which have more than one variable arise very commonly. Simple examples are

- formula for the area of a triangle $A = \frac{1}{2}bh$ is a function of the two variables, base ***b*** and height ***h.***
- formula for electrical resistors in parallel:

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

is a function of three variables $\boldsymbol{R_1}$, $\boldsymbol{R_2}$ and $\boldsymbol{R_3}$, the resistances of the individual resistors.

Let's talk about functions of two variables here. You should be used to the notation *y = f(x)* for a function of one variable, and that the graph of *y = f(x)* is a curve. For functions of two variables the notation simply becomes

z = f(x,y)

Where the two independent variables are x and y, while z is the dependent variable. The graph of something like ***z = f (x, y)*** is a surface in three-dimensional space. Such graphs are usually quite difficult to draw by hand. Since *z = f(x,y)* is a function of two variables, if we want to differentiate we have to decide whether we are differentiating with respect to ***x*** or with respect to ***y*** (the answers are different). A special notation is used. We use the symbol ***∂*** instead of ***d*** and introduce the partial derivatives of ***z***, which are:

- $\frac{\partial z}{\partial x}$ is read as "partial derivative of z (or f) with respect to x", and means differentiate with respect to y holding x constant.
- $\frac{\partial z}{\partial y}$ means differentiate with respect to y holding x constant

Another common notation is the subscript notation:

$$z_x \; means \; \frac{\partial z}{\partial x}$$

$$z_y \; means \frac{\partial z}{\partial y}$$

Note that we cannot use the dash $'$ symbol for partial differentiation because it would not be clear what we are differentiating with respect to.

Example 1

Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = x^2 + 3xy + y - 1$.

Solution. To find $\frac{\partial z}{\partial x}$ treat y as a constant and differentiate with respect to x. We have $z = x^2 + 3xy + y - 1$ so

$$\frac{\partial z}{\partial x} = 2x + 3y$$

Similarly

$$\frac{\partial z}{\partial y} = 3x + 1$$

Example 2

Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = 1 - x - \frac{1}{2}y$. Interpret your answers and draw the graph.

Solution. The graph of $z = 1 - x - \frac{1}{2}y$ is a plane passing through the points $(x, y, z) = (1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$. The partial derivatives are:

$$\frac{\partial z}{\partial x} = -1, \qquad \frac{\partial z}{\partial y} = -\tfrac{1}{2}$$

Interpretation: $\frac{\partial z}{\partial x}$ is the slope you will notice if you walk on the surface in a direction keeping your y coordinate fixed. $\frac{\partial z}{\partial y}$ is the slope you will notice if you walk on the surface in such a direction that your x coordinate remains the same. There are, of course, many other directions you could walk, and the slope you will notice when walking in some other direction can be worked out knowing both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. It's like when you walk on a mountain, there are many directions you could walk and each one will have its own slope.

4.3 Functions of 3 or more variables

The general notation would be something like

$$w = f(x, y, z)$$

where x, y and z are the independent variables. For example, $w = x\sin(y + 3z)$. Partial derivatives are computed similarly to the two variable case. For example, $\partial w/\partial x$ means differentiate with respect to x holding both y and z constant and so, for this example, $\partial w/\partial x = \sin(y + 3z)$. Note that a function of three variables does not have a graph.

4.4 Second order partial derivatives

Again, let $z = f(x, y)$ be a function of x and y.

- $\frac{\partial^2 z}{\partial x^2}$ means the second derivative with respect to x holding y constant
- $\frac{\partial^2 z}{\partial y^2}$ means the second derivative with respect to y holding x constant
- $\frac{\partial^2 z}{\partial x \partial y}$ means differentiate first with respect to y and then with respect to x.

The "mixed" partial derivative $\frac{\partial^2 z}{\partial x \partial y}$ is as important in applications as the others. It is a general result that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

i.e. you get the same answer whichever order the differentiation is done.

Example 3

Let $z = 4x^2 - 8xy^4 + 7y^5 - 3$. Find all the first and second order partial derivatives of z.

Solution.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 8x - 8y^4 \\
\frac{\partial z}{\partial y} &= -8x(4y^3) + 35y^4 = -32xy^3 + 35y^4 \\
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 8 \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \\
&= \frac{\partial}{\partial y}(-32xy^3 + 35y^4) = -32x(3y^2) + 140y^3 \\
&= -96xy^2 + 140y^3 \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-32xy^3 + 35y^4) = -32y^3 \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(8x - 8y^4) = -32y^3
\end{aligned}$$

-

Example 4

Find all the first and second order partial derivatives of the function $z = \sin xy$.

Solution.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= y\cos xy \\
\frac{\partial z}{\partial y} &= x\cos xy \\
\frac{\partial^2 z}{\partial x^2} &= -y^2 \sin xy \\
\frac{\partial^2 z}{\partial y^2} &= -x^2 \sin xy \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(x\cos xy) = x(-y\sin xy) + \cos xy = -xy\sin xy + \cos xy \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(y\cos xy) = y(-x\sin xy) + \cos xy = -xy\sin xy + \cos xy
\end{aligned}$$

4.5 Subscript notation for second order partial derivatives

If z = f(x,y) then

$$z_{xx} \text{ means } \frac{\partial^2 z}{\partial x^2}$$

$$z_{yy} \text{ means } \frac{\partial^2 z}{\partial y^2}$$

$$z_{xy} \text{ means } \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 z}{\partial x \partial y}$$

4.6 Important point

Unlike ordinary derivatives, partial derivatives do not behave like fractions, in particular

$$\frac{\partial x}{\partial z} \neq \frac{1}{\partial z / \partial x}$$

4.7 Chain rule for partial derivatives

Recall the chain rule for ordinary derivatives:

$$\text{if } y = f(u) \text{ and } u = g(x) \text{ then } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

In the above we call u the **intermediate variable** and x the **independent variable**. For partial derivatives the chain rule is more complicated. It depends on how many intermediate variables and how many independent variables are present. Below three formulae are given which it is hoped indicate the general points. Essentially, every intermediate variable has to have a term corresponding to it in the right hand side of the chain rule formula. For example in the second one below there are three intermediate variables x, y and z and three terms in the RHS.
Formula 3 below illustrates a case when there are 2 intermediate and 2 independent variables.

(1) if $z = f(x, y)$ and x and y are functions of t ($x = x(t)$ and $y = y(t)$) then z is ultimately a function of t only and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

(2) if $w = f(x, y, z)$ and $x = x(t)$, $y = y(t)$, $z = z(t)$ then w is ultimately a function of t only and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

(3) if $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$ then z is a function of u and v and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Example 1

Let $z = x^2 y$, $x = t^2$ and $y = t^3$. Calculate dz/dt by (a) the chain rule, (b) expressing z as a function of t and finding dz/dt directly.
Solution. (a) by the chain rule

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2xy)(2t) + (x^2)(3t^2) \\ &= 4xyt + 3x^2t^2 \\ &= 4t^2t^3t + 3t^4t^2 \\ &= 7t^6\end{aligned}$$

(b) $z = x^2 y$ and $x = t^2$, $y = t^3$ so $z = t^4 t^3 = t^7$. Differentiating gives $dz/dt = 7t^6$.