Question one

(a) Let $(e_j)_{j=1}^n$ be the standard basis for $\mathbb{R}^n$. Verify that $\left\|\sum_{j=1}^n \lambda_j e_j\right\| = \sum_{j=1}^n |\lambda_j|$ is norm.

(b) Let $(e_j)_{j=1}^n$ be the standard basis for $\mathbb{R}^n$. Verify that $\left\|\sum_{j=1}^n \lambda_j e_j\right\| = \sup_{1\le j\le n}|\lambda_j|$ is norm.

Question Two

(a) Let $\ell^p(\mathbb{N})$ be the set of all sequences (a_k) in the field $\mathbb{C}$ such that $\sum_{k=1}^\infty |a_j|^p < \infty$, with norm $\|(a_k)\| = \left(\sum_{k=1}^\infty |a_k|^p\right)^{1/p}$, where $1 \le p \le \infty$. Show that $\left(\ell^2, \|(a_k)\|\right)$ is complete space.

(b) Let $\ell^\infty(\mathbb{N})$ be the set of all bounded sequences (a_k) in the field $\mathbb{C}$ with norm $\|(a_1, a_2, \cdots)\|_{\ell^\infty} = \sup_n |a_n|$. Show that $(\ell^\infty, \|(a_1, a_2, \cdots)\|)$ is complete space.

Question Three

(a) Suppose $\mathcal{V}$ is an inner product space over the field $\mathbb{F}$ and that $x, y \in \mathcal{V}$. Prove that $|\langle x, y\rangle| \le \|x\|\|y\|$.

(b) Let $\phi : H \mapsto \mathbb{C}$ be the linear operator on a complex Hilbert space H define by $\phi_u(x) = \langle x, u\rangle$. Show that ϕ_u is a bounded linear operator with $\|\phi\| = \|u\|$.

Question Four

(a) Let $\{v_k\}_{k=1}^\infty$ be a sequence of unit vector in a complex Hilbert space H, define an operator $\phi : H \mapsto \mathbb{C}$ as $\phi(v) = \sum_{k=1}^\infty \frac{1}{k^2}\langle v, v_k\rangle$. Show that ϕ is a bounded linear operator.

(b) Show that $\int_0^{\pi/2} e^x cos(x)\, dx \le \frac{\sqrt{\pi}}{2\sqrt{2}}(e^x - 1)^{1/2}$.

(c) Show that $\int_0^\pi 3t\sqrt{sin(t)}\, dt \le \pi\sqrt{6\pi}$.

Question Five

(a) State the definition of equivalent norm. Let $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ and $\|f\|_1 = \int_0^1 |f(x)|dx$ be two norms on $C([0,1];\mathbb{R})$. Show that they are not equivalent.

(b) State the definition of equivalent norm. Show that $\|x\|_\infty = \max_{i=1,2,\cdots,n} |x_i|$ is equivalent to $\|x\|_1 = \sum_{i=1}^n |x_i|$.

Question Six

(a) Show that a linear operator $\varphi : D(\varphi) \subset C[0,1] \mapsto C[0,1]$, where $D(\varphi) = C^1[0,1]$ given by $= f'$ is unbounded.

(b) Let $A : \ell^2 \mapsto \ell^2$ be a linear operator given by $A(a_1, a_2, a_3, \cdots) = (a_2, \sqrt[3]{2}\, a_3, \sqrt[3]{3}\, a_4, \cdots)$. is unbounded.

Question Seven

(a) Let $A : \ell^2 \mapsto \ell^2$ be a bounded linear operator given by $A(a_1, a_2, a_3, \cdots) = (0, a_1, a_2/2, a_3/3, \cdots)$. State the definition of adjoint A^* of A and compute the adjoint of A

(b) Show that A is not self-adjoint operator.

Question Eight

(a) Let $A : H \mapsto H$ be a bounded linear operator on a complex Hilbert space H. such that . State the definition of an orthogonal projection. Show that $P = \frac{1}{2}(I - A)$ is an orthogonal projection if and only if $A^2 = I$ and $A^* = A$ where I is the identity operator

(b) Let $A : \ell^2 \mapsto \ell^2$ be a bounded linear operator given by $A(a_1, a_2, a_3, \cdots) = (\frac{a_1+a_2}{2}, \frac{a_3+a_4}{2}, \cdots)$. State the definition of compact operator.

Question Nine

(a) Let $A : \ell^2 \mapsto \ell^2$ be a bounded linear operator given by $A(a_1, a_2, a_3, \cdots) = (\frac{a_1+a_2}{2}, \frac{a_3+a_4}{2}, \cdots)$. Show that A is not self-adjoint and compute $||AA^*||$.

(b) Show that A is not compact operator.

Question Ten

(a) Let $A : \ell^2 \mapsto \ell^2$ be a bounded linear operator given by $A(a_1, a_2, a_3, \cdots) = (0, a_1, a_2/2, a_3/3, \cdots)$. Show that the spectrum of A is equal to zero. $\sigma(A) = \{\lambda \in \mathbb{C} | |\lambda| \leq 1\}$.

(b) Let $S : \ell^2 \mapsto \ell^2$ be a bounded linear operator given by $S_r(a_1, a_2, a_3, \cdots) = (0, \frac{a_1}{3}, \frac{a_2}{3}, \frac{a_3}{3}, \cdots)$. Determine the operator norm $||S_r||$. Show that S_r has no eigenvalue. Show that the spectrum of S_r is equal to $[-1, 1]$.

Question Tweleve

(a) Let $A : \ell^2 \mapsto \ell^2$ be a bounded linear operator given by $A(a_1, a_2, a_3, \cdots) = ((a_1, 0, (a_2, 0, (a_3, 0, \cdots)$. Show that A is not compact . $A^*A = I$.

(b) Let $A : \ell^2 \mapsto \ell^2$ be a bounded linear operator given by $A(a_1, a_2, a_3, \cdots) = (a_1, a_2/2, a_3/3, \cdots)$. Show that the operator A is self-adjoint operator.

Question Thirteen

(a) Assume that E is finite dimensional vector space and that $A : E \mapsto E$ is a linear operator. Then show that the continuous spectrum of A is equal to empt set.

(b) Assume that E is finite dimensional vector space and that $A : E \mapsto E$ is a linear operator. Then show that the residual spectrum of A is equal to empt set.

Question Fourteen

(a) Let H be a complex Hilbert space. If $u \in H$ and $f(v) =< v, u >$ for each $v \in H$, then show that f is a bounded linear functional.

(b) If f is a bounded linear functional on H, then prove that there exist $u \in H$ such that $f(v) =< v, u >$ for each $v \in H$ and $||f|| = ||u||$.

Question Fifteen

(a) Let $H = \ell^2(\mathbb{N})$ and let $m = \{x \in \ell^2(\mathbb{N}) |\ x_1 = 0 = x_3\ \}$. Find the orthogonal complement of m.

(b) Define the Banach space. Let $C[a, b]$ be the set of all real valued continuous functions defined on interval $[a, b]$, with the norm $\|f\|_\infty = \max_{t\in[0,1]} |f(t)|$. Show that $(C[a, b], \|.\|_\infty)$ is a Banach space.

Question Sixteen

(a) Define the complete normed linear space. Let $C[-1, 1]$ be the set of all real valued continuous functions defined on interval $[-1, 1]$, with the norm $\|f\|_1 = \int_0^1 |f(x)|dx$ Show that $(C[-1, 1], \|.\|_\infty)$ is not a complete space.

(b) State the definition of bounded linear operator. Let $A : \ell^p \mapsto \ell^p$ given by $A(a_1, a_2, a_3, \cdots) = (a_1, a_1, a_2, a_3 \cdots)$. Show that $A : \ell^p \mapsto \ell^p$ is a bounded linear operator, and compute $\|A\|$.

Question Seventeen

(a) Let $\ell^2(\mathbb{N})$ be the set of all sequences (a_k) in the complex number such that $\sum_{k=1}^\infty |a_j|^2 < \infty$, with inner product $\langle (a_j)_{j=1}^\infty, (b_j)_{j=1}^\infty \rangle = \sum_{j=1}^\infty a_j \bar{b}_j$. Show that $\big(\ell^2, \langle ., . \rangle\big)$ is a Hilbert space.

(b) Let (e_k) be an orthonormal sequence in an inner product space X, show that for any $x, y \in X$ $\sum_{j=1}^\infty |\langle x, e_j\rangle\langle y, e_j\rangle| \le \|x\|\|y\|$.

Question Eighteen

(a) For any x belongs to the Hilbert space H. Let $u = \sum_{j=1}^n \langle e_j, x\rangle e_j$ where $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal set. Prove that $\|u\|^2 = \sum_{j=1}^n |\langle e_j, x\rangle|^2$.

(b) Let $A : \ell^2 \mapsto \ell^2$ be the linear operator given by $A(a_1, a_2, a_3, \cdots) = (a_1, 0, a_2, 0, a_3, \cdots)$.

- State the definition of adjoint A^* of A and compute the adjoint of A
- Show that A is not self-adjoint operator.

Question Ninteen

(a) State the definition of equivalent norm. Let $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ and $\|f\|_1 = \int_0^1 |f(x)|dx$ be two norms on $C([0, 1]; \mathbb{R})$. Show that they are not equivalent.

(b) State the definition of bounded operator. Define $A : C[0, 1] \mapsto C[0, 1]$ by $(Af)(t) = t \int_0^t f(s)ds$. Prove that this is a bounded linear operator, and compute $\|A\|$.

Question Twenty Let A be a bounded self-adjoint linear operator on a complex Hilbert space H:

(a) Prove that $||A|| = \sup_{||u||=1} | < Au, u > |$, for each $u \in H$.

(b) If $< Au, u >= 0$ then $A = 0$.

(c) Prove that $||Au|| = ||A^*u||$ for each $u \in H$.

(d) Eigenvectors corresponding to eigenvalues are orthogonal to each other.

Question Twenty one [0.5+6 marks] Let S be a closed subspace of the Hilbert space H then:

(a) State the definition of projection operator $P_S : H \mapsto S$.

(b)Show that projection operator is bounded.

(c)Show that $P_R + P_S$ is projection operator if $P_R P_S = 0$.

(d)Show that projection operator is positive operator.

Question Twenty two

(a) Let X and Y are Banach spaces, and let $T : \mathcal{D}(T) \mapsto Y$ be closed where $\mathcal{D}(T) \subseteq X$. Show that if $\mathcal{D}(T)$ is closed then T is bounded operator.

(b) Let T be an operator acting on the Banach space $X = C[0,1]$ defined by $Au = u'$ with domain $\mathcal{D}(T) = C^1[0,1]$. Show that T is closed operator. Show that T is unbounded operator. Show that $\mathcal{D}(T)$ is not closed operator.

Question Twenty Three

(a) State bounded inverse mapping theorem. Let $(X_1, ||.||_1)$ and $(X_2, ||.||_2)$ are Banach spaces, if there is a constant α such that $||x||_1 \leq \alpha \; ||x||_2$. Show that there is a constant β such that $||x||_2 \leq \beta \; ||x||_1$.

(b) Consider $A : C[0,1] \mapsto C[0,1]$ defined by

$$(Af)(x) = \int_0^1 \sinh{(x-t)} \, f(t) dt$$

where $\sinh{(x-t)} : [0,1] \times [0,1] \mapsto C$ and $\sinh{(x-t)} \in C([0,1] \times [0,1])$

- Show that A is a compact self-adjoint operator.
- Compute the spectral radius $r(A)$ of A.

Question Twenty four

(a) Let $X \neq \{0\}$ be a complex Banach space and let $T \in B(X)$ define the spectrum $\sigma(T)$ and point spectrum $\sigma_p(T)$ of the operator T. Let $K(X,X)$ be the set of all compact operator. Consider $T \in K(X,X)$ then show that $\sigma(T) = \sigma_p(T) \cup \{0\}$.

(b) Let $B : \ell^2 \mapsto \ell^2$ be a bounded linear operator given by $B(a_1, a_2, a_3, \cdots) = (0, a_1/2, a_2/3, \cdots)$. Compute the spectrum $\sigma(T)$ of the operator B.

Consider $T : L_2(0,1) \mapsto L_2(0,1)$ defined by

$$(Tf)(t) = e^t \, f(t)$$

where $e^t \in C[0,1]$ and $f \in L_2(0,1)$;

- Compute the operator norm $||T||$ of T.
- Show that the resolvent set $\rho(T) = (-\infty, 1) \cup (e, \infty)$.

Question Twenty five

Let $A : C[a,b] \mapsto C[a,b]$ be the bounded linear operator given by

$$(Af)(x) = \int_a^b k(x,t) f(t) dt$$

where $k : [0,1] \times [0,1] \mapsto C$ and $k \in ([0,1] \times [0,1])$State the definition of compact operator. Prove that A is a compact operator.