Chapter 5

Introduction to Factorial Experimental

Factorial Experiments

- **In some experiments we want to draw conclusions about more than one factor.**
- **The factorial experiments are an experiment whose design consists of two or more factors, each factor taking two or more levels.**
- **The term factorial is used because the experimental conditions include all possible combinations of the factors.**
- **In factorial experiments they studying the effect of each factor alone and the interaction between them.**

Note :

- **The factorial experiments is not design ,but it experiment which used with all design .**

Notation

- **1) Factor: The factor denote by capital letters .**

  For example A,B,C

- **2) Levels: The levels denote by small letters, and putting the digit (number) with the letter .**

  For example (a_0 , a_1 ,a_2 ,b_0, b_1 ,b_2).

- ***3)* Treatment Combination: Consist the combinations of various level for the factors. For example (a_0b_0,a_0b_1, a_0b_2).**

An Example Factorial Experiment

- **If we were looking at Gender and Time of Exam.**

1)We have two factor : Gender denoted by (A), and Time of Exam denoted by (B).

2) Gender would only have two levels: Male or Female

(a_0, a_1)

3) Time of Exam might have multiple levels, morning, noon ,and night (b_0, b_1, b_2)

4) Number of Treatment Combination(T.C.) =The multiplicative result of the levels in the experiment=(2*3)=6

Example

B / A	b_0	b_1	b_2
a_0	a_0b_0	a_0b_1	a_0b_2
a_1	a_1b_0	a_1b_1	a_1b_2

Effects in Factorial Experiments

1) Simple effect :The difference between two levels of factors under a fixed level of another factor .

B / A	b_0	b_1	
a_0	a_0b_0	a_0b_1	$(a_0b_1-a_0b_0)$
a_1	a_1b_0	a_1b_1	$(a_1b_1-a_1b_0)$
	$(a_1b_o-a_0b_0)$	$(a_1b_1-a_0b_1)$	

2) Main effect: The additive average of two simple effect.

$$=((a_0b_1-a_0b_0)+(a_1b_1-a_1b_0))/2$$

$$=((a_1b_0-a_0b_0)+(a_1b_1-a_0b_1))/2$$

3) Interaction: The difference average of two simple effect.

$$=((a_0b_1-a_0b_0)-(a_1b_1-a_1b_0))\ /2$$

$$=((a_1b_0-a_0b_0)-(a_1b_1-a_0b_1))\ /2$$

Factorial experiment: investigate all possible combinations of the levels for the factors

Factor A	Factor B	
	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	40

Main effect: the change in the response produced by a change in the level of the factor.

- Main effect of A (B) : the difference between the average response at the high level of A (B) and the average response at the low level of A (B)

$$A = \frac{30+40}{2} - \frac{10+20}{2} = 20, B = \frac{20+40}{2} - \frac{10+30}{2} = 10$$

- Interaction: the dependency between factors. The difference in response between the levels of one factor is not the same at all level of the other factors.

	Factor *B*	
Factor *A*	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	0

- At low level of B, the A effect is $A=30-10=20$
- At high level of B, the A effect is A=0-20=-20

Very different depends on the level of B

- When an interaction is large, the corresponding main effects have very little meaning.

$$A=\frac{30+0}{2}-\frac{10+20}{2}=0$$

There is no factor A effect

- At different level of B, we can see the different effect of A.

The knowledge of AB interaction is more useful than the main effect

Factorial Experiment in a CRD

- The factorial CRD used when we have two or more than two factors and when the experimental units are homogenous
- In the factorial CRD treatment combination randomly distributed on the experimental units .
- When an interaction is large, the corresponding main effects have little practical meaning.
- A significant interaction will often mask(Hidden) the significance of main effects.

Fixed Effects Model

- Factor A: Effects are fixed constants and sum to 0
- Factor B: Effects are fixed constants and sum to 0
- Interaction: Effects are fixed constants and sum to 0 over all levels of factor B, for each level of factor A, and vice versa
- Error Terms: Random Variables that are assumed to be independent and normally distributed with mean 0, variance σ^2

$$\sum_{i=1}^{a} A_i = 0, \quad \sum_{j=1}^{b} \beta_j = 0 \quad \sum_{i=1}^{a} A\beta_{ij} = 0 \ \forall j \quad \sum_{j=1}^{b} A\beta_{ij} = 0 \ \forall i \quad \varepsilon_{ijk} \sim N\left(0, \sigma_\varepsilon^2\right)$$

Completely Randomized in a Factorial Experiments

Statistical (Effects) Model:

$y_{ijk} : \mu + A_i + B_j + AB_{ij} + \varepsilon_{ijk} \quad i = 1,\ldots,a \quad j = 1,\ldots,b \quad k = 1,\ldots,r$

y_{ijk} : the value of experiment al unit that take(ith) level at Factors A ,(jth) level at B

μ : Overall Mean

A_i : (Factor A effect) orEffect of i^{th} level of factor A

B_j : (Factor B effect) or Effect of j^{th} level of factor B

AB_{ij} : Interactio n effect whe n i^{th} level of A and j^{th} level of B are combined

ε_{ijk} : Random error terms

Testing Hypotheses:

$H_0 : A_1 = \cdots = A_a = 0$ v.s. H_1 : at least one $A_i \neq 0$

$H_0 : \beta_1 = \cdots = \beta_b = 0$ v.s. H_1 : at least one $\beta_j \neq 0$

$H_0 : (A\beta)_{ij} = 0 \; \forall i, j$ v.s. H_1 : at least one $(A\beta)_{ij} \neq 0$

- Model depends on whether all levels of interest for a factor are included in experiment:
  - **Fixed Effects:** All levels of factors A and B included
  - **Random Effects:** Subset of levels included for factors A and B
  - **Mixed Effects:** One factor has all levels, other factor a subset

Completely randomized design: *a* levels of factor A, *b* levels of factor B, *r* replicates

Table General Arrangement for a Two-Factor Factorial Design

		Factor *B*			
		1	2	…	*b*
Factor *A*	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

The ANOVA table

- Total Variation (SST) is partitioned into 4 components:
  - Factor A: Variation in means among levels of A
  - Factor B: Variation in means among levels of B
  - Interaction: Variation in means among combinations of levels of A and B that are not due to A or B alone
  - Error: Variation among subjects within the same combinations of levels of A and B (Within SS)

Source	df	SS	MS	F
Treat. Comb.	ab-1	SSTreat=AB-C.F	MStreat=SStreat/(ab-1)	FTreat=MStreat/MSE
Factor A	a-1	SSA=A-C.F	MSA=SSA/(a-1)	F_A=MSA/MSE
Factor B	b-1	SSB=B-C.F	MSB=SSB/(b-1)	F_B=MSB/MSE
Interaction	(a-1)(b-1)	SSAB=AB-A-B+C.F	MSAB=SSAB/[(a-1)(b-1)]	F_{AB}=MSAB/MSE
Error	ab(r-1)	SSE=RAB-AB	MSE=SSE/[ab(r-1)]	
Total	abr-1	SST=RAB-C.F.		

$$C.F. = \frac{(Y_{..})^2}{abr}$$

$$AB = \frac{\sum_{i,j=1}^{ab} Y_{ij}^2}{r}$$

$$A = \frac{\sum_{i=1}^{a} Y_{i..}^2}{br}$$

$$RAB = \sum Y_{ijk}^2$$

$$B = \frac{\sum_{j=1}^{b} Y_{.j}^2}{ar}$$

Test for Interaction:

$H_0: A\beta_{11} = ... = A\beta_{ab} = 0$

$H_a:$ Not all $A\beta_{ij} = 0$

$$TS: F_{AB} = \frac{MSAB}{MSE}$$

$RR: F_{AB} \geq F_{\alpha,(a-1)(b-1),ab(r-1)}$

Test for Factor A

$H_0: A_1 = ... = A_a = 0$

$H_a:$ Not all $A_i = 0$

$$TS: F_A = \frac{MSA}{MSE}$$

$RR: F_A \geq F_{\alpha,(a-1),ab(r-1)}$

Test for Factor B

$H_0: \beta_1 = ... = \beta_b = 0$

$H_a:$ Not all $\beta_j = 0$

$$TS: F_B = \frac{MSB}{MSE}$$

$RR: F_B \geq F_{\alpha(b-1),ab(r-1)}$

Blocking in a Factorial Design

- 2-Factor ANOVA can be conducted in a Randomized Block Design, where each block is made up of *ab* experimental units. Analysis is direct extension of RCBD with 1-factor ANOVA
- Factorial Experiments can be conducted with any number of factors. Higher order interactions can be formed (for instance, the *AB* interaction effects may differ for various levels of factor *C*).
- When experiments are not balanced, calculations are immensely messier and you must use statistical software packages for calculations

Completely Randomized Block Design in a Factorial Experiments

Statistical (Effects) Model:

$y_{ijk} : \mu + A_i + B_j + AB_{ij} + R_k + \varepsilon_{ijk} \quad i = 1,\dots,a \quad j = 1,\dots,b \quad k = 1,\dots,r$

y_{ijk} : the value of experiment al unit that take(ith) level at Factors A ,(jth) level at B

μ : Overall Mean

A_{i} : (Factor A effect) or Effect of i^{th} level of factor A

B_{j} : (Factor B effect) or Effect of j^{th} level of factor B

AB_{ij} : Interactio n effect whe n i^{th} level of A and j^{th} level of B are combined

R_k : Effect of Blocks

ε_{ijk} : Random error term s

The ANOVA table

Source	df	SS	MS	F
Blocks	r-1	SSr=R-C.F	MSr =SSr/(r-1)	Fr = MSr/MSE
Treat. Comb.	ab-1	SS_{Treat}=AB-C.F	MS_{Treat}=SStreat/(ab-1)	F_{Treat}=MStreat/MSE
Factor A	a-1	SSA=A-C.F	MSA=SSA/(a-1)	F_A =MSA/MSE
Factor B	b-1	SSB=B-C.F	MSB=SSB/(b-1)	F_B =MSB/MSE
Interaction	(a-1)(b-1)	SSAB=AB-A-B+C.F	MSAB=SSAB/[(a-1)(b-1)]	F_{AB} =MSAB/MSE
Error	(r-1)(ab-1)	SSE=RAB-R-AB+C.F	MSE=SSE/[(r-1)(ab-1)]	
Total	abr-1	SST=RAB-C.F.		

$$C.F. = \frac{(Y_{..})^2}{abr}$$

$$R = \frac{\sum Y_{..k}^2}{ab}$$

$$A = \frac{\sum_{i=1}^{a} Y_{i..}^2}{br}$$

$$AB = \frac{\sum_{i,j=1}^{ab} Y_{ij}^2}{r}$$

$$B = \frac{\sum_{j=1}^{b} Y_{.j}^2}{ar}$$

$$RAB = \sum Y_{ijk}^2$$

Latin Square in a Factorial Experiments

Mathematical model:

$$Y_{ijkl} = \mu + A_i + B_j + AxB_{ij} + R_k + C_l + e_{ijk}$$

Examples : of a 2 x 3 factorial arrangement of A and B (2 levels of A and 3 levels of B):

In a Latin Square with 6 rows and 6 columns:

Source	df	
Total	35	tr - 1
Treatment	(5)	t - 1
A	1	a - 1
B	2	b - 1
A x B	2	(a - 1)(b - 1)
Row	5	r - 1 Same as block in RCBD
Column	5	c - 1
Error (BxT)	20	(t - 1)(r - 2) Column takes another 5 df from Error

Total and Treatment df are same for all designs.