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Random Variable

Chapter Five

Higher Diploma
First Semester

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2023 - 2024

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Random Variables

Def' : A random variable X is a real valued function defined sample space, which takes values from the sample space (S) to a space real numbers R is a random experiment.

$X : S \rightarrow R$

The r.v. X may be either discrete or continuous , as with sample space.

Types of Random variables :

1- Discrete Random Variable:

Def' : If (X) be (r.v.) then (X) is said to be a discrete (r.v.) if then range of (X) (R(x)) (sample space) is countable set, whether finite or infinite set.

For Example: The number of students in statistics dept., then

$S=\{ x: x= 1,2,3, \dots , 400 \}$

For Example: A coin is a tossed until a head occur then. $S=\{x:x=1,2,\dots\}$

Probability (mass or density) function:

If (x) be a discrete r.v. with different values (x_1 , x_2 , ... , x_n), then the function

$$P(x) = \begin{cases} P(X = xi) & if\ \ i = 1,2,3,\dots,n \\ 0 & o.w\ (other\ wise) \end{cases}$$

is defined to be probability mass function of x (p.m.f).

and it is a real – value function, and satisfies following properties : -

1- $0 \le P(x) \le 1$, For all x

2- $\sum_{\forall x} P(x) = 1$

Ex1/ Let three coins are tossed , let x be the number of heads that occur then,

1 - find P(x) of each value of (x).

2 - check P(x) is (p.m.f) of (x).

3 - Graph P(x).

Ex2/ Two dice are tossed , let x be sum of the two number shown by the two dice.

1 - find P(x) of each value of (x).

2 - check P(x) is (p.m.f) of (x).

3 - Graph P(x).

Ex3/ A coin is tossed three times, a coin weighted so that p(H)=2/3 and p(T)=1/3 . let x be the number of heads that occur then,

1 - find P(x) of each value of (x).

2 - check P(x) is (p.m.f) of (x).

3 - Graph P(x).

Cumulative distribution function: (C.d.f)

Let be X a discrete (r.v.) , then the cumulative distribution function (c.d.f)

Of x (F(x)) is defined by :

F(X) = P(X≤ x)

If r.v. (x) is discrete .Then

$$\boldsymbol{F(x) = \sum_{x=-\infty}^{x} P(x)}$$

Properties of (c.d.f):

1- F(a) **≤ F(b) if a ≤ b**

i.e F(x) is increasing function

2- $\lim_{x\to\infty} F(x) = 1 \quad and \lim_{x\to-\infty} F(x) = zero$

Point & interval (Discrete r.v.)

Without point & interval

$\Pr(a \le x \le b) = F(b) - F(a-1)$

$\Pr(a < x \le b) = F(b) - F(a)$

$\Pr(a \le x < b) = F(b-1) - F(a-1)$

$\Pr(a < x < b) = F(b-1) - F(a)$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{15} & 1 \le x < 2 \\ \frac{3}{15} & 2 \le x < 3 \\ \frac{6}{15} & 3 \le x < 4 \\ \frac{10}{15} & 4 \le x < 5 \\ 1 & 5 \le x \end{cases}$$

$$Pr(2 \le x \le 4) = F(4) - F(1) = \frac{10}{15} - \frac{1}{15} = \frac{9}{15}$$

$$Pr(2 < x \le 4) = F(4) - F(2) = \frac{10}{15} - \frac{3}{15} = \frac{7}{15}$$

$$Pr(2 \le x < 4) = F(3) - F(1) = \frac{6}{15} - \frac{1}{15} = \frac{5}{15}$$

$$Pr(2 < x < 4) = F(3) - F(2) = \frac{6}{\ } - \frac{3}{\ } - \frac{3}{\ }$$

Ex4/ Let three coins are tossed , let x be the number of heads that occur then,

1 - find P(x) of each value of (x).

2 - check P(x) is (p.m.f) of (x).

3 - Graph P(x).

2- continuous Random Variable:

Def' if (x) be a continuous r.v., defined on an interval, then the function f(x) is defined to be probability density function (P.d.f)of (x), and it is a mathematical f^n, and satisfies, the following properties :

1- $0 \le f(X) \le 1$

2- $\int_{-\infty}^{\infty} f(x)dx = 1$

3- For any interval (a,b) , $p(a \le x \le b) = \int_a^b f(x)dx$

Ex5/ Let X be a r.v. with p.d.f f(x) is:

$$f(x) = \begin{cases} \frac{1}{x^2} & 1 < x < \infty \\ 0 & o\,w \end{cases}$$

Find 1- check of f(X) is a p.d.f of (x).

2- $A = p(1 < x < 2)$.

3- $B = p(4 < x < 5)$.

properties of C.d.f :

$$1 - F(x) = \int_{-\infty}^{X} f(t)dt$$

2- $f(x) = \frac{\partial F(x)}{\partial x}$

3- P(a < x < b) = F(b) – F(a) = $\int_a^b f(x)dx$

Ex7/ Let X be a r.v. with p.d.f f(x) is:

$$f(x) = \begin{cases} 3(1-x)^2 & 0 < x < 1 \\ 0 & o.w \end{cases}$$

Find the C.d.f & graphical?

Ex8/ Let

$$F(x)=\begin{cases}0 & x\le 0\\ 1-e^{-5x} & 0<x<\infty\\ 1 & x\to\infty\end{cases}$$

1) *find the p.d.f. of x.*

Ex9/ Let X be a r.v. with p.d.f f(x) is:

$$f(x)=\begin{cases}\frac{1}{6}x+k & 0\le x\le 3\\ 0 & o.w\end{cases}$$

Find: 1- value of k.

2- pr (1 ≤ x ≤ 2) .

Questions about (p.m.f) , (p.d.f) , (c.d.f) for (D.r.v. & C.r.v.)

Q1/ Find the probability distribution of the D.r.v. (the number of sixes) in two tosses of a die. Find graph of (x) .

Q2/ 3 bad articles are mixed with 7 good ones. Find the probability distribution of the number of the bad articles if 3 articles are drawn at random without replacement?

Q3/ Find the probability distribution of the number of boys in a family of three children. Then find c.d.f and graphical of (p.mf) and (c.d.f)?

Q4/ Let x a r.v. with p.d.f :

$$p(x) = \frac{2}{x^3} \qquad 1 < x < \infty$$

Find 1- check the p.d.f . 2- c.d.f and graphical.

Q5/ Let x a r.v. with p.d.f :

$$f(x) = \begin{cases} c\,x\,e^{-x} & 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

Find value of the constant c ?

Q6/ Let x a r.v. with p.d.f :

$$f(x) = e^{-x} \qquad x > 0$$

$$\qquad 0 \qquad\qquad \text{o.w}$$

Find the c.d.f and graphical?

Q7/ Let x be a r.v. is p.m.f. of P(x);

$$P(x) = \frac{cx}{5} \qquad\qquad x = 1,2,3,4,5$$

$$\qquad 0 \qquad\qquad\qquad \text{o.w}$$

Find 1- the value of the constant c?

2- c.d.f and graph?

3- Mathematical Expectation:

Let x be a random variables and let g(x) be a function of x, then the mathematical expectation of g(x) is denoted by E[g(x)],

$$E[\,g(x)] = \begin{cases} \sum_{\forall x} g(x)P(X) & \textit{if } x \textit{ is discrete } r.v. \\ \int_{-\infty}^{\infty} g(x)f(x)\, d(x) & \textit{if } x \textit{ is continuous } r.v. \end{cases}$$

properties :

1) $Ea = a$ if a is a constant (real number), then

2) If a is a constant, and $Q(x)$ is a fun.of x, then

$$E[a\{Q(x)\}] = a\,E[Q(x)]$$

3) If a_1 & a_2 are two constants, then;

$$E(a_1x + a_2) = E\,a_1 . E(x) + E\,a_2$$

$$E(a_1x + a_2) = a_1\,Ex + a_2$$

4) If a_1 & a_2 are two constants, and $Q_1(x)$ & $Q_2(x)$ are two functions of x then;

a) $E[Q_1(x) \mp Q_2(x)] = E[Q_1(x)] \mp E[Q_2(x)]$

b) $E[aQ_1(x) \mp bQ_2(x)] = a\,E[Q_1(x)] \mp b\,E[Q_2(x)]$

5) If $y = x_1 + x_2 + x_3 \Rightarrow Ey = Ex_1 + Ex_2 + Ex_3$

6) If x and y are two random variables then;

$E(xy) = E(x).E(y)$ if x & y are independent

$E(xy) \neq E(x).E(y)$ if x & y are dependent

7) Let $y = \sum x_i \Rightarrow E(y) = \sum E(x_i) \Rightarrow E(y) = nE(x)$

Ex10/ Let x have the p.m.f:

$$P(x) = \begin{cases} \frac{x}{6} & x = 1,2,3 \\ 0 & o.w \end{cases}$$

Find 1- E(X) ? 2- $E(\times^2)$?

Ex11/ Let x be ar.v. with f(X) = 2(1-X) 0 < X < 1

Find : 1- M1 . 2- M2 . 3- $E(6X + 3\times^2)$

Ex12/ let x be a r.v. of f(x) is a p.d.f ;

$$f(x) = \frac{1}{2}(x+1) \qquad -1 < x < 1$$

Find mean of this dist. Of x?

4- Variance & Standard Deviation of the r.v.:

It's denoted by σ^2 , S^2 and is given by :

$$\boldsymbol{S^2(x) = Var\,(x) = E\left(x^2\right) - \left(E(X)\right)^2 = E\left(x^2\right) - \mu^2}$$

Variance of the r.v. is:

$$S^2(x) = \sum x^2 P(x) - \left[\sum x\, P(x)\right]^2 \qquad \ldots\ldots Discrete\ \ r.v.$$

$$S^2(x) = \int_{-\infty}^{\infty} x^2\, f(x)dx - \left[\int_{-\infty}^{\infty} x\, f(x)\, dx\right]^2 \quad \ldots Continuous\ r.v.$$

The standard deviation of X, denoted by σ , s is the (non negative) square root of Var(x):

$$S(x) = \sqrt{S^2(x)} \qquad \ldots Standard\ Deviation\ of\ the\ r.v.$$

Ex13/ Find Variance and standard deviation of Example(10) ?

Ex14/ Find Variance and standard deviation of Example(11) ?

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5- Moment:

i-Non - Central Moment :

In special case of non – central moment is the moment about the origin i.e (when a=0, then the r-th non – central moment about origin is a=0)

$$M_r = E(x)^r = \begin{cases} \sum x^r \, P(x) & x \text{ is } disc.r.v. \\ \int x^r \, f(x)dx & x \text{ is } cont.r.v \end{cases}$$

ii- Central Moment :

let x be a r.v. with p.d.f f(x), then the r-th central moment of x about mean (M) is denoted by , and is defined as:

$$Mr' = E(x-m)^r = \begin{cases} \sum_{\forall x} (x-m)^r p(x) & for\ dis.r.v. \\ \int_{R_x} (x-m)^r f(x)\,dx & for\ cont.r.v. \end{cases}$$

$$r = 2 \rightarrow M_2' = E(x-M)^2 = \sigma_x^2 \Rightarrow \text{variance}$$

$$r = 3 \rightarrow M_3' = E(x-M)^3 = Skewnes$$

$$r = 4 \rightarrow M_4' = E(x-M)^4 = kurtosis$$

Relation between non –central moment (Mr) & central moment (Mr')

$$r = 2 \Rightarrow M_2' = E(x - M)^2 = \sigma^2$$

$$M_2' = E(x)^2 - 2M\,E(x) + M^2$$

$$M_2' = E(x)^2 - M^2$$

$$M_2' = M_2 - M_1^2$$

$$\therefore M_2 = M_2' + M_1^2$$

Ex15/ Let x be a r.v. with

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & o.w \end{cases}$$

$Find:\ M_1, M_2, E(3x + 5x^2)\quad and\ \sigma^2, \sigma\,?$

Ex16/ Let x be a r.v. with $p(x) = \begin{cases} \frac{x}{6} & x = 1,2,3 \\ 0 & o.w \end{cases}$

$Find:\ M_1, M_2, E(2x + 3x^2)\quad and\ \sigma^2, \sigma\,?$

6- The Moment Generating function (m.g.f):

Let x be a r.v. with p.d.f of f(x), the moment generating function (m.g.f) of x denoted by [ ], and is defined for all real value of [where h is a positive number]. Such that the mathematical expectation exist, thus ;

$$M_x(t) = Ee^{tx} = \begin{cases} \sum e^{tx} P(x) & x \text{ is } disc.r.v. \\ \int e^{tx} f(x)dx & x \text{ is } cont.r.v \end{cases}$$

Definition : - Let be the moment generation function of a r.v. x that is existed. Then the moment of r-th order about origin at (t = 0) is define as ;

$$Ex^r = M_x^r(0) = \frac{\partial^r M_x(t)}{\partial t^r}\Big|_{t=0} = M_r \qquad r = 0,1,2,\ldots$$

$$M_x(t) = Ee^{tx}$$

Ex17/ Let x be a r.v. with $f(x) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta \\ 0 & o.w \end{cases}$

Find: 1- (m.g.f) of X ? 2- M_x , σ^2 ?

Ex18/ A fair coin is tossed twic let X be the number of heads the occur

Find: 1- (m.g.f) of X ? 2- M_x , σ^2 ?

Ex19/ Let x be a r.v. with $f(x) = \begin{cases} 2e^{-2x} & x \geq o \\ 0 & o.w \end{cases}$

Find: 1- (m.g.f) of X ? 2- M_x , σ^2 ?

7- Joint Probability Distribution Function (J.p.d.f) :

Joint Distribution:

IF are r.v's defined on the sample probability space , then the is called p-dimensional r.v's .

Joint probability density function [J.p.d.f] :

Def^n: IF are a p-dimensional r.v's ; Then the Joint pro.density fun. Of $(x_1, x_2, \dots, x_p)$ is defined to be:

$$f(x_1, x_2, \dots, x_p) = \mathrm{P}(X_1 = x_1, X_2 = x_2, \dots, X_p = x_p)$$

Or

$$f(x_1, x_2, \dots, x_p) = \begin{cases} \sum\limits_{\forall x_1} \sum\limits_{\forall x_2} \cdots \sum\limits_{\forall x_p} P(x_1, x_2, \dots, x_p) & disc.r.v's \\ \int\limits_{Rx_1} \int\limits_{Rx_2} \cdots \int\limits_{Rx_p} f(x_1, x_2, \dots, x_p) dx_p \dots dx_2 dx_1 & \end{cases}$$

Def^n.: IF p=2 , let x and y are two r.v's then

$$f(x,y) = p(X = x \quad , Y = y)$$

$$= \sum_{\forall x}\sum_{\forall y} p(x,y) \qquad \text{if} \quad x \,\&\, y \quad \text{are} \quad \text{discrete} \quad r.v\text{'}s$$

$$= \int_{Rx}\int_{Ry} f(x,y)dydx \qquad \text{if} \quad x \,\&\, y \quad \text{are} \quad \text{continuous} \quad r.v\text{'}s$$

1) $$0 \leq f(x,y) \leq 1 \qquad \forall \quad x, y$$

2)

$$\sum_{\forall x}\sum_{\forall y} f(x,y) = 1$$

$$\int_{Rx}\int_{Ry} f(x,y)dydx = 1$$

Ex20/ Let the j.pd.f of two r.v's $x_1 and x_2$ is definded

$$p(x) = \begin{cases} \frac{x_1+x_2}{21} & x_1 = 1,2,3. \quad x_2 = 1,2 \\ 0 & o.w \end{cases}$$

Find: 1- (j.p.d.f) for each values of X_1 and X_2?   2- p(X_1=2 ,X_2=1)?

3- prove p(X_1,X_2) is j.p.d.f of (X_1,X_2)?

Ex21/ Let $f(x,y) = \begin{cases} e^{-(x+y)} & x,y>0 \\ 0 & o.w \end{cases}$

Find 1- P(X=3 , Y=4).   2-P(0<X<2 , 1<Y<3)

3- check of the j.p.d.f of (x,y)?

Ex22/ $f(x) = \begin{cases} 6x^2y & 0 < x < 1 ,0 < y < 1 \\ 0 & o.w \end{cases}$

Find:$p(o < x < \frac{3}{4} , \frac{1}{3} < y < 2)$.

Ex23/ A certain market has both an express checkout line and a super express checkout line. Let X denote the number of customers in line at the express checkout line at a particular time of day and let Y denote the number of customers in line at the super express checkout at the same time. Suppose the joint p.m.f of X and Y is as given in the following table:

X \ Y	0	1	2	3
0	0.08	0.07	0.04	0.00
1	0.06	0.15	0.05	0.04
2	0.05	0.04	0.10	0.06
3	0.00	0.03	0.04	0.07
4	0.00	0.01	0.05	0.06

Find:

1-What is P (X = 1; Y = 1), that is, the probability that there is exactly one customer in each line?

2-What is P (X = Y), that is, the probability that the number of customers in the two lines are identical?

3-Let A denote the event that there are at least two more customers in one line than in the other line. Express A in terms of X and Y and calculate the probability of this event

4-What is the probability that the total number of customers in the two lines is exactly four? At least four?

Ex24/ Let X and Y have the joint probability density function given by:

$$f(x,y) = \begin{cases} k(1-y) & 0 \le x \le y \le 1 \\ 0 & o.w \end{cases}$$

Find the value of **k** that makes this a probability density function?

8- Marginal Probability Distribution Function (m.p.d.f) :

Def^n.: let x & y are two r.v's with (j.p.d.f) , then are called the marginal p.d.f of x & y respectively, which can be defined as follows:-

If (m.p.d.f) for f(X):

$$f(x) = \begin{cases} \sum_{\forall y} P(x,y) & if\ X\ \&\ y\ are\ D.r.v. \\ \int_{Ry} f(x,y)dy & if\ X\ \&\ y\ are\ C.r.v. \end{cases}$$

If (m.p.d.f) for f(y):

$$f(y) = \begin{cases} \sum_{\forall x} P(x,y) & if\ X\ \&\ y\ are\ D.r.v. \\ \int_{Rx} f(x,y)dx & if\ X\ \&\ y\ are\ C.r.v. \end{cases}$$

Note: X and Y are independent iff $f(x,y) = f(x) \times f(y)$

Ex25/ Let X and Y have the joint probability density function given by:

$$f(x,y) = \begin{cases} x + y & 0 < x < 1 , 0 < y < 1 \\ 0 & o.w \end{cases}$$

Find:1- The marginal p.d.f of X? **2-** The marginal p.d.f of y?

3- Are X and Y are independent? **4-** E(X) and E(y) ?

Ex26/ Let X and Y have the joint probability density function given by:

$$P(x,y) = \begin{cases} \frac{xy^2}{30} & x = 1,2,3. \quad , y = 1,2. \\ 0 & o.w \end{cases}$$

Find: **1-**m.p.d.f of X? **2-** m.p.d.f of y? **3-** Are X and Y are independent?

4- E(X) and E(y) ?

Ex27/If X & Y having the j.p.d.f as:

$$P(x,y) = \begin{cases} \frac{xy}{96} & o < x < 4 \ , 1 < y < 5. \\ 0 & o.w \end{cases}$$

Find: **1-**f(X)? 2- f(y)? 3- F(X) and F(y)? 4- Var(X) and Var(y)?

Questions about chapter Five

Q9/ Two refills for a ballpoint pen are selected at random from a box that contains **3 blue** refills, **2 red** refills and **3 green** refills. If **X** is the number of **blue** refill and **Y** is the number of **red** refills selected, find

1-The joint probability mass function p(x, y)?

2-If {(X,y)Є A} , where A is the region {(X,Y) : X+Y≤1}?

3-Var(x) and Var(Y)? **4-** C.d.f for (x) and (Y)?

Q10/ Find g(X) and h(Y) for the joint density function :

$$f(x,y)=\begin{cases}\frac{2}{5}(2x+3y), & 0\le x\le 1, 0\le y\le 1\\ 0, & \text{elsewhere}\end{cases}$$

Q11/ For example (23) find:

1-Determine the marginal p.m.f of X and then calculate the expected number of customers in line at the express checkout?

2-Determine the marginal p.m.f of Y ?

3-By inspection of the probabilities P (X = 4), P (Y = 0), and P(X = 4;Y =0) are X and Y independent random variables? Explain.

Q12/ Two components of a minicomputer have the following joint p.d.f for their useful lifetimes X and Y :

$$f(x,y) = \begin{cases} xe^{[-x(1+y)]} & x \geq 0\,, y \geq o \\ o & o.w \end{cases}$$

1-What is the probability that the lifetime X for the first component exceeds 3? **2-**What are the marginal pdf 's of X and Y ? Are the two lifetimes independent?

3- What is the probability that the lifetime of at least one component exceeds 3?

Q13/ A fair coin is tossed three times independently: let X denote the number of heads on the first toss and Y denote the total number of heads. Find the joint probability mass function of X and Y ?And find marginal mass function of X and y?

Q14/ Consider the pdf for X and Y:

$$f(x,y) = K(x^2 + xy) \qquad 0 \le x \le 1, 0 \le y \le 1$$

Find : 1- The value of K? 2- P(X>Y)? 3- f(x) and g(y)?