Kurdistan Regional Government-Iraq
Ministry of Higher Education and Scientific Research
Salahaddin University – Erbil
College of Engineering
Department of Architecture Engineering

Mathematic-I

Spring Semester
Academic year (2023-2024)
Four hours per week
Five Credits

Prepared by: Ali A. Mahmod
Email: ali.mahmod@su.edu.krd

1

Chapter 2 Derivatives

We defined the slope of a curve at a point as the limit of secant slope. This limit is called a derivative. The process of calculating a derivative is called differentiation.

2.1 The Derivative as a Function

- The slope of a curve $y = f(x)$ at the point where $x = x_1$ is

$$\lim_{h\to 0}\frac{f(h+x_1)-f(x_1)}{h}, \qquad where \quad h \neq 0$$

- We called this limit, when it existed, the derivative of f at x_1.

- The derivative of the function $f(x)$ with respect to the variable of x is the function f' whose value at x:

$$f'(x) = \lim_{h\to 0}\frac{f(h+x_1)-f(x_1)}{h}$$

2

Chapter 2 Derivatives
2.1 The Derivative as a Function

Example: Differentiate $f(x) = \frac{x}{x-1}$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(h + x_1) - f(x_1)}{h}$$

Here we have $f(x) \frac{x}{x-1}$ and

$$f(x+h) = \frac{(x+h)}{(x+h)-1} \quad so$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{(x+h)}{(x+h)-1} - \frac{x}{x-1}}{h}$$

$$f'(x) = \frac{1}{h} \lim_{h \to 0} \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)^2}$$

3

Chapter 2 Derivatives
2.2 Notation

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and dependent variable is y.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

Note:

- ❖ The symbol d/dx and D indicate the operation of differentiation and are called differentiation operator.
- ❖ We read dy/dx as "the derivative of y with respect to x"
- ❖ The "prime" notation y' and f' come from notation that **Newton used for derivatives**.
- ❖ A function is continuous at every point where it has a derivative.
- ❖ We can differentiate f' to second derivative or higher order derivative, it is denoted

$$f''(x) = y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D^2(f)(x) = D_x^2 f(x)$$

And so on…

4

Chapter 2 Derivatives
2.3 Differentiation Rules

Rule 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

Rule 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}$$

Rule 3 Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

5

Chapter 2 Derivatives
2.3 Differentiation Rules

Rule 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable, At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Rule 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

6

Chapter 2 Derivatives
2.3 Differentiation Rules

Rule 6 Derivative Quotient Rule

If u and v are differentiable functions of x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Rule 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

7

Chapter 2 Derivatives
2.3 Differentiation Rules

Example: Find the derivative of the followings with respect to x:

(a). $f(x) = 3x^2 - 8z$ (b). $f(x) = x^4 + 12x$ (c). $f(x) = (x^2+1)(x^3+3)$

(d). $f(x) = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$ (e). $f(x) = \frac{x^2-1}{x^2+1}$ (f). $f(x) = x^3 + \frac{4}{3}x^2 - 5x + 1$

Solution:

(a). $\frac{d}{dx}(3x^2 - 8z) = 6x$

(b). $\frac{d}{dx}(x^4 + 12x) = 4x^3 + 12$

(c). $\frac{d}{dx}\left((x^2+1)(x^3+3)\right) = (x^2+1)(3x^2) + (x^3+3)(2x) = 5x^4 + 3x^2 + 6x$

(d). $\frac{d}{dx}\left(\frac{1}{x}\left(x^2 + \frac{1}{x}\right)\right) = \frac{d}{dx}\left(x + \frac{1}{x^2}\right) = 1 + \frac{-2}{x^3} = 1 - \frac{2}{x^3}$

(e). $\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$

(f). $\frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right) = 3x^2 + \frac{8}{3}x - 5$

8

Chapter 2 Derivatives
2.4 Derivative as Rate of Change

Example: How fast does the area change with respect to diameter for diameter $\mathbf{10\ m}$?

Solution:
- Area of circle:

$$A(D) = \frac{\pi}{4}D^2$$

- The rate of change of the area:

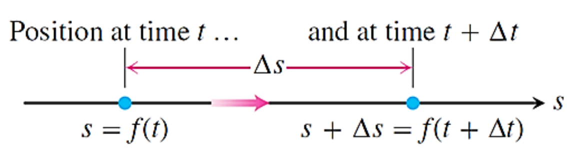
$$\frac{dA}{dD} = \frac{d}{dD}\left(\frac{\pi}{4}D^2\right) = \frac{\pi}{2}D$$

- When $D = 10\ m$, the area is changed at rate $\mathbf{5\pi\ m^2/m}$.

9

Chapter 2 Derivatives
2.5 Motion Along a Line

Suppose that an object is moving along a coordinate line (say an $s - axis$), its position. On that line as a function of t is $s = f(t)$.



- The Displacement of the object over the time interval from t to $t + \Delta t$ is:

$$\Delta s = f(t + \Delta t) - f(t)$$

- The velocity of the object over that time interval:

$$v_{av} = \frac{Displacement}{Time\ Travel} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}f(t)$$

10

Chapter 2 Derivatives
2.5 Motion Along a Line

- Speed is the absolute value of velocity

$$\boldsymbol{speed} = |\boldsymbol{v}(\boldsymbol{t})| = \left|\frac{\boldsymbol{ds}}{\boldsymbol{dt}}\right|$$

- Acceleration

$$\boldsymbol{a}(\boldsymbol{t}) = \frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \frac{\boldsymbol{d^2 s}}{\boldsymbol{dt^2}}$$

- Jerk

$$\boldsymbol{j}(\boldsymbol{t}) = \frac{\boldsymbol{da}}{\boldsymbol{dt}} = \frac{\boldsymbol{d^2 v}}{\boldsymbol{dt^2}} = \frac{\boldsymbol{d^3 s}}{\boldsymbol{dt^3}}$$

11

Chapter 2 Derivatives
2.5 Motion Along a Line

Note:

- Near the surface of the Earth, all bodies fall with the same constant acceleration. Galileo's experiments with free fall (there is no air resistance and closely models the fall of dense, heavy objects, such rock or steel tool, for the first few seconds of their fall, before air resistance starts to slow them down) lead to the equation:

$$\boldsymbol{s} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{g}\boldsymbol{t^2}$$

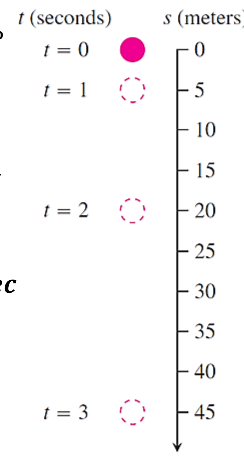
Where $\boldsymbol{s}$ is distance and $\boldsymbol{g}$ is the acceleration due the Earth's gravity

- The value of $\boldsymbol{g}$ in the equation depends on the units used to measure $\boldsymbol{t}$ and $\boldsymbol{s}$.
- With $\boldsymbol{t}$ in seconds
  $\boldsymbol{g = 32\ ft/sec^2}$ (feet per second square) or
  $\boldsymbol{g = 9.8\ m/sec^2}$ (meters per second square)

12

Chapter 2 Derivatives
2.5 Motion Along a Line

Example: Figure below shows the free fall of a heavy ball bearing released from rest at time $\boldsymbol{t = 0}$ sec.
(a) How many meters does the ball fall in the first **2** sec?
(b) What is the instantaneous velocity, speed, acceleration then?



Solution: The equation of free fall is $\boldsymbol{s = \frac{1}{2} g t^2}$

(a). During the first **2** sec

$$\boldsymbol{s(2) = \frac{1}{2}(9.8)(2)^2 = 19.6\, m}$$

(b). Velocity at $\boldsymbol{t = 2\ sec}$

$$\boldsymbol{v(t) = s'(t) = gt = 9.8t}$$
$$\boldsymbol{v(2) = (9.8)(2) = 19.6\ \ m/sec}$$

Acceleration at t= **2** sec

$$\boldsymbol{a(t) = v'(t) = 9.8\ \ m/sec^2}$$

Speed at $\boldsymbol{t = 2}$ sec

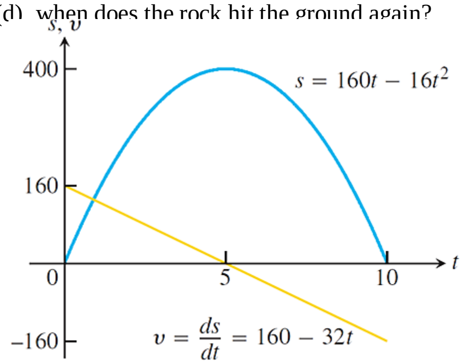
$$\boldsymbol{speed = |v(t)| = 19.6\ \ m/sec}$$

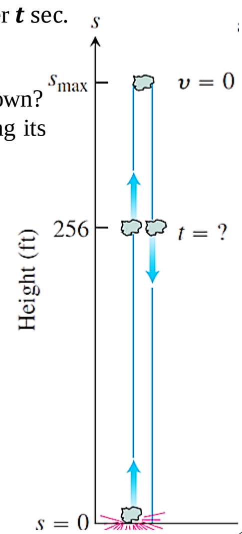
13

Chapter 2 Derivatives
2.5 Motion Along a Line

Example: A dynamite blast blows a heavy rock straight up with a launch velocity of $\boldsymbol{160\ ft}$/sec (figure). It reaches a height of $\boldsymbol{s = 160t - 16t^2}$ after $\boldsymbol{t}$ sec.

(a) How high does the rock go?
(b) what are the velocity and speed of the rock when it is $\boldsymbol{256\ ft}$ above the ground on the way up? On the way down?
(c) what is the acceleration of the rock at any time t during its flight (after the blast)?
(d) when does the rock hit the ground again?





14

Chapter 2 Derivatives
2.5 Motion Along a Line

Solution:

(a) To find the maximum height, all we need to do is to find when $\boldsymbol{v = 0}$ and evaluate $\boldsymbol{s}$ *at* this time.

At any Time $\boldsymbol{t}$, the velocity is:

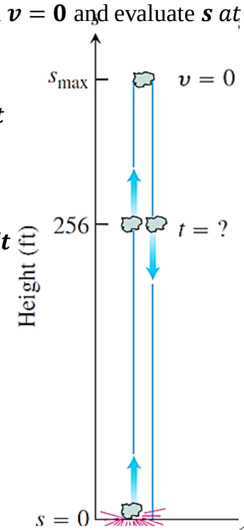
$$v = \frac{ds}{dt} = \frac{d}{dt}\left(160t - 16t^2\right) = 160 - 32t$$

The velocity is zero when

$$160 - 32t = 0 \quad or \quad t = 5\ sec$$

The rock's height at $\boldsymbol{t = 5}$ sec is

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 400\ ft$$



(b)

$$s(t) = 160t - 16t^2 = 256$$

To solve this equation, we write

$$16t^2 - 160t + 256 = 0$$
$$16\left(t^2 - 10t + 16\right) = 0$$
$$(t - 2)(t - 8) = 0$$
$$t = 2\ sec, t = 8\ sec.$$

15

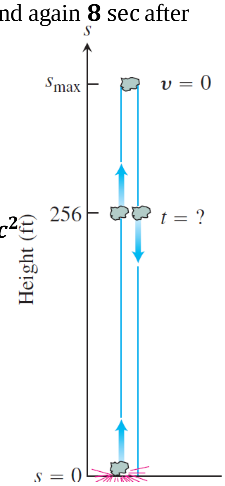
Chapter 2 Derivatives
2.5 Motion Along a Line

Solution:

(b)

The rock is $\boldsymbol{256\ ft}$ above the ground $\boldsymbol{2}$ sec after the explosion and again $\boldsymbol{8}$ sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 96\ ft/sec$$
$$v(8) = 160 - 32(8) = -96\ ft/sec$$



(c)

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32\ ft/sec^2$$

(d) The rock hits the ground at positive time $\boldsymbol{t}$ for which $\boldsymbol{s = 0}$

$$160t - 16t^2 = 0 \Rightarrow t = 0, t = 10$$

The time that the rock hits the ground at $\boldsymbol{t = 10}$ sec

16

Chapter 2 Derivatives
2.6 Derivative of Trigonometric Functions

The Derivative of Six Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

17

Chapter 2 Derivatives
2.7 Chain Rules

To differentiate a composite function like $F(x) = f(g(x)) = \sin(x^2 - 4)$ or the derivative of $F = f \circ g$, we use the chain rule.

Theorem 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)).g'(x)$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

Where dy/du is evaluated at $u = g(x)$.

18

Chapter 2 Derivatives
2.7 Chain Rules

Example: Differentiate the followings:

$(a).\ \sin(x^2+x)$ $(b).\ \tan(5-\sin 2x)$ $(c).\ (5x^3-x^4)^7$ $(d).\ \dfrac{1}{3x-2}$

$(e).\ \sin^5 x$ $(f).\ (1-2x)^{-3}$

Solution:

(a). $\frac{d}{dx}(\sin(x^2+x)) = \cos(x^2+x)\frac{d}{dx}(x^2+x) = (2x+1)\cos(x^2+x)$
(b). $\frac{d}{dx}(\tan(5-\sin 2x)) = \sec^2(5-\sin 2x)\frac{d}{dx}(5-\sin 2x)$ $= -(\cos 2x)\sec^2(5-\sin 2x)\frac{d}{dx}(2x)$ $= -2(\cos 2x)\sec^2(5-\sin 2x)$
(c). $\frac{d}{dx}(5x^3-x^4)^7 = 7\left(5x^3-x^4\right)^6\frac{d}{dx}(5x^3-x^4) = 7\left(5x^3-x^4\right)^6(15x^2-4x^3)$
(d). $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} = -(3x-2)^{-2}\frac{d}{dx}(3x-2) = -3(3x-2)^{-2}$
(e). $\frac{d}{dx}(\sin^5 x) = 5(\sin^4 x)\frac{d}{dx}\sin x = 5(\cos x)\sin^4 x$
(f). $\frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x) = 6(1-2x)^{-4}$

19

.

Chapter 2 Derivatives
2.8 Parametric Equation

Used to describe the curve by expressing both coordinates as functions of third variable t. For example $x = f(t)$, and $y = g(t)$ over an interval of $t-value$, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve.

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

2.9 Linearization

We can approximate complicated functions with simpler ones that the accuracy we want for specific applications and are easier to work with by linearization L.

DEFINITION Linearization, Standard Linear Approximation

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x-a)$$

is the linearization of f at a. The approximation $f(x) \approx L(x)$ of f by L is the Standard Linear Approximation of f at a. the point $x = a$ is the center of the approximation.

20

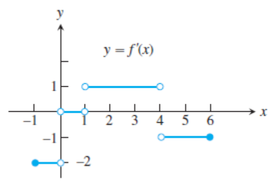
Chapter 2 Derivatives

Problem: Use the following information to graph the function $y = f(x)$ for $[-1,6]$

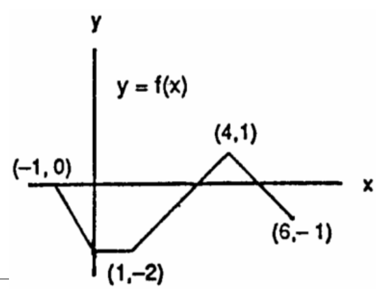
i. The graph of f is made of line segments joined end to end.

ii. The graph starts at the point (-1, 0).

iii. The derivative of f, where defined, agrees with the step function shown here.



Solution:



Chapter 2 Derivatives

Example: Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$

Solution:

The standard linear approximation equation is:

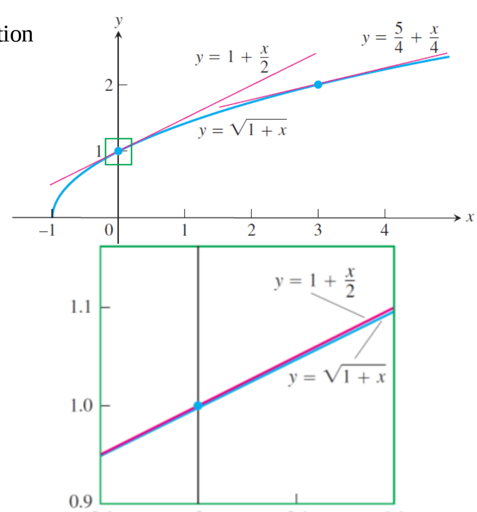
$$L(x) = f(a) + f'(a)(x-a)$$

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$f(0) = \sqrt{1+0} = 1$$

$$f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2}$$

$$\therefore L(x) = 1 + \frac{1}{2}x$$



Chapter 2 Derivatives

Approximation	True value	|True value − approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$< 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$< 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$< 10^{-5}$

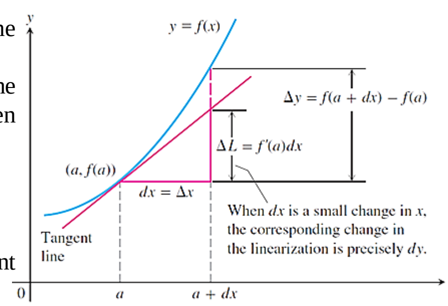
23

Chapter 2 Derivatives
2.10 Differential

- The function graphed $y = f(x)$ in the figure, let $x = a$, and set $dx = \Delta x$
- Geometrically, the differential dy is the change ΔL in the linearization of f when changes by an amount $dx = \Delta x$.

$$\Delta y = f(a + dx) - f(a)$$

- The corresponding change in the tangent line L *is:*



$$\Delta L = L_2 - L_1 \Rightarrow L_1 = f(a), L_2 = f(a) + f'(a)dx$$

$$\Delta L = L(a + dx) - L(a)$$

$$= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)}$$

$$\Delta L = f'(a)dx$$

24

Chapter 2 Derivatives
2.10 Differential

DEFINITION Differential
Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is

$$dy = f'(x)dx$$

Example: Find the value of dy when $x = 1$, and $dx = 0.2$ if $y = x^5 + 37x$
Solution:

$$\frac{dy}{dx} = 5x^4 + 37 \Rightarrow dy = (5x^4 + 37)dx = 8.4$$

Example: Find differential of functions

$a.$ $d(\tan 2x) = \sec^2 2x\, d(2x) = 2\sec^2 2x\, dx$

$b.$ $d\left(\frac{x}{x+1}\right) = \frac{(x+1)dx - x\, d(x+1)}{(x+1)^2} = \frac{(x+1)dx - x\, dx}{(x+1)^2} = \frac{dx}{(x+1)^2}$

25

Chapter 2 Derivatives
2.11 Estimating with Differential

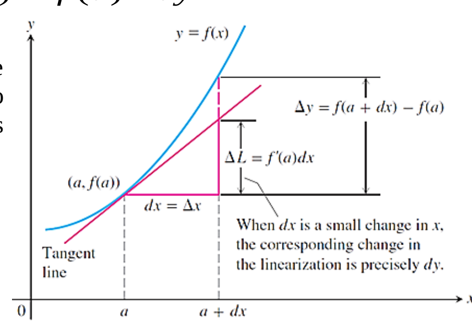
If $f(x)$is a differentiable at a point a and want to predict how much this value will change if we move to a nearby point $a + dx$. If dx is small, then we can see from Figure, that Δy is approximately equal to the differential dy. Since

$$f(a + dx) = f(a) + \Delta y$$

The differential approximation gives

$$f(a + dx) \approx f(a) + dy$$

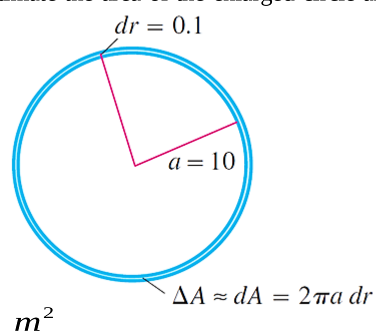
where $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to calculate $f(a + dx)$ when $f(x)$ is known and dx is small.



26

Chapter 2 Derivatives
2.11 Estimating with Differential

Example: The radius r of a circle increases from $a = 10\ m$ *to* $10.1\ m$ (Figure). Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area.



Solution:

$$A = \pi r^2 \Rightarrow dA = A'(r).dr$$

$$A'(r) = 2\pi r \ \ and$$

$$dr = r_2 - r_1 = 10.1 - 10 = 0.1$$

$$dA = 2\pi(10) * 0.1 = 2\pi$$

Thus the approximated area

$$A(r_1 + \Delta r) \approx A(10) + dA$$

$$\approx \pi(10)^2 + 2\pi = 102\pi \ \ m^2$$

The true area

$$A(10.1) = \pi(10.1)^2 = 102.01\pi \ \ m^2$$

The error of our estimate

$$\Delta A - dA = 102.01\pi - 102\pi = 0.01\pi \ \ m^2$$

Chapter 2 Derivatives
2.12 Implicit Differentiation

Implicit Differentiation

1. **Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.**
2. **Collect the terms with dy/dx on one side of the equation.**
3. **Solve for dy/dx.**

Example: Find dy/dx if $y^2 = x$

Solution:

$$2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Example: Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution:

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} = slope$$

The slope at $(3, -4)$ $= -\dfrac{3}{-4} = \dfrac{3}{4}$

Chapter 2 Derivatives
2.12 Implicit Differentiation

Example: Find $\boldsymbol{dy/dx}$ if $\boldsymbol{y^2 = x^2 + \sin xy}$
Solution:

$$2y\frac{dy}{dx} = 2x + \cos xy(x\frac{dy}{dx} + y)$$

$$\Rightarrow 2y\frac{dy}{dx} = 2x + x\cos xy\frac{dy}{dx} + y\cos xy$$

$$\Rightarrow 2y\frac{dy}{dx} - x\cos xy\frac{dy}{dx} = 2x + y\cos xy$$

$$\Rightarrow (2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - y\cos xy}{2y + x\cos xy}$$

29

Chapter 2 Derivatives
2.12 Implicit Differentiation

Problem: Find the slope of the curve $\boldsymbol{x = f(t)}$, $\boldsymbol{y = g(t)}$ at the given value of $\boldsymbol{t}$.

$$x^2 - 2tx + 2t^2 = 4,\ \ 2y^3 - 3t^2 = 4,\ \ t = 2$$

Solution:

$$x^2 - 2tx + 2t^2 = 4$$

$$\Rightarrow 2x\frac{dx}{dt} - 2t\frac{dx}{dt} - 2x + 4t = 0$$

$$\Rightarrow 2(x - t)\frac{dx}{dt} = 2(x - 2t)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x - 2t}{x - t}$$

At t=2

$$x^2 - 2(2)x + 2(2)^2 = 4$$

$$\Rightarrow x^2 - 4x + 4 = 0$$

$$\Rightarrow (x - 2)(x - 2) = 0 \Rightarrow x = 2$$

$$2y^3 - 3t^2 = 4$$

$$\Rightarrow 6y^2\frac{dy}{dt} - 6t = 0 \Rightarrow \frac{dy}{dt} = \frac{t}{y^2}$$

At t=2

$$2y^3 - 3(2)^2 = 4 \Rightarrow y^3 = 8$$

$$y = 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{t}{y^2} \times \frac{x - t}{x - 2t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{t(x - t)}{y^2(x - 2t)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(2 - 2)}{2^2(2 - 2(2))} = 0$$

30

Chapter 2 Derivatives
2.13 Related Rate

The problem of finding a rate you cannot measure easily from some other rates that you can is called a related rates problem.

Related Rates Problem Strategy

1. **Draw a picture and name the variables and constants. Use t for time. Assume that all variables are differentiable functions of t.**
2. **Write down the numerical information (in terms of the symbols you have chosen).**
3. **Write down what you are asked to find (usually a rate, expressed as a derivative).**
4. **Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate want to the variables whose rates you know.**
5. **Differentiate with respect to t. Then express the rate you want in terms of the rate and variables whose values you know.**
6. **Evaluate. Use known values to find the unknown rate.**

31

Chapter 2 Derivatives
2.13 Related Rate Class C

Example: How rapidly will the fluid level inside a vertical cylindrical tank of radius 1-meter drop if we pump the fluid out at the rate of $3000\ L/\text{min}$?
Solution:

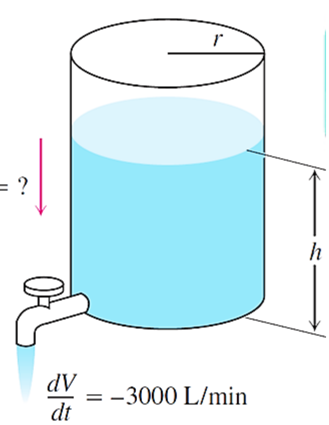
- Let r is the radius of the tank and h is the height and V is the volume of the fluid in the tank.
- At time passes, r remains constant, and h and V change.
- dh/dt =?

$$V = \pi r^2 h \quad \text{(r is a constant)}$$

$$\frac{dV}{dt} = -3000L/\min = -3m^3/\min$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-3}{\pi r^2} m/\min$$

$$if \quad r = 1m, \quad \frac{dh}{dt} \approx -0.95m/\min$$



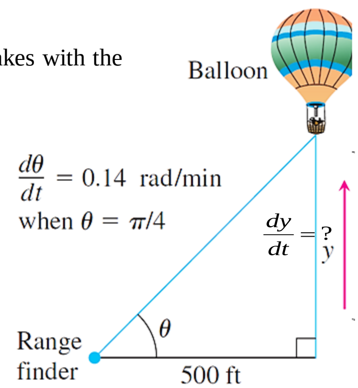
32

Chapter 2 Derivatives
2.13 Related Rate

Example: A hot air balloon rising straight up from a level field is tracked by a range finder $\mathbf{500\ ft}$ from the liftoff point. At the moment the range finder's elevation angle is $\boldsymbol{\pi/4}$, the angle is increasing at the rate of $\mathbf{0.14\ rad/min}$. How fast is the balloon rising at that moment?

Solution:

- Let $\boldsymbol{\theta}$ is the angle in radians the range finder makes with the ground, $\boldsymbol{y}$ is the height in feet of the balloon.



$$\tan\theta = \frac{y}{500} \Rightarrow y = 500\tan\theta$$

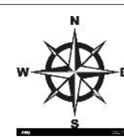
$$\frac{dy}{dt} = 500\sec^2\theta\frac{d\theta}{dt}$$

When

$$\theta = \frac{\pi}{4},\quad \frac{d\theta}{dt} = 0.14 rad/\min$$

$$\frac{dy}{dt} = 500\sec^2(\frac{\pi}{4})(0.14) = 140\ ft/\min$$

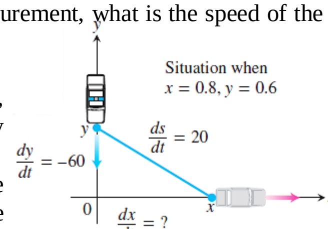
33



Chapter 2 Derivatives
2.13 Related Rate Class B

Example: A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is $\mathbf{0.6\ mi}$ north of the intersection and the car is $\mathbf{0.8\ mi}$ to the east, the police determine with radar that the distance between them and the car is increasing at $\mathbf{20\ mph}$. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution:



- We picture the car and cruiser in the coordinate plane, using the positive $\boldsymbol{x-axis}$ as the eastbound highway and the positive $\boldsymbol{y-axis}$ *as the southbound highway*.
- Let $\boldsymbol{x}$ is the position of the car at time $\boldsymbol{t}$, $\boldsymbol{y}$ is the position of the cruiser at time $\boldsymbol{t}$ and $\boldsymbol{s}$ is the distance between them.

$$s^2 = x^2 + y^2 \Rightarrow s = \sqrt{x^2+y^2} \Rightarrow \frac{ds}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{x}[\sqrt{x^2+y^2}\frac{ds}{dt} - y\frac{dy}{dt}]$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{0.8}[\sqrt{0.8^2+0.6^2}(20) - 0.6\times(-60)] \Rightarrow \frac{dx}{dt} = 70mph$$

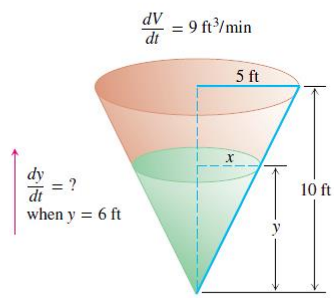
34

Chapter 2 Derivatives
2.13 Related Rate Class A

Example: Water runs into a conical tank at the rate of $9\ ft^3/min$. The tank stands point down and has a height of $10\ ft$ and a base radius of $5\ ft$. How fast is the water level rising when the water is $6\ ft$ deep?

Solution:

- Let V is the volume of the water in the tank at time t, x is the radius of the surface of the water tank at time t, and y is the depth of water in tank at time t.



- $\frac{dy}{dt}=?$ $\qquad V=\frac{1}{3}\pi x^2 y$
- In the similar triangle $\frac{x}{5}=\frac{y}{10}\Rightarrow x=\frac{y}{2}$

$$V=\frac{1}{3}\pi(\frac{y}{2})^2 y=\frac{\pi}{12}y^3\Rightarrow\frac{dV}{dt}=\frac{\pi}{4}y^2\frac{dy}{dt}$$

$$\Rightarrow\frac{dy}{dt}=\frac{9\times4}{\pi(6)^2}\approx 0.32\ ft/\min$$

35