*Kurdistan Regional Government-Iraq*
*Ministry of Higher Education and Scientific Research*
*Salahaddin University – Erbil*
*College of Engineering*
*Department of Architectural Engineering*

# Mathematic-I

**Spring Semester**
**Academic year (2023-2024)**
Four hours per week
Five Credits

**Prepared by: Ali A. Mahmod**
**Email: ali.mahmod@su.edu.krd**

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## Chapter 3 Application of Derivative

This chapter studies some of the important applications of derivatives. We learn how derivatives are used to find **extreme values of functions**.

### 3.1 Extreme Value of Functions

**Definition**

Let $f$ be a function with domain $D$. Then $f$ has an absolute maximum value on $D$ at a point $c$ if

$$f(x) \leq f(c) \quad for\ all\ x\ in\ D$$

And an absolute minimum value on $D$ at c if

$$f(x) \geq f(c) \quad for\ all\ x\ in\ D,$$

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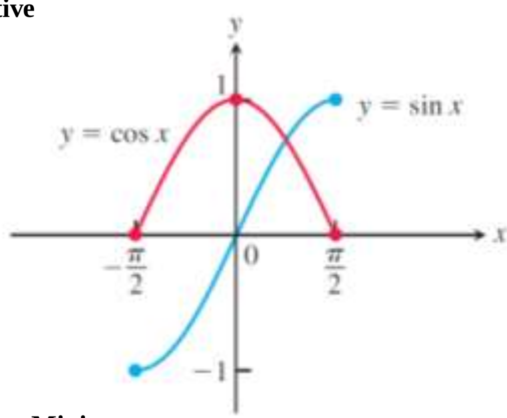
**Chapter 3 Application of Derivative**
**3.1 Extreme Value of Functions**

For example, on the closed interval $[\pi/2, -\pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of $\mathbf{1}$ (once) and an absolute minimum value of $\mathbf{0}$ (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of $\mathbf{1}$ and a minimum value of $-\mathbf{1}$ (Figure).




**3.1.1 Absolute Maximum, Absolute Minimum**

**Theorem 1 The Extreme Value Theorem**

**If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains both an absolute maximum value $M$ and an absolute minimum value $m$ in $[a, b]$. That is, there numbers $x_1$ and $x_2$ in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other $x$ in $[a, b]$.**

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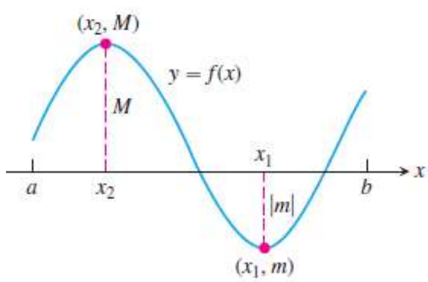
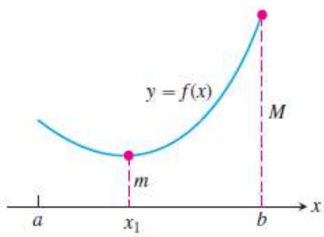
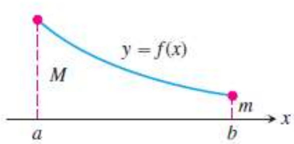
**Chapter 3 Application of Derivative**
**3.1.1 Absolute Maximum, Absolute Minimum**



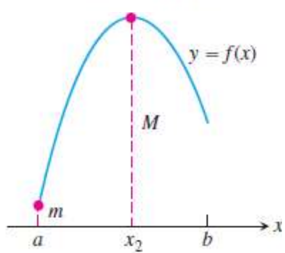

Maximum and minimum at interior points




Minimum at interior point, maximum at endpoint




Maximum and minimum at endpoints




Maximum at interior point, minimum at endpoint

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**Chapter 3 Application of Derivative**
**3.1.1 Absolute Maximum, Absolute Minimum**

Note:

- ❖ Absolute maximum value is the greatest value of $f$ on its interval.
- ❖ Absolute minimum value is the smallest value of $f$ on its interval.
- ❖ Absolute maximum and minimum values are called absolute **Extrema** (plural of the Latin extremum). Absolute extrema are also called **Global Extrema**.

For example

| Function Rule | Domain $D$ | Absolute Extrema on $D$ |
|---|---|---|
| $(a)\ y = x^2$ | $(-\infty, \infty)$ | No absolute maximum<br>Absolute minimum of $0$ at $x = 0$. |
| $(b)\ y = x^2$ | $[0, 2]$ | Absolute maximum of $4$ at $x = 2$.<br>Absolute minimum of $0$ at $x = 0$. |
| $(c)\ y = x^2$ | $(0, 2]$ | Absolute maximum of $4$ at $x = 2$.<br>No absolute minimum. |
| $(d)\ y = x^2$ | $(0, 2)$ | No absolute extrema. |

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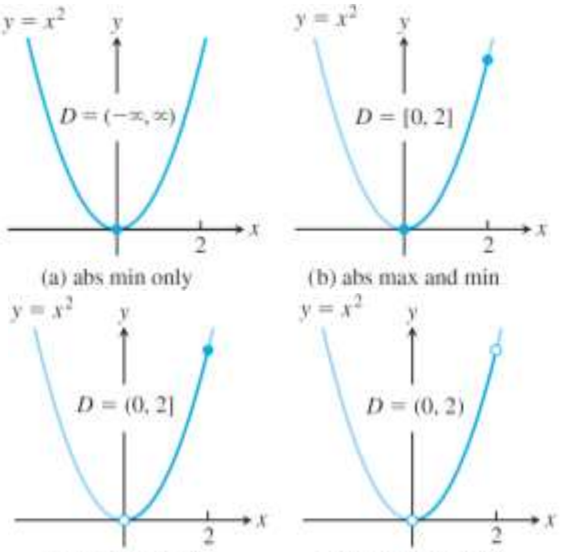
**Chapter 3 Application of Derivative**
**3.1.1 Absolute Maximum, Absolute Minimum**




(a) abs min only (b) abs max and min (c) abs max only (d) no max or min

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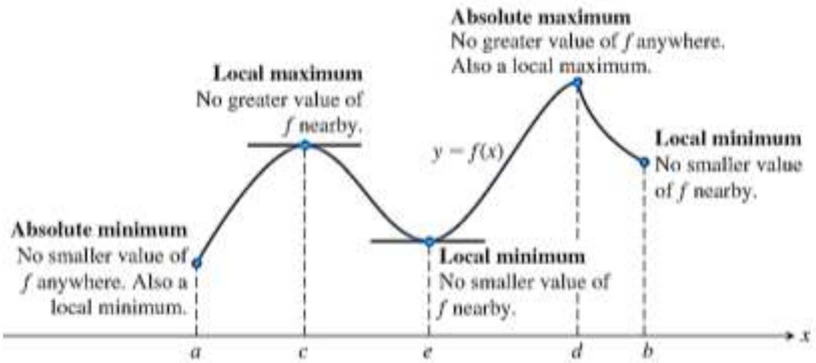
**Chapter 3 Application of Derivative**
**3.1.2 Local (Relative) Extreme Values**

**Definitions Local Maximum, Local Minimum**

A function $f$ has a local maximum value at an interior point $c$ of its domain if
$$f(x) \le f(c) \quad \text{for all } x \text{ in some open interval containing } c.$$

A function $f$ has a local minimum value at an interior point $c$ of its domain if
$$f(x) \ge f(c) \quad \text{for all } x \text{ in some open interval containing } c.$$

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**Chapter 3 Application of Derivative**
**3.1.3 Finding Extrema**

**Theorem 2 The First Derivative Theorem for Local Extreme Values**

If $f$ has a local maximum or minimum value at an interior point $c$ of its domain and if $f'$ is defined at $c$, then
$$f'(x) = 0$$

**Definition Critical Point**

An interior point of the domain of a function $f$ where $f'$ is <u>zero</u> or <u>undefined</u> is a critical point of $f$.

**Finding the Absolute Extrema of a Continuous Function $f$ on a Finite Closed Interval**

1. Evaluate $f$ at all critical points and endpoints.
2. Take the largest and smallest of these values.

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**Chapter 3 Application of Derivative**

Example: Find the absolute maximum and minimum values of $\boldsymbol{f(x) = x^2}$ on $[\boldsymbol{-2, 1}]$

Solution:

$$f(x) = x^2$$

$$f'(x) = 2x$$

To find the critical point, let $f'(x) = 0$

$$x = 0$$

The critical points value: $f(0) = 0$

Endpoint Values: $f(-2) = 4$

$$f(1) = 1$$

The function has an absolute maximum value of $\mathbf{4}$ at $\boldsymbol{x = -2}$ and an absolute minimum value of $\mathbf{0}$ at $\boldsymbol{x = 0}$.

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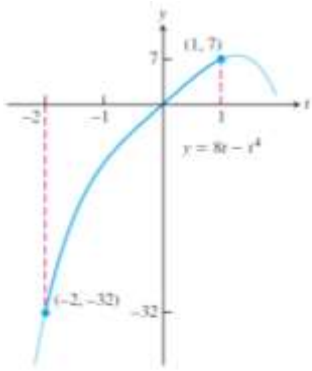
**Chapter 3 Application of Derivative**

Example: Find the absolute maximum and minimum values of $\boldsymbol{g(t) = 8t - t^4}$ on $[\boldsymbol{-2, 1}]$.




Solution:

$$g(t) = 8t - t^4$$

To find the critical point, let $g'(t) = 0$

$t = \sqrt[3]{2} > 1$ Out of the given domain

Endpoint Values: $g(-2) = -32$

$$g(1) = 7$$

The function has an absolute maximum value of $\mathbf{7}$ at t$=$ $\mathbf{1}$ and an absolute minimum value of $\mathbf{-32}$ at t$=$ $\mathbf{-2}$.

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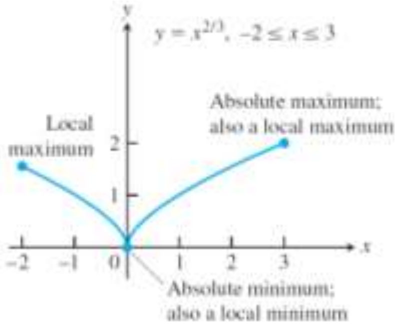
## Chapter 3 Application of Derivative

Example: Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on $[-\mathbf{2}, \mathbf{3}]$.

Solution:

$$f(x) = x^{2/3}$$

$$f'(x) = \frac{2}{3\sqrt[3]{x}}$$




To find the critical point, let $f'(x) = 0$

The derivative is undefined, and function has a critical point at $\boldsymbol{x = 0}$

The critical points value: $f(0) = 0$

Endpoint Values: $f(-2) = \sqrt[3]{4}$

$$f(3) = \sqrt[3]{9}$$

The function has an absolute maximum value of $\sqrt[3]{\mathbf{9}}$ at $\boldsymbol{x = 3}$ and an absolute minimum value of $\mathbf{0}$ at $\boldsymbol{x = 0}$.

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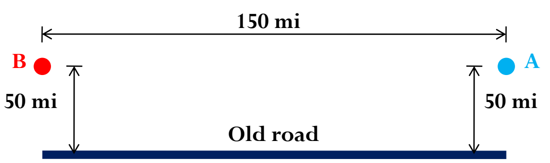
## Chapter 3 Application of Derivative

Example: A highway must be constructed to connect Village $\boldsymbol{A}$ with Village $\boldsymbol{B}$. There is a rudimentary roadway that can be upgraded $\mathbf{50}\ \boldsymbol{mi}$ south of the line connecting the two villages. The cost of upgrading the existing roadway is $\$\mathbf{300{,}000}\ \boldsymbol{per\ mile}$, whereas the cost of constructing a new highway is $\$\mathbf{500{,}000}\ \boldsymbol{per\ mile}$. Find the cost of connecting the two villages.

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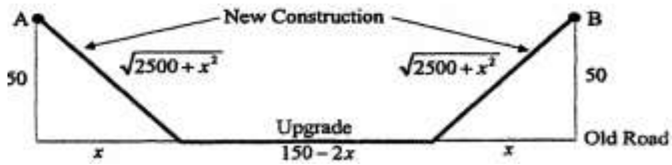
## Chapter 3 Application of Derivative

Solution:

1- Construct a new road between ***A*** and ***B*** directly.

$$C(x) = 500000 \times 150 = 75 \times 10^6 \ \ dallar$$

2- Combination of the two road is shown in the figure below:




$$0 \le x \le 75$$

$$C(x) = 300000(150 - 2x) + 500000(2\sqrt{2500 + x^2})$$

To find the minimum cost, we use local extrema

$$C'(x) = 300000(-2) + 500000(\frac{2x}{\sqrt{2500 + x^2}}) \Rightarrow C'(x) = 0$$

$$\frac{10}{6}x = \sqrt{2500 + x^2} \Rightarrow \frac{100}{36}x^2 = 2500 + x^2 \Rightarrow x = \pm 37.5mi \Rightarrow x = 37.5mi$$

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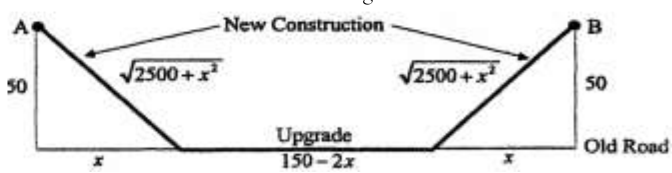
## Chapter 3 Application of Derivative

Solution:

2- Combination of the two road is shown in the figure below:




$$C(37.5) = 85 \times 10^6 \ \ dollar$$

The minimum cost to connect the two villages is **$75** million construction a new road between them.

$$C(75) = 90.139 \times 10^6 \ \ dollar$$

$$C(0) = 95 \times 10^6 \ \ dollar$$

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**Chapter 3 Application of Derivative**
**3.2 Monotonic Functions and The First Derivative Test**
**3.2.1 Increasing and Decreasing Function**

**Definitions Increasing, Decreasing Function**

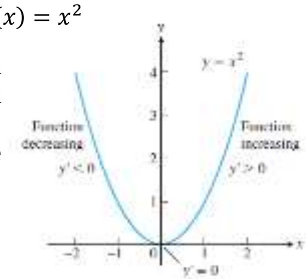
**Let $f$ be a function defined on an interval $I$ and let $x_1$ and $x_2$ by any two points in $I$.**

1. **If $f(x_1) < f(x_2)$ wherever $x_1 < x_2$, then $f$ is said to be increasing on $I$.**
2. **If $f(x_2) < f(x_1)$ wherever $x_1 < x_2$, then $f$ is said to be decreasing on $I$.**

**A function that is increasing or decreasing on $I$ is called monotonic on $I$.**

For example, in figure below, is the function graphed of $f(x) = x^2$

- The function decreases on $(-\infty, \mathbf{0}]$ and increases on $[\mathbf{0}, \infty)$ and the function is monotonic on $(-\infty, \mathbf{0}]$ and $[\mathbf{0}, \infty)$.
- On the interval $(-\infty, \mathbf{0}]$, the tangents have negative slope, so the first derivative is always negative.
- On the interval $[\mathbf{0}, \infty)$, the tangents have positive slope, so the first derivative is always positive.

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**Chapter 3 Application of Derivative**
**3.2 Monotonic Functions and The First Derivative Test**
**3.2.1 Increasing and Decreasing Function**

**Corollary 3 First Derivative Test for Monotonic Functions**

**Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.**

**If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ is increasing on $[a, b]$.**
**If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.**

Using first derivative for monotonic functions as follow:

1. Find the domain.
2. Determine the first derivative of the function.
3. Let first derivative $= \mathbf{0}$. And find the critical points.
4. Identify the intervals according to the critical points and test with the first derivative by substitution the value of $x$.

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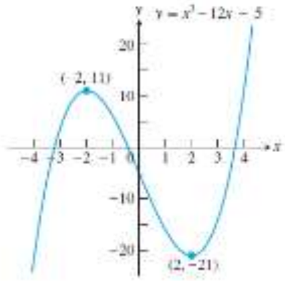
**Chapter 3 Application of Derivative**
**3.2 Monotonic Functions and The First Derivative Test**

Example: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which $f$ is increasing and decreasing?

Solution: the domain is $(-\infty, \infty)$

1. $f'(x) = 3x^2 - 12$
2. $3x^2 - 12 = 0$, to find the critical points
3. These critical points subdivide the domains of $f$ into intervals $x = \pm 2$




| Intervals | $x < -2$ | $-2 < x < 2$ | $x > 2$ |
|---|---|---|---|
| $f'$ Evaluated | $f'(-3) = 15$ | $f'(0) = -12$ | $f'(3) = 15$ |
| Sign of $f'$ | $+$ | $-$ | $+$ |
| Behavior of $f$ | increasing | decreasing | increasing |

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**Chapter 3 Application of Derivative**
**3.2 Monotonic Functions and The First Derivative Test**

**First Derivative Test for Local Extrema**

**Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself. Moving across $c$ from left to right,**

1. **If $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$,**
2. **If $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$,**
3. **If $f'$ does not change sign at $c$ (that is, $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local extremum at $c$.**

**Example:** Find the critical point of $f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$. Identify the intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.

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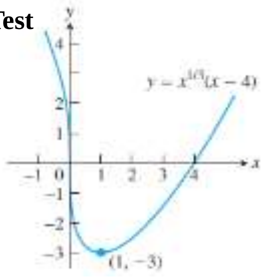
## Chapter 3 Application of Derivative
### 3.2 Monotonic Functions and The First Derivative Test



Solution:

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}}$$

The critical points $x = 0$ and $x = 1$.

| Intervals | $x < 0$ | $0 < x < 1$ | $x > 1$ |
|---|---|---|---|
| Sign of $f'$ | $-$ | $-$ | $+$ |
| Behavior of $f$ | decreasing | decreasing | increasing |

- The function is decreasing on the interval $(-\infty, 0)$, and is decreasing on the interval $(0, 1)$, and function is increasing on the interval $(1, \infty)$.
- The function $f$ does not have an extreme value at $x = 0$. ($f'$ does not change sign)
- The function $f$ has a local minimum at $x = 1$ ($f'$ changes from $-ve$ to $+ve$).
- The value of the local minimum is $f(1) = -3$ .

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## Chapter 3 Application of Derivative
### 3.3 Concavity

The curve of the function graphed $f(x) = x^3$ rises as $x$ increases. As we approaches the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slope of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slope of the tangents are increasing on the interval $(0, \infty)$.

**Definition Concave Up, Concave Down**

**The graph of a differentiable function $y = f(x)$ is**

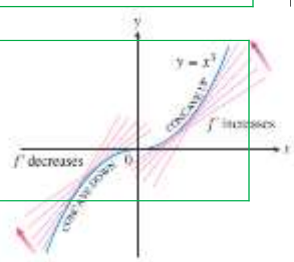
**a) Concave up on an open interval $I$ if $f'$ is increasing on $I$.**

**b) Concave down on an open interval $I$ if $f'$ is decreasing on $I$.**

**The Second Derivative Test for Concavity**

**Let $y = f(x)$ be twice-differentiable on an interval $I$,**

1. **If $f'' > 0$ on $I$, the graph of $f$ over $I$ is concave up.**
2. **If $f'' < 0$ on $I$, the graph of $f$ over $I$ is concave down.**

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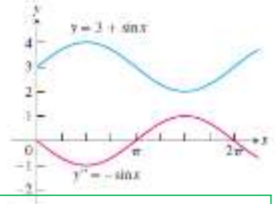
**Chapter 3 Application of Derivative**
**3.4 Point of Inflection**

The curve in the figure below changes concavity at $(\pi, 3)$.We call the point $(\pi, 3)$ a point of inflection of the curve.

**Definition Point of Inflection**

**A point where the graph of a function has a tangent line and where the concavity changes is a point of inflection**

Note:
If $y$ is a twice-differentiable function, $y'' = 0$ at a point of inflection and has a local maximum or minimum.




**Theorem 5 Second Derivative Test for Local Extrema**

**Suppose $f''$ is a continuous on an open interval that contains $x = c$.**

1. **If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.**
2. **If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $x = c$.**
3. **If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The inflection $f$ may have a local maximum, a local minimum, or neither.**

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Absolute Extrema: Absolute Max. and Absolute Min.
If first **derivative equals to zero**, the function has local max or min.
There is a **critical point if f' is zero or undefined**.
If **f'** changes from **neg to pos**, local **min**
If f' changes from **pos to neg**, local **max**
If f'=0, f"<0, local max and f'=0, f">0, local min
Monotonic function means increasing and decreasing fun.
If f'>0 means f increasing, and f'<0 means function is decreasing.
If f' is increasing means concave up, and decreasing means concave down
If second derivative < 0 concave down and >0 concave up
If f"=0, inflection point

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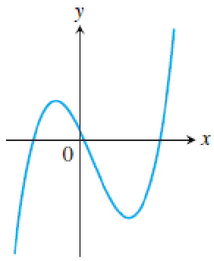
## Chapter 3 Application of Derivative

Problem: Identify the inflection points and local maxima and minima of the function. And identify the intervals on which the function is concave up and concave down.

$$y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$$




Solution:

$$y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$$

$$y' = x^2 - x - 2$$

$$y'' = 2x - 1$$

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## Chapter 3 Application of Derivative

The critical point: $y' = x^2 - x - 2 = 0 \Rightarrow x = 2, x = -1$

| Interval | $x < -1$ | $-1 < x < 2$ | $x > 2$ |
|---|---|---|---|
| Sign of $y'$ | + | - | + |
| Behavior of f | Increasing | Decreasing | Increasing |
| | | Local maximum at x=-1 | Local minimum at x=2 |

The inflection point: $y'' = 2x - 1 \Rightarrow x = \frac{1}{2}$

| Interval | $x < 1/2$ | $x > 1/2$ |
|---|---|---|
| Sign of $y''$ | - | + |
| Behavior of y | Concave down | Concave up |

| $x < -1$ | $-1 < x < 1/2$ | $1/2 < x < 2$ | $x > 2$ |
|---|---|---|---|
| Increasing | Decreasing | Decreasing | increasing |
| Concave down | Concave down | Concave up | Concave up |

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**Chapter 3 Application of Derivative**
**3.6 Applied Optimization Problems**

- Optimization something means to maximize or minimize some aspect of it.
- We use The differential calculus is a powerful tool for solving problems that call for maximizing or minimizing a function.
- For example, we can determine what are the dimensions of a rectangle with fixed perimeter having maximum area.

**Solving applied optimization problems**

1. Read the problem. What is given? What is the unknown quantity to be optimized.
2. Draw an illustrated picture for the problem.
3. Introduce all known and unknown variables, list every relation in the picture.
4. Write an equation for the unknown quantity.
5. Test the critical points and endpoints of the domain of unknown.
6. Use the first and second derivative to identify and classify the function's critical points.

**Example:** An open-top box is to be made by cutting small congruent squares from the corners of a $\mathbf{12cm}$ by $\mathbf{12cm}$ sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

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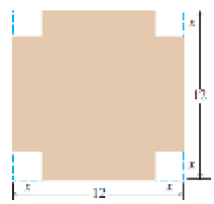
**Chapter 3 Application of Derivative**
**3.6 Applied Optimization Problems**

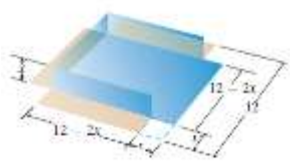
**Solution:** Let $\boldsymbol{x}$ is the cutout square on a side, and $\boldsymbol{V}(\boldsymbol{x})$ is volume of the box.



$$V(x) = x(12-2x)(12-2x) = 144x - 48x^2 + 4x^3$$

- The domain: $0 \le x \le 6$
- First derivative and second derivative:

$$V' = 144 - 96x + 12x^2$$
$$V'' = -96 + 24x$$



- Find the critical point:

$$V' = 0 \Rightarrow 144 - 96x + 12x^2 = 0 \Rightarrow x = 2, x = 6$$

- Substitute the values of $\boldsymbol{x}$ in the second derivative equation

$$V''(2) = -96 + 24(2) = -48$$
$$V''(6) = -96 + 24(6) = 48$$

The function has local maximum at $\boldsymbol{x} = \mathbf{2}$. the cutout square should be 2 cm. on a side.

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**Chapter 3 Application of Derivative**
**3.6 Applied Optimization Problems**

Example: You have been asked to design a $\mathbf{1}$ $\boldsymbol{liter}$ can shaped like a right circular cylinder. What dimensions will use the least material?

Solution:

- $\boldsymbol{r}$ and $\boldsymbol{h}$ are the radius and height of the cylinder respectively in $\boldsymbol{cm}$.
- To design a cylinder with least material, we ignore the thickness.
- Let A is the total surface of the cylinder.

$$A = Circular\ \ wall(2\pi rh) + Circular\ \ ends(2\pi r^2)$$

$$A = 2\pi rh + 2\pi r^2$$

$$Volume = \pi r^2 h \Rightarrow 1000cm^3 = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$$

$$\therefore A = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 \Rightarrow A = \frac{2000}{r} + 2\pi r^2$$

- First and second derivative:

$$\frac{dA}{dr} = \frac{-2000}{r^2} + 4\pi r \quad \text{and} \quad \frac{d^2A}{dr^2} = \frac{4000}{r^3} + 4\pi$$

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**Chapter 3 Application of Derivative**
**3.6 Applied Optimization Problems**

Example: You have been asked to design a 1-liter can shaped like a right circular cylinder (Figure 4.34). What dimensions will use the least material?

Solution:

- Critical points:

$$\frac{dA}{dr} = 0 \Rightarrow \frac{-2000}{r^2} + 4\pi r = 0 \Rightarrow r = \sqrt{\frac{500}{\pi}} \approx 5.42$$

- Check for local minimum:

$$\frac{d^2A}{dr^2}(5.42) = +ve$$

The function has local minimum value at $\boldsymbol{r} = \mathbf{5.42}$

$$\therefore h = \frac{1000}{\pi r^2} = 10.84$$

To design a cylinder satisfying the least material, we use the dimensions of the cylinder are.

$$h = 10.84\ \ cm,\ \ r = 5.42\ \ cm$$

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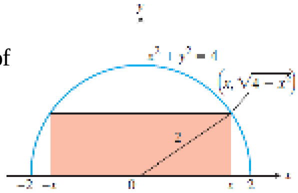
**Chapter 3 Application of Derivative**
**3.6 Applied Optimization Problems**

Example: A rectangle is to be inscribed in a semicircle of radius **2**. What is the largest area the rectangle can have, and what are its dimensions?

Solution:




- Let length $= 2x$, height $= \sqrt{4-x^2}$ and $A$ is the area of rectangular.
- The domain is $0 \le x \le 2$.

$$A = 2x \cdot \sqrt{4-x^2} = 2\sqrt{4x^2 - x^4}$$

- First and second derivative:

$$\frac{dA}{dx} = \frac{(8x-4x^3)}{\sqrt{4x^2-x^4}} \quad \text{and} \quad \frac{d^2A}{dx^2} = \frac{(4x^2-x^4)(8-12x^2)-(8x-4x^3)^2}{(4x^2-x^4)^{3/2}}$$

- Critical point:

$$\frac{dA}{dx} = 0 \Rightarrow \frac{(8x-4x^3)}{\sqrt{4x^2-x^4}} 0 \Rightarrow 8x - 4x^3 = 0 \Rightarrow x = \pm\sqrt{2} \Rightarrow x = \sqrt{2}$$

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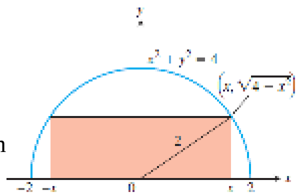
**Chapter 3 Application of Derivative**
**3.6 Applied Optimization Problems**

Example: A rectangle is to be inscribed in a semicircle of radius **2**. What is the largest area the rectangle can have, and what are its dimensions?

Solution:




- Check for local maximum:

$$\frac{d^2A}{dx^2}(\sqrt{2}) = -ve$$ The function has local maximum value at $x = \sqrt{2}$.

- Rectangular dimensions

$$long = 2x = 2\sqrt{2}\ unit$$

$$height = \sqrt{4-2} = \sqrt{2}\ unit$$

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