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Ministry of Higher Education and Scientific Research
Salahaddin University – Erbil
College of Engineering
Department of Architecture Engineering

Mathematic-I

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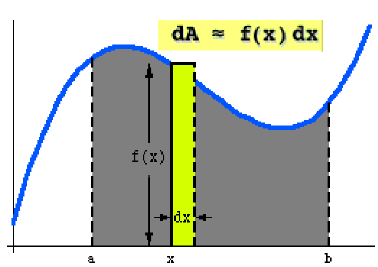
1

Chapter 4 Integral

- Is a tool for calculating much more than areas and volumes.
- Has many applications in statistics, economics, the sciences, and engineering.
- Allows us to calculate quantities ranging from probabilities and averages to energy consumption and the forces against a dam’s floodgates.
- The idea behind integration is that we can effectively compute many quantities by breaking them into small pieces, and then summing the contributions from each small part.



2

Chapter 4 Integral
4.1 Antiderivative

> **If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is**
>
> $$F(x)+C$$
>
> **Where C is an arbitrary constant.**

Example: Find an antiderivative for each of the following functions.

a. $f(x)=2x$

b. $g(x)=\cos x$

c. $h(x)=2x+\cos x$

Solution:

a. $F(x)=x^2+c$

b. $G(x)=\sin x+c$

c. $H(x)=x^2+\sin x+c$

3

Chapter 4 Integral

Antiderivative Formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1}+C, \quad n\neq -1, \quad n \text{ rational}$
2.	$\sin kx$	$-\frac{\cos kx}{k}+C, \quad k \text{ a constant}, k\neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k}+C, \quad k \text{ a constant}, k\neq 0$
4.	$\sec^2 x$	$\tan x+C$
5.	$\csc^2 x$	$-\cot x+C$
6.	$\sec x\tan x$	$\sec x+C$
7.	$\csc x\cot x$	$-\csc x+C$

4

Chapter 4 Integral
4.2 Indefinite Integral

Definition　Indefinite Integral, Integrand

The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)\,dx$$

The symbol $\int$ is an integral sign. The function f is the integrand of the integral, and x is the variable of integration.

Example: Evaluate

$$\int (x^2 - 2x + 5)dx$$

Solution:

$$\int (x^2 - 2x + 5)dx = \underbrace{\frac{x^3}{3} - x^2 + 5x}_{\text{Antiderivative}} + \underset{\text{Arbitrary Constant}}{C}$$

5

Chapter 4 Integral
4.2 Indefinite Integral

Theorem1　The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

6

Chapter 4 Integral
4.2 Indefinite Integral

$\cos 2\theta = 1 - 2\sin^2\theta$

Example: Evaluate

$$\int \sin^2 x \, dx$$

Solution:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \Rightarrow \int \sin^2 x \, dx = \frac{1}{2}\int (1-\cos 2x)dx$$

$$= \frac{x}{2} - \frac{1}{2}\int \cos 2x)dx$$

$u = 2x$

$1/2du = dx$

$$\therefore \frac{x}{2} - \frac{1}{2}\int (\cos 2x)dx = \frac{x}{2} - \frac{1}{2}\int \cos u \ (\frac{1}{2} \ du)$$

$$= \frac{x}{2} - \frac{1}{4}\sin u + c$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$$

7

Chapter 4 Integral
4.2 Indefinite Integral

$\cos 2\theta = 2\cos^2\theta - 1$

Example: Evaluate

$$\int \cos^2 x \, dx$$

Solution:

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \Rightarrow \int \cos^2 x \, dx = \frac{1}{2}\int (1+\cos 2x)dx$$

$$= \int \frac{1}{2} \, dx + \frac{1}{2}\int \cos 2x)dx$$

$u = 2x$

$1/2du = dx$

$$\therefore \int \frac{1}{2} \, dx + \frac{1}{2}\int (\cos 2x)dx = \int \frac{1}{4} \, du + \frac{1}{4}\int (\cos u)du$$

$$= \frac{u}{4} + \frac{1}{4}\sin u + c$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + c$$

8

Chapter 4 Integral
4.2 Indefinite Integral

Example: Evaluate

$$a.\ \int \sqrt{1+y^2}.2y\ dy \quad \Rightarrow \quad let\ u = 1+y^2$$

$$du = 2y\ dy$$

$$\therefore \int \sqrt{1+y^2}.2y\ dy = \int \sqrt{u}\ du$$

$$= \int u^{\frac{1}{2}}\ du$$

$$= \frac{(u)^{(\frac{1}{2}+1)}}{(\frac{1}{2}+1)} + c$$

$$= \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(1+y^2)^{\frac{3}{2}} + c$$

9

Chapter 4 Integral
4.2 Indefinite Integral

Example: Evaluate

$$b.\ \int \sqrt{4t+1}\ dt \quad \Rightarrow \quad let\ u = 4t+1$$

$$du = 4dt \Rightarrow dt = \frac{1}{4}du$$

$$\therefore \int \sqrt{4t+1}\ dt = \int \sqrt{u}\ (\frac{1}{4})du$$

$$= \frac{1}{4}\int u^{\frac{1}{2}}\ du$$

$$= \frac{1}{4}.\frac{(u)^{(\frac{1}{2}+1)}}{(\frac{1}{2}+1)} + c$$

$$= \frac{1}{6}u^{\frac{3}{2}} + c = \frac{1}{6}(4t+1)^{\frac{3}{2}} + c$$

10

Chapter 4 Integral
4.2 Indefinite Integral

Example: Evaluate

$$c.\ \int \cos(7t+5)\ dt \quad \Rightarrow \quad let\ u = 7t+5$$

$$du = 7dt \Rightarrow dt = \frac{1}{7}du$$

$$\therefore \int \cos(7t+5)\ dt = \int \cos u\ (\frac{1}{7})du$$

$$= \frac{1}{7}\int \cos u\ du$$

$$= \frac{1}{7}\sin u + c = \frac{1}{7}\sin(7t+5) + c$$

11

Chapter 4 Integral
4.2 Indefinite Integral

Example: Evaluate

$$d.\ \int x^2 \sin(x^3)\ dx \quad \Rightarrow \quad let\ u = x^3$$

$$du = 3x^2dx \Rightarrow x^2dx = \frac{1}{3}du$$

$$\therefore \int x^2 \sin(x^3)\ dx = \int \sin u\ (\frac{1}{3})du$$

$$= \frac{1}{3}\int \sin u\ du$$

$$= -\frac{1}{3}\cos u + c = -\frac{1}{3}\cos(x^3) + c$$

12

Chapter 4 Integral
4.2 Indefinite Integral

Example: Evaluate

$$e.\ \int \frac{1}{\cos^2 2x}\ dx = \int \sec^2 2x\ dx$$

$u = 2x$
$1/2du = dx$

$$= \frac{1}{2}\int \sec^2 u\ du$$

$$= \frac{1}{2}\tan u + c = \frac{1}{2}\tan(2x) + c$$

13

Chapter 4 Integral
4.2 Indefinite Integral

Example: Evaluate

$$f.\ \int \frac{2x}{\sqrt[3]{x^2+1}}\ dx \quad \Rightarrow \quad let\ u = x^2 + 1$$

$$du = 2xdx$$

$$\therefore \int \frac{2x}{\sqrt[3]{x^2+1}}\ dx = \int u^{-\frac{1}{3}} du$$

$$= \frac{3}{2}\ u^{\frac{2}{3}} + c = \frac{3}{2}\ (x^2+1)^{\frac{2}{3}} + c$$

14

Chapter 4 Integral
4.2 Indefinite Integral

Example: The acceleration of gravity near the surface of earth is **9.8** meter per second square. Find velocity?

Solution:

$$a(t) = \frac{dv}{dt} = 9.8\ m/\sec^2 \Rightarrow dv = 9.8\ dt$$

- Take integration for both sides: $\int dv = \int 9.8\ dt$

$$v = 9.8t + c$$

$$v(t) = 9.8t + c$$

- where $\boldsymbol{t = 0}, \boldsymbol{v = 0}$

$$0 = 9.8(0) + c \Rightarrow c = 0$$

- the velocity at time $\boldsymbol{t}$: $v(t) = 9.8t$

15

Chapter 4 Integral
4.2 Indefinite Integral

Example: Solve $\frac{dy}{dx} = \sqrt{\frac{x+1}{y-1}}$

$$\sqrt{y-1}\ dy = \sqrt{x+1}\ dx$$

$$(y-1)^{1/2} dy = (x+1)^{1/2}\ dx$$

Let $u = y-1$ $\quad z = x+1$

$du = dy$ $\quad dz = dx$

$$\therefore (y-1)^{1/2} dy = (x+1)^{1/2}\ dx \Rightarrow u^{1/2} du = z^{1/2}\ dz$$

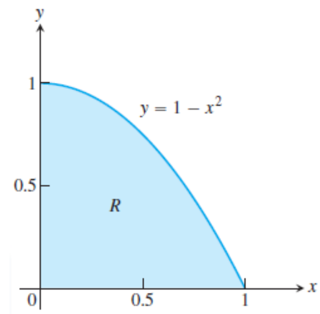
$$\int u^{1/2} du = \int z^{1/2}\ dz \Rightarrow \frac{2}{3} u^{3/2} = \frac{2}{3} z^{3/2} + c$$

$$(y-1)^{3/2} = (x+1)^{3/2} + c$$

16

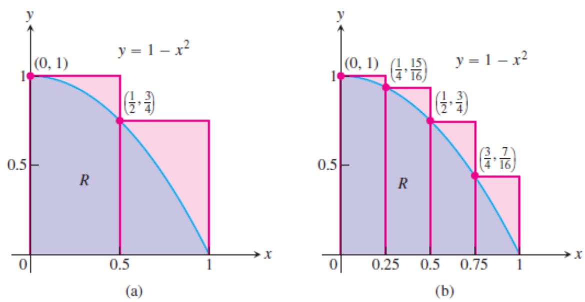
Chapter 4 Integral
4.3 Estimating with Finite Sums

Example: What is the area of the shaded region $\boldsymbol{R}$ that lies above the **x-axis**, below the graph of $\boldsymbol{y = 1 - x^2}$ and between the vertical lines $\boldsymbol{x = 0}$ and $\boldsymbol{x = 1}$? (See Figure)



Solution:

➢ We can calculate the area of the region $\boldsymbol{R}$ approximately by dividing the region into rectangular.



17

Chapter 4 Integral
4.3 Estimating with Finite Sums

Solution:

➢ If $\boldsymbol{w}$ is the width of rectangle, and $\boldsymbol{n}$ is the number of subinterval, so

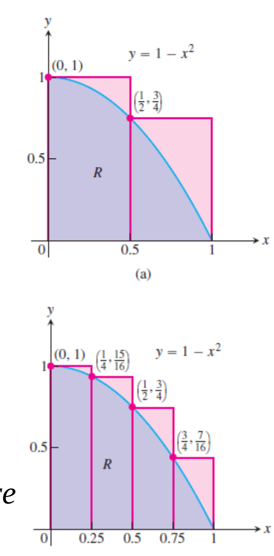
$$w = \frac{x_n - x_0}{n}$$

➢ For figure ($\boldsymbol{a}$), $\boldsymbol{w = 0.5}$, $\boldsymbol{n = 2}$. The area is:

$$A = 1 \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2} = \frac{7}{8} = 0.875 \ unit \ square$$

➢ For figure ($\boldsymbol{b}$), $\boldsymbol{w = 0.25}$, $\boldsymbol{n = 4}$. The area is:

$$A = \frac{1}{4}\left(1 + \frac{15}{16} + \frac{3}{4} + \frac{7}{16}\right) = \frac{25}{32} = 0.78125 \ unit \ square$$



18

Chapter 4 Integral
4.4 Sigma Notation and Limits of Finite Sums

Sigma Notation

- Writing a sum with many terms in the compact form, **sigma notation** used:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

- The Greek letter Σ (capital sigma, corresponding to our letter **S**), stands for **"sum"**.
- k is the index of summation and tells where the sum begins.

$$\sum_{k=1}^{n} a_k$$

The summation symbol (Greek letter sigma)

The index k ends at $k = n$

a_k is a formula for kth term

The index k starts at $k = 1$

19

Chapter 4 Integral
4.4 Sigma Notation and Limits of Finite Sums

Example: Using Sigma Notation

The sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1+2+3+4+5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1+2-3=-2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

20

Chapter 4 Integral
4.4 Sigma Notation and Limits of Finite Sums

Algebra Rules for Finite Sums

1. ***Sum Rule***: $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. ***Difference Rules***: $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. ***Constant Multiple Rule***: $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (***Any number c***)

4. ***Constant Value Rule***: $\sum_{k=1}^{n} c = n \cdot c$ (***c is any constant value***)

5. ***The first n squares***: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

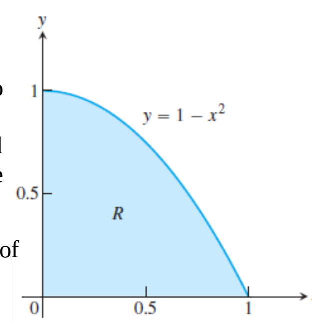
6. ***The first n cubes***: $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

21

Chapter 4 Integral
4.4 Sigma Notation and Limits of Finite Sums

Example:

Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x-axis using equal width rectangles whose widths approach zero and whose number approaches infinitely.



Solution:

- If w is the width of rectangle, and n is the number of subinterval, so

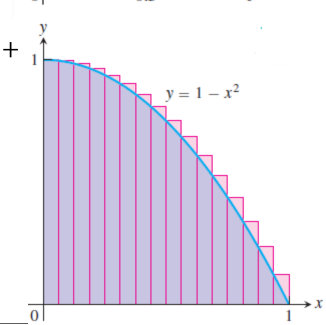
$$w = \frac{\Delta x}{n} = \frac{1}{n}$$

- So, the area can be calculated as: $A = A_1 + A_2 + A_3 + \cdots + A_{n-1} + A_n$

$$A_1 = y_1 \cdot w = \left(1 - \left(\frac{1}{n}\right)^2\right) \cdot \frac{1}{n}$$

$$A_2 = y_2 \cdot w = \left(1 - \left(\frac{2}{n}\right)^2\right) \cdot \frac{1}{n}$$

$$A_n = y_n \cdot w = \left(1 - \left(\frac{n}{n}\right)^2\right) \cdot \frac{1}{n}$$



22

Chapter 4 Integral
4.4 Sigma Notation and Limits of Finite Sums

Solution:

- It can be expressed this in sigma notation as follow:

$$\sum_{k=1}^{n} f(x)\cdot w = \sum_{k=1}^{n}\left(1-\left(\frac{k}{n}\right)^2\right)\left(\frac{1}{n}\right)$$

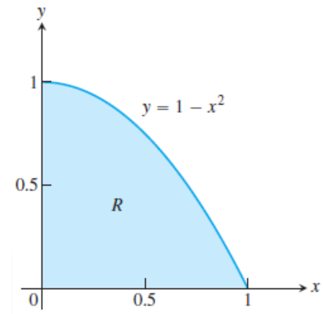
$$= \sum_{k=1}^{n}\left(1-\left(\frac{k}{n}\right)^2\right)\left(\frac{1}{n}\right)$$

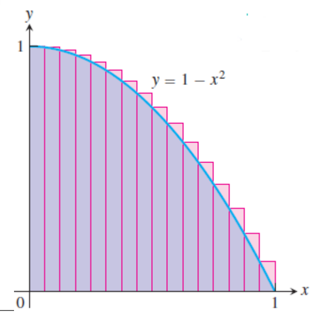
$$= \sum_{k=1}^{n}\left(\frac{1}{n}-\frac{k^2}{n^3}\right) = \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$= n\cdot\frac{1}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= 1 - \left(\frac{1}{n^3}\right)\frac{n(n+1)(2n+1)}{6}$$

$$= 1 - \frac{2n^3+3n^2+n}{6n^3}$$





Chapter 4 Integral
4.4 Definite Integral

- Suppose the function $y = f(x)$, is nonnegative and continuous. The area of region enclosed $[a,b]$ by the graph of $f(x)$, the $x - axis$ and the vertical lines $x = a$, and $x = b$.
- If we take an element as shown in the figure,

$$h = y_2 - y_1$$

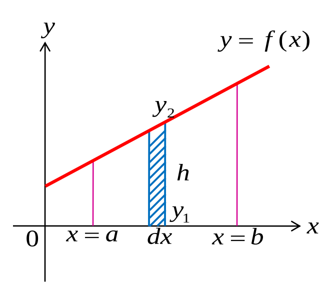
$$dA = h\ dx$$

$$dA = (y_2 - y_1)\ dx$$

$$= F(x)\ dx$$

$$A = \int dA = \int F(x)\ dx$$

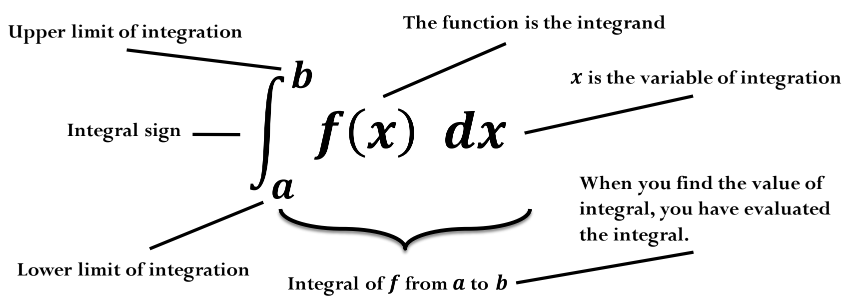
$$A = \int_a^b f(x)\ dx$$



Chapter 4 Integral
4.4 Definite Integral

Theorem 2 The Existence of Definite Integrals
A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

5.4.1 Notation and Existence of The definite Integral



Chapter 4 Integral
4.4 Definite Integral

Theorem 3
When f and g are integrable, the definite integral satisfies Rules 1 to 7

1. ***Order of Integration***:	$\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. ***Zero With Interval***:	$\int_a^a f(x)dx = 0$
3. ***Constant Multiple***:	$\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. ***Sum and Difference***:	$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Chapter 4 Integral
4.4 Definite Integral

Theorem 4

When f and g are integrable, the definite integral satisfies Rules $\mathbf{1}$ to $\mathbf{7}$

5. ***Additivity***: $$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

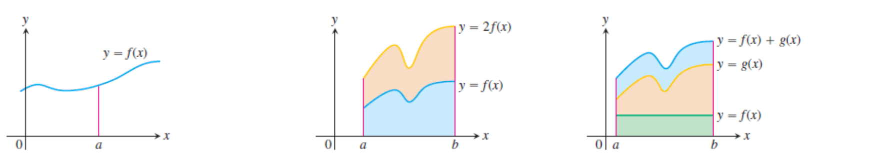
6. ***Max − Min Inequality***: If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)dx \le \max f \cdot (b-a)$$

7. ***Domination***: $$f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)dx$$

27

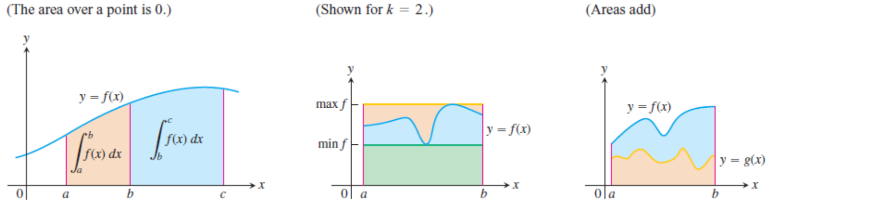
Chapter 4 Integral
4.4 Definite Integral



(a) *Zero Width Interval:* $\int_a^a f(x)\,dx = 0.$ (The area over a point is 0.)

(b) *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$ (Shown for $k = 2$.)

(c) *Sum:* $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ (Areas add)



(d) *Additivity for definite integrals:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(e) *Max-Min Inequality:* $\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$

(f) *Domination:* $f(x) \ge g(x)$ on $[a, b]$ $\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

28

Chapter 4 Integral
4.4 Definite Integral

Theorem 5 The Mean Value Theorem for Definite Integrals

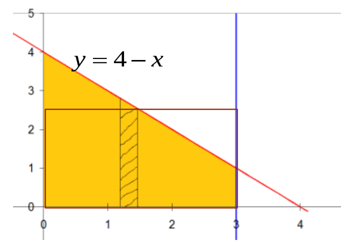
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example: Find the average value of $f(x) = 4 - x$ on $[0, 3]$.

Solution:

$$av(f) = \frac{1}{3}\int_0^3 (4-x)dx$$
$$= \frac{1}{3}\left|\left[4x - \frac{x^2}{2}\right|_0^3\right| = \frac{5}{2}$$



29

Chapter 4 Integral
4.4 Definite Integral

Theorem 6 Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

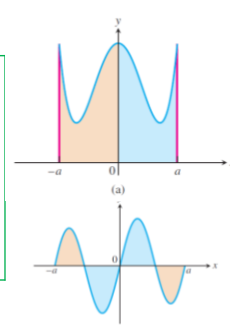
$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) *if f is even, then* $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.

(b) *if f is odd, then* $\int_{-a}^{a} f(x)\, dx = 0$.



30