Kurdistan Regional Government-Iraq
Ministry of Higher Education and Scientific Research
Salahaddin University – Erbil
College of Engineering
Department of Architecture Engineering

Mathematic-I

Spring Semester
Academic year (2023-2024)
Four hours per week
Five Credits

Prepared by: Ali A. Mahmod, M.Sc.
Email: ali.mahmod@su.edu.krd

Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Definition Area Under a Curve as a Definite Integral

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx$$

Example: Find the area under the line $y = x$ over the interval $[0, b]$, $b > 0$.

Solution:

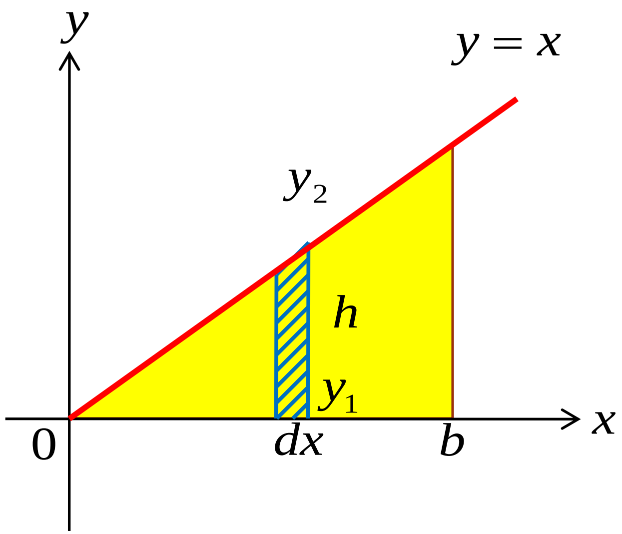
- Graph the function.

$$dA = h\ dx$$

$$dA = \int_0^b (y_2 - y_1)\ dx$$

$$A = \int_0^b (x - 0)\ dx$$

$$A = \left.\frac{x^2}{2}\right|_0^b = \frac{b^2}{2} - 0 = \frac{1}{2}b^2$$

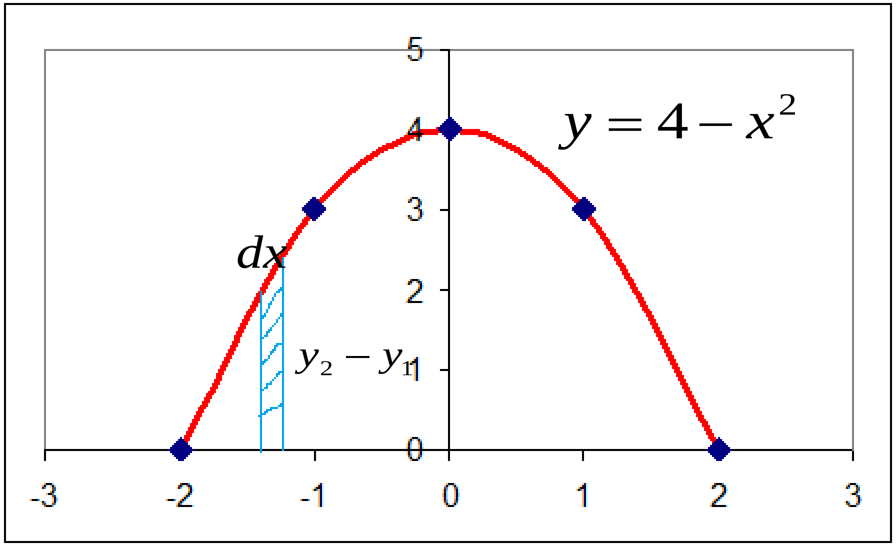


Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Example: Find the area between $\boldsymbol{x-axis}$ and the curve $\boldsymbol{y=4-x^2}$ for $\boldsymbol{-2 \le x \le 2}$.

Solution:

- Graph the function.
- The Area



$$dA=(y_2-y_1)\ dx$$

$$dA=(4-x^2)\ dx$$

$$A=\int_{-2}^{0}(4-x^2)\ dx+\int_{0}^{2}(4-x^2)\ dx\ or$$

$$A=\int_{-2}^{2}(4-x^2)\ dx\ or$$

$$A=2\int_{0}^{2}(4-x^2)\ dx\ or$$

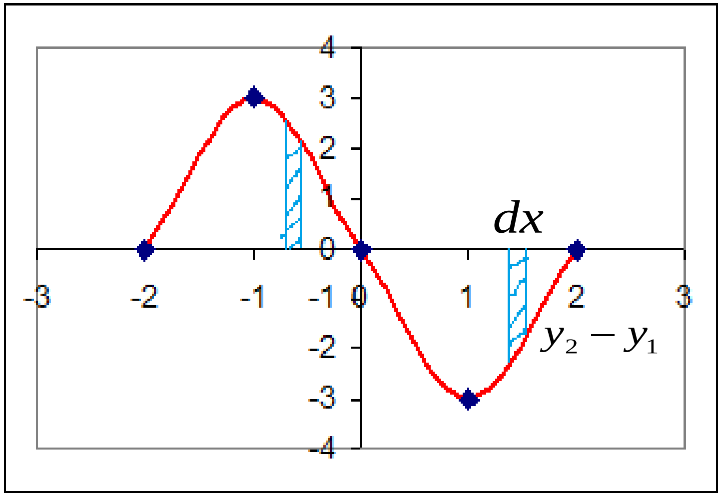
$$A=2\left|[4x-\frac{x^3}{3}]\Big|_0^2\right|=2\left|[4(2)-\frac{2^3}{3}]\right|=\frac{32}{3}\ unit\ square$$

Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Example: Find the area between $\boldsymbol{x-axis}$ and the curve $\boldsymbol{y = x^3 - 4x}$ for $\boldsymbol{-2 \le x \le 2}$.

Solution:

- Graph the function.
- The Area



$$dA = (y_2 - y_1)\,dx$$

$$dA = (x^3 - 4x)\,dx$$

$$A = 2\int_0^2 (x^3 - 4x)dx$$

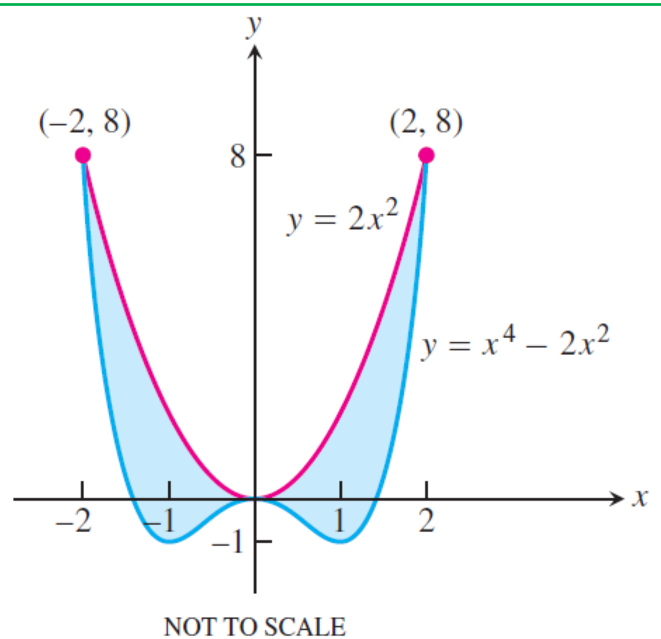
$$A = 2\left|\left[\frac{x^4}{4} - 2x^2\right]\Big|_0^2\right| = 2\left|\left[\frac{2^4}{4} - 2(2)^2\right] - 0\right| \quad \Bigg| = 8\,unit\ square$$

Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Definition Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx$$



Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Example: Find the area bounded by $\boldsymbol{y = 2 - x^2}$ and $\boldsymbol{y = -x}$.

Solution:

- Find point of intersection.

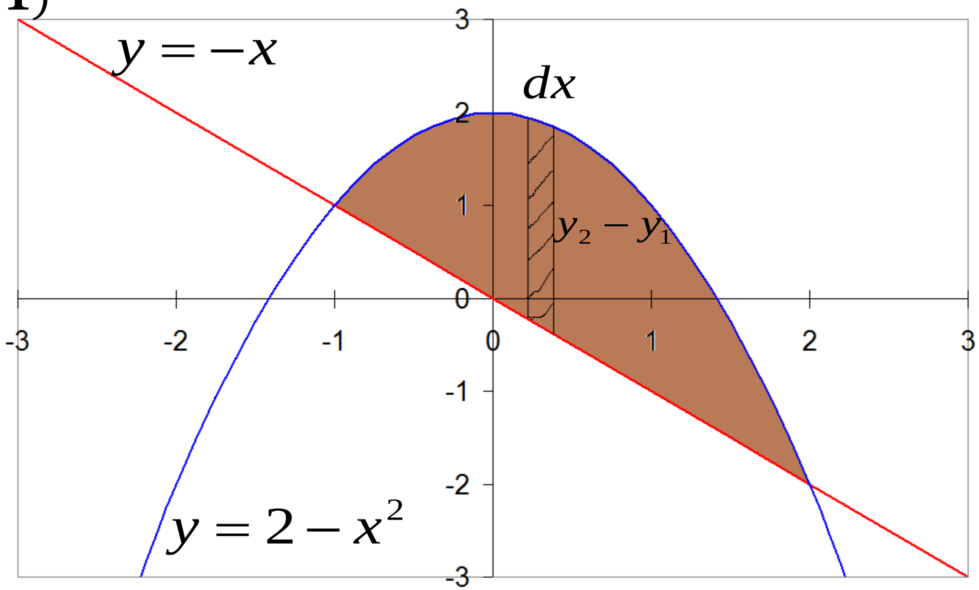
$$2 - x^2 = -x \Rightarrow x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$\boxed{x = 2, \quad x = -1}$$

$$(2,-2), \quad (-1,1)$$

- Points intersections: $(\mathbf{2}, -\mathbf{2})$ and $(-\mathbf{1}, \mathbf{1})$
- Graph the functions

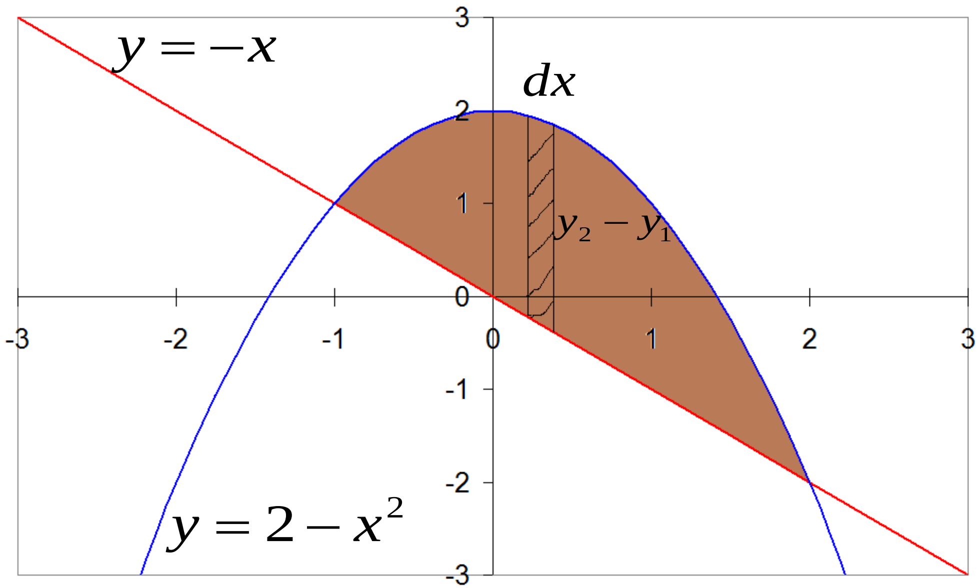


Chapter 5 Application of Definite Integral

5.1 Area Between Curves

Solution:

➢ The Area.



$$dA=(y_2-y_1)\ dx$$

$$dA=(2-x^2-(-x))\ dx$$

$$A=\int_{-1}^{2}(2-x^2+x)\ dx$$

$$A=\left|\left[2x-\frac{x^3}{3}+\frac{x^2}{2}\right|_{-1}^{2}\right]\right|=\left|[2(2)-\frac{(2)^3}{3}+\frac{(2)^2}{2}]-[2(-1)-\frac{(-1)^3}{3}+\frac{(-1)^2}{2}]\right|$$

$$=\left|4-\frac{8}{3}+2+2-\frac{1}{3}-\frac{1}{2}]\right|=\frac{9}{2}\ \ unit\ \ square$$

Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Example: Find the area bounded on the right by $\boldsymbol{y = x - 2}$ on the left by $\boldsymbol{y^2 = x}$ and below by $\boldsymbol{x - axis}$.

Solution:

- Find point of intersection.

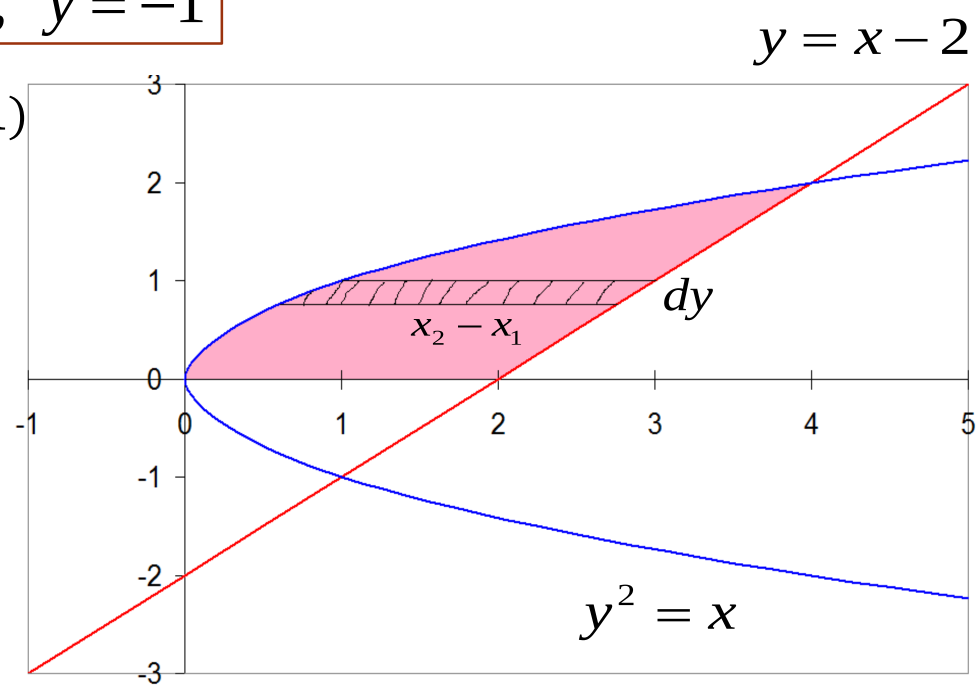
$$y + 2 = y^2 \Rightarrow y^2 - y - 2 = 0$$

$$(4,2), \ (1,-1) \qquad (y-2)(y+1) = 0$$

$$\boxed{y = 2, \ y = -1}$$

- Points intersections: $(\mathbf{4}, \mathbf{2})$ and $(\mathbf{1}, -\mathbf{1})$

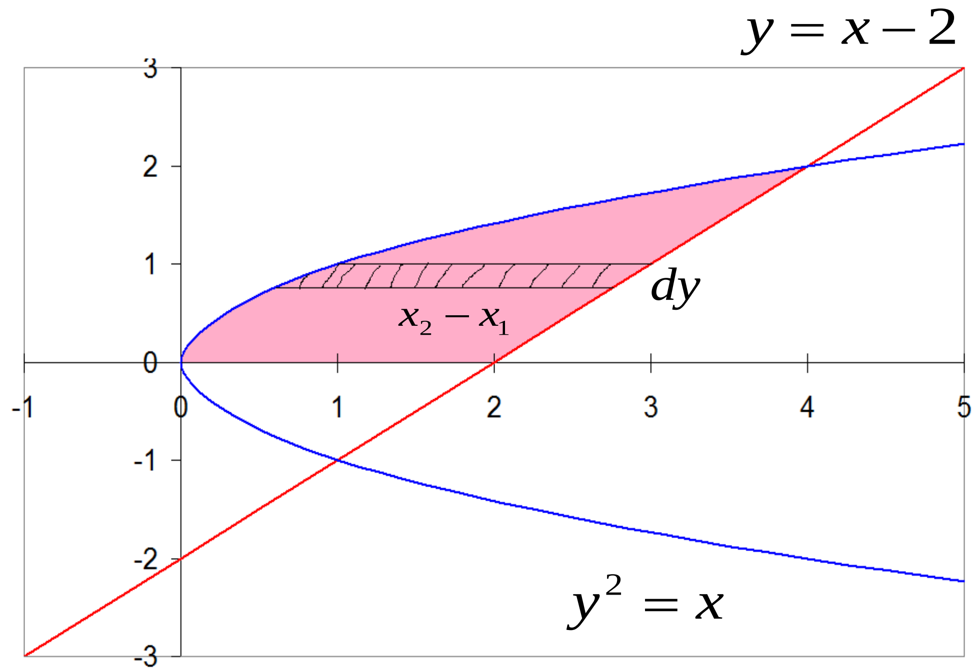
- Graph the functions



Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Solution:

- The Area



$$dA = (x_2 - x_1)\ dy$$

$$dA = (y + 2 - y^2)\ dy$$

$$A = \int_0^2 (y + 2 - y^2)\ dy$$

$$A = \left| \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2] \right| = \left| [\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}] - [\frac{(0)^2}{2} + 2(0) - \frac{(0)^3}{3}] \right|$$

$$= \left| 2 + 4 - \frac{8}{3}] \right| = \frac{10}{3}\ \ unit\ \ square$$

Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Example: Find the area bounded by $\boldsymbol{y = 3 - x}$ and $\boldsymbol{x = 3y - y^2}$.

Solution:

- Find point of intersection.

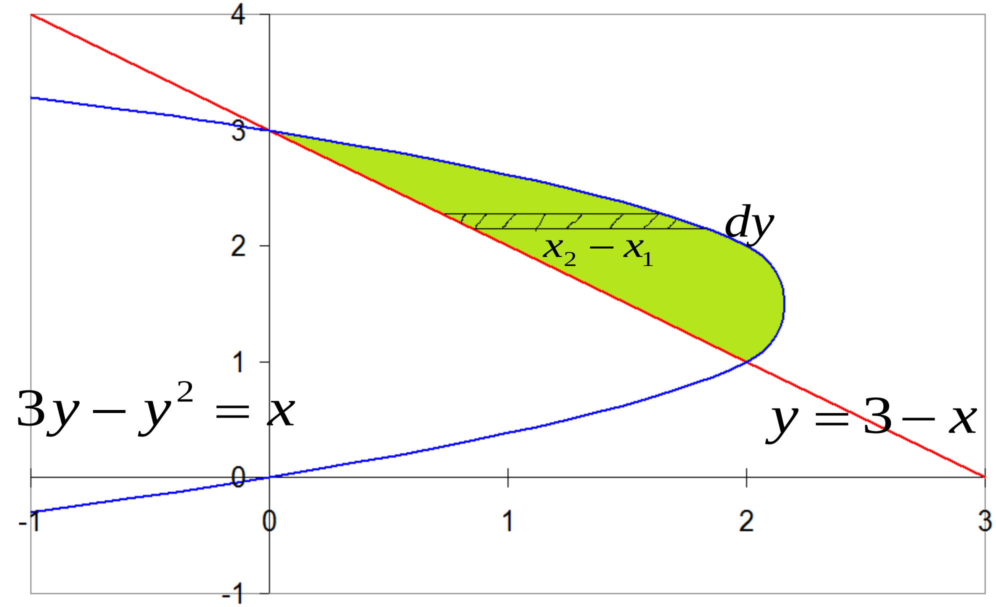
$$3 - y = 3y - y^2 \Rightarrow y^2 - 4y + 3 = 0$$

$$(y-3)(y-1) = 0$$

$$\boxed{y = 3, \quad y = 1}$$

- Points intersections: $(\mathbf{0}, \mathbf{3})$ and $(\mathbf{2}, \mathbf{1})$

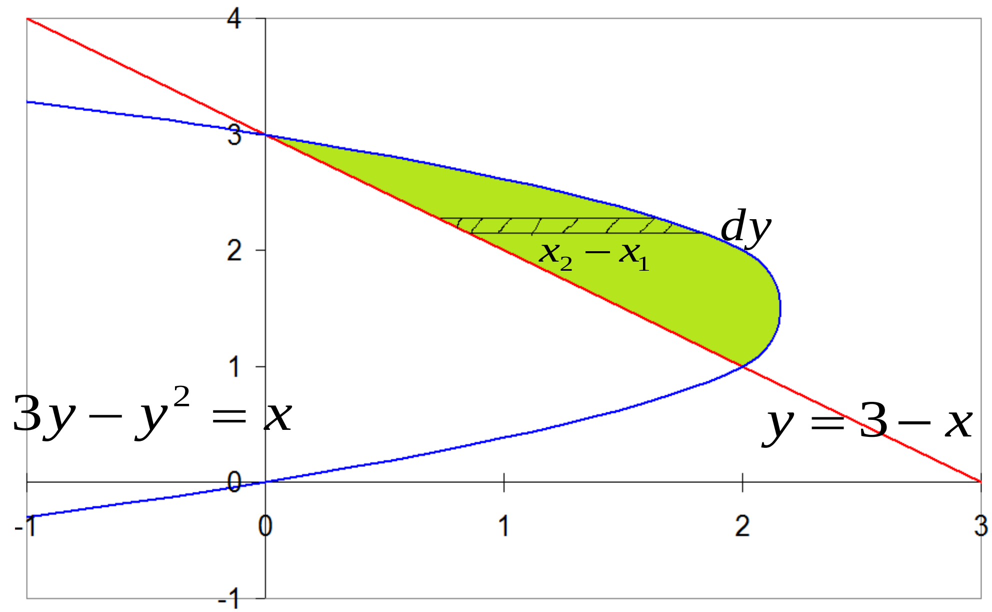
- Graph the functions



Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Solution:

- The Area.



$$dA = (x_2 - x_1)\ dy$$

$$dA = (3y - y^2 - (3 - y))\ dy$$

$$dA = (4y - y^2 - 3)\ dy$$

$$A = \int_1^3 (4y - y^2 - 3)\ dy$$

$$A = \left| 2y^2 - \frac{y^3}{3} - 3y \Big|_1^3 \right| = \left| [2(3)^2 - \frac{(3)^3}{3} - 3(3)] - [2(1)^2 - \frac{(1)^3}{3} - 3(1)] \right|$$

$$= \left| 18 - 9 - 9 - 2 + \frac{1}{3} + 3] \right| = \frac{4}{3}\ \ unit\ \ square$$

Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Example: Find the area bounded by $\boldsymbol{y = \sin\frac{\pi x}{2}}$ and $\boldsymbol{y = x}$.

Solution:

- Find point of intersection. $$x = \sin\frac{\pi x}{2}$$
- at $\boldsymbol{x = 0}$ $$0 = \sin\frac{\pi(0)}{2} \Rightarrow 0 = 0 \;\; ok$$
- at $\boldsymbol{x = 0.5}$ $$0.5 = \sin\frac{\pi(0.5)}{2} \Rightarrow 0.5 \neq 0.7 \;\; not \;\; ok$$
- at $\boldsymbol{x = 1}$ $$1 = \sin\frac{\pi(1)}{2} \Rightarrow 1 = 1 \;\; ok$$
- at $\boldsymbol{x = -1}$ $$-1 = \sin\frac{\pi(-1)}{2} \Rightarrow -1 = -1 \;\; ok$$

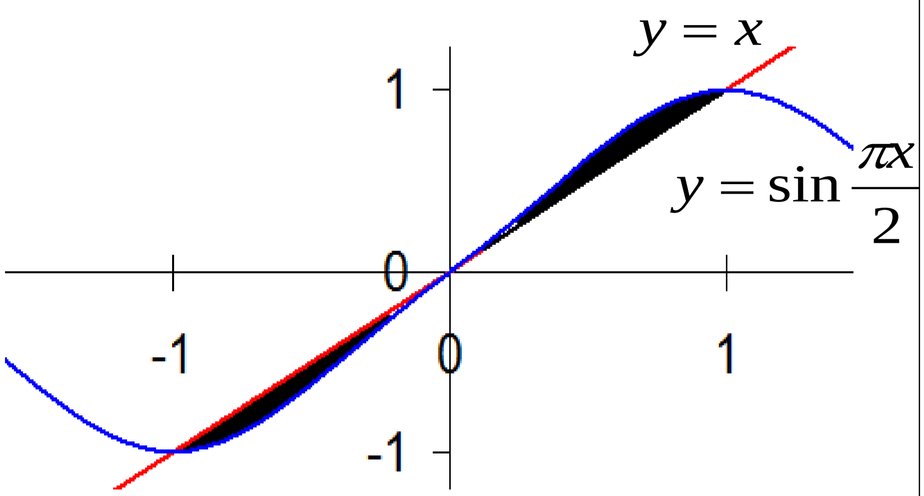
- Points of intersection: $(\mathbf{0}, \mathbf{0})$, $(\mathbf{1}, \mathbf{1})$, and $(\mathbf{-1}, \mathbf{-1})$

Chapter 5 Application of Definite Integral

5.1 Area Between Curves

Solution:

➢ Graph the functions.



$$dA = (y_2 - y_1)\ dx$$

$$dA = 2[(\sin\frac{\pi x}{2} - x)]\ dx$$

$$A = 2\int_0^1 (\sin\frac{\pi x}{2} - x)\ dx$$

$$A = \left| 2\ [\frac{-2}{\pi}\ \cos\ \frac{\pi\ x}{2}\ -\ \frac{x^2}{2}\ ]\Big|_0^1 \right| = \left| 2\ [(\frac{-2}{\pi}\ \cos\ \frac{\pi(1)}{2}\ -\ \frac{(1)^2}{2})\ -\ (\frac{-2}{\pi}\ \cos\ \frac{\pi(0)}{2}\ -\frac{(0)^2}{2}\ )] \right|$$

$$= \left| 2[0 - \frac{1}{2} + \frac{2}{\pi} + 0] \right| = \frac{4-\pi}{\pi}\ \ unit\ \ square$$

Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Example: Find the area between $\boldsymbol{y = 2x - x^2}$ and $\boldsymbol{y = -3}$.

Solution:

- Find point of intersection.

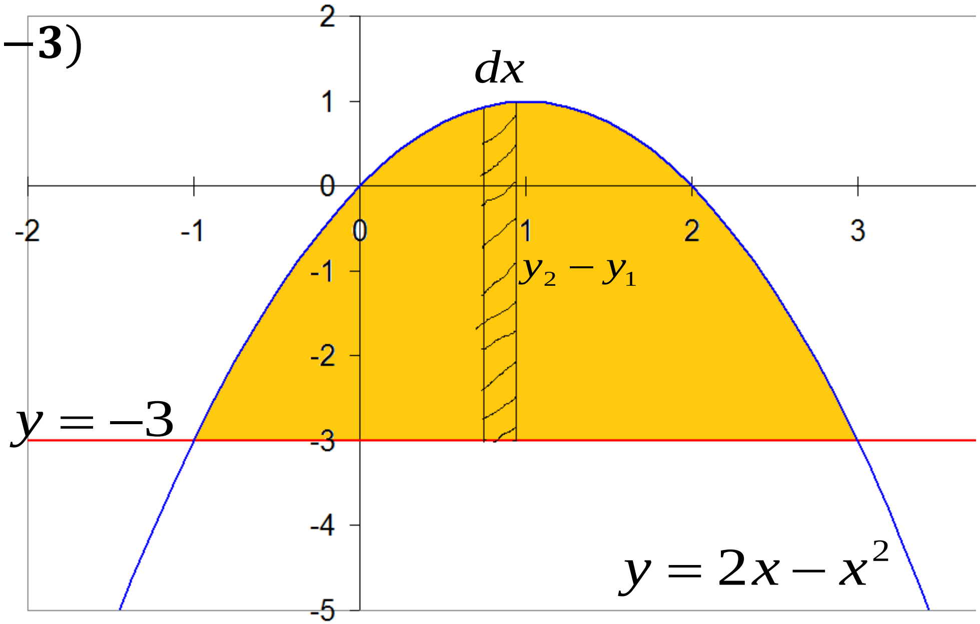
$$2x - x^2 = -3 \Rightarrow x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$\boxed{x = 3, \;\; x = -1}$$

- Points intersections: $(\mathbf{3}, \mathbf{-3})$ and $(\mathbf{-1}, \mathbf{-3})$

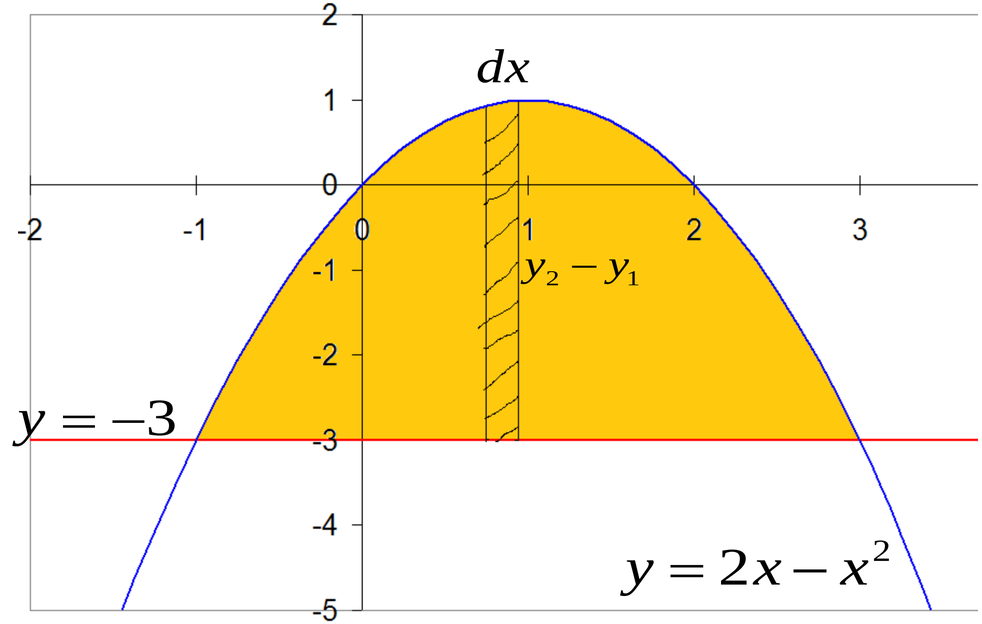
- Graph the functions



Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Solution:

- The Area.



$$dA = (y_2 - y_1)\ dx$$

$$dA = (2x - x^2 - (-3))\ dx$$

$$A = \int_{-1}^{3} (2x - x^2 + 3)\ dx$$

$$A = \left| x^2 - \frac{x^3}{3} + 3x \Big|_{-1}^{3} \right] = \left| [(3)^2 - \frac{(3)^3}{3} + 3(3)] - [(-1)^2 - \frac{(-1)^3}{3} + 3(-1)] \right|$$

$$= \left| 9 - 9 + 9 - 1 - \frac{1}{3} + 3] \right| = \frac{32}{3}\ \ unit\ \ square$$

Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Example: Find the area bounded by $\boldsymbol{y = \sin x}$, $\boldsymbol{y = \cos x}$, and $\boldsymbol{y - axis}$ in the first quadrant.

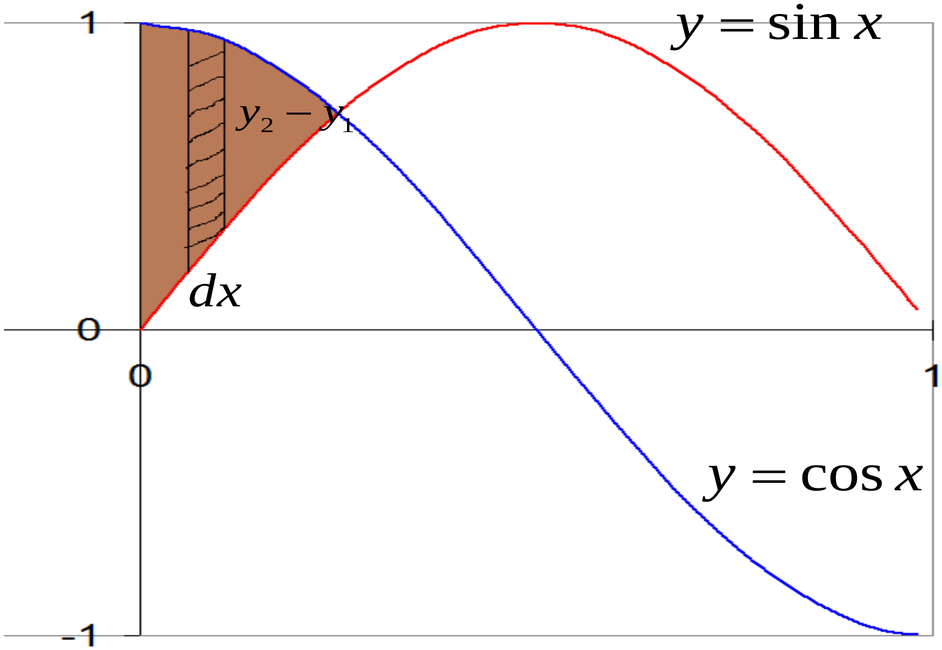
Solution:

- Find point of intersection.

$$\sin x = \cos x \Rightarrow \tan x = 1$$

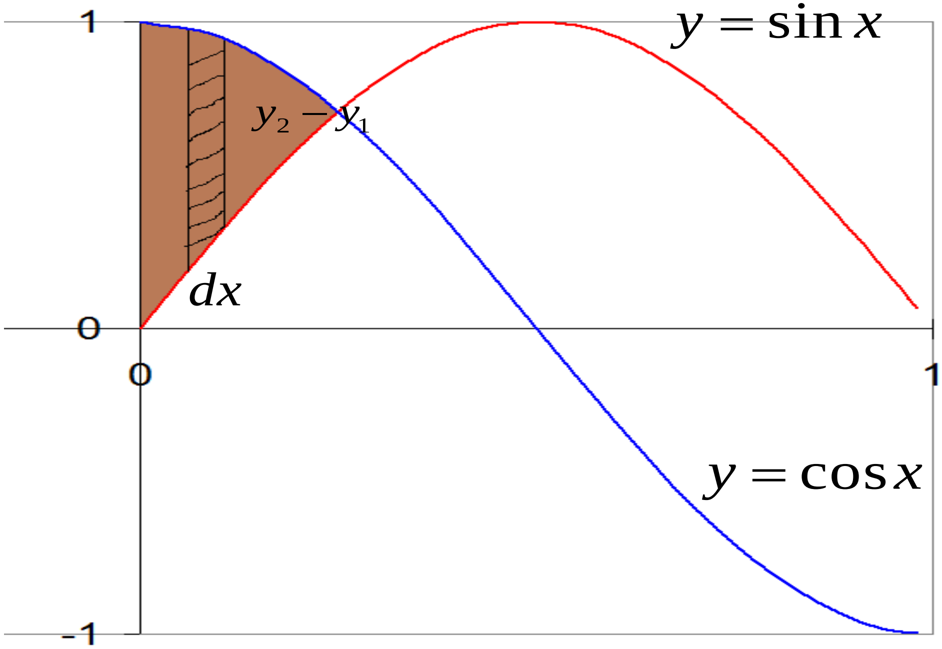
$$\boxed{x = \frac{\pi}{4},\ \ x = \frac{5\pi}{4} \ldots}$$

- Points intersections: $(\boldsymbol{\frac{\pi}{4}, 0.707})$
- Graph the functions



Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Solution:
- The Area.



$$dA = (y_2 - y_1)\ dx$$

$$dA = (\cos x - \sin x)\ dx$$

$$A = \int_0^{\pi/4} (\cos x - \sin x)\ dx$$

$$A = \left| \sin x + \cos x \Big|_0^{\pi/4} \right| = \left| [\sin\frac{\pi}{4} + \cos\frac{\pi}{4}] - [\sin 0 + \cos 0] \right|$$

$$= \left| \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 \right| = 0.414\ \textit{unit square}$$

Chapter 5 Application of Definite Integral

5.1 Area Between Curves

Example: Find the area, on the right by $\boldsymbol{x+y=2}$ on the left by $\boldsymbol{y=x^2}$ and below by $\boldsymbol{x-axis}$.

Solution:

- Find point of intersection.

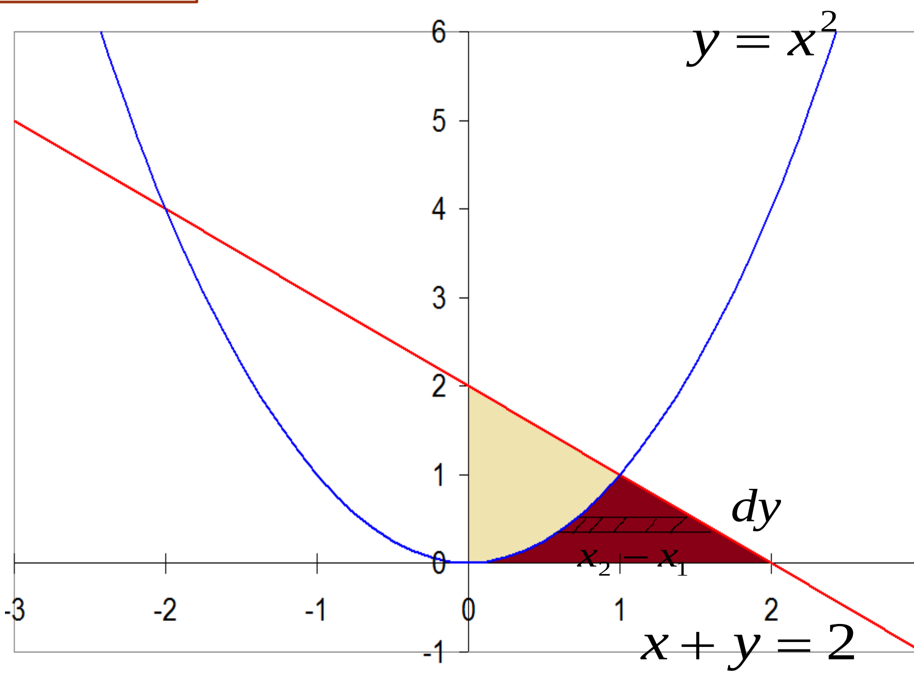
$$2-x=x^2 \Rightarrow x^2+x-2=0$$

$$(x+2)(x-1)=0$$

$$\boxed{x=-2,\quad x=1}$$

- Points intersections: $(\mathbf{1},\mathbf{1})$ and $(-\mathbf{2},\mathbf{4})$

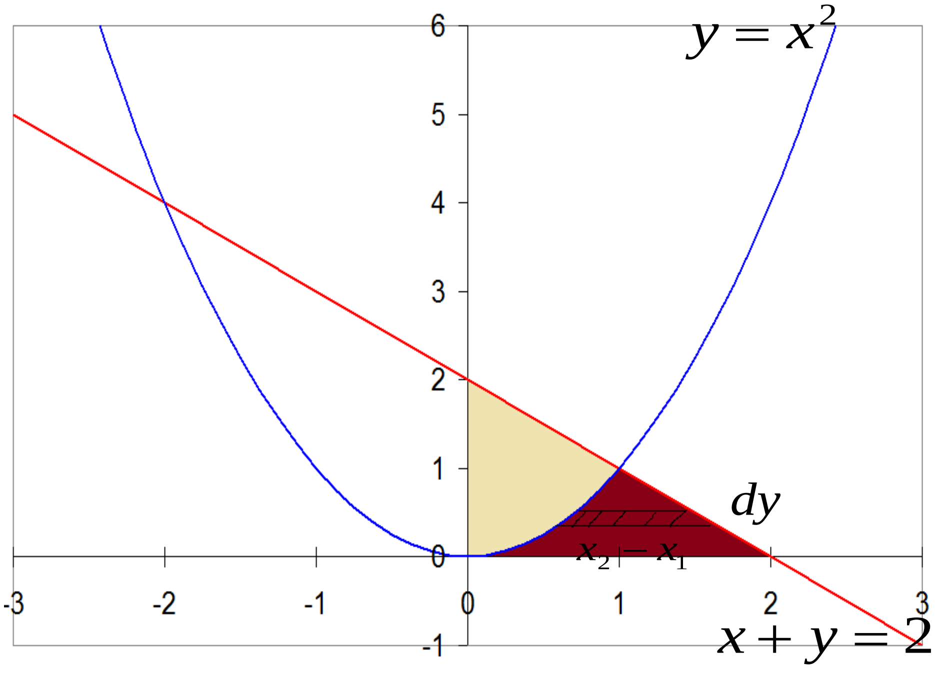
- Graph the functions



Chapter 5 Application of Definite Integral
5.1 Area Between Curves

Solution:

- The Area.



$$dA = (x_2 - x_1)\ dy$$

$$dA = (2 - y - \sqrt{y})\ dy$$

$$A = \int_0^1 (2 - y - y^{1/2})\ dy$$

$$A = \left| 2y - \frac{y^2}{2} - \frac{2}{3} y^{3/2} \Big|_0^1 \right| = \left| [2(1) - \frac{(1)^2}{2} - \frac{2}{3}(1)^{3/2}] - [2(0) - \frac{(0)^2}{2} - \frac{2}{3}(0)^{3/2}] \right|$$

$$= \left| 2 - \frac{1}{2} - \frac{2}{3} \right| = \frac{5}{6}\ \ unit\ \ square$$

Chapter 5 Application of Definite Integral
5.2 Volume Using Cross-Section

- Define volumes of solids whose cross-sections are plane regions.
- To find the volume of a solid $\boldsymbol{S}$ like the one in Figure 5.1, we begin by extending the definition of a cylinder from classical geometry to cylindrical solids with arbitrary bases Figure 5.2.
- The volume of cylindrical solid is:

$$\boldsymbol{Volume = area \times height = A.h}$$

- This equation is the basis for defining the volumes of many solids that are not cylindrical by the method of slicing.
- If we cut the solid by a plane which is perpendicular to $\boldsymbol{x-axis}$, we get a cross-section of the solid $\boldsymbol{S}$, slice $\boldsymbol{S}$ into thin “slabs” (like thin slices of a loaf of bread). at each point in the interval $[\boldsymbol{a},\boldsymbol{b}]$ is a region $\boldsymbol{R}(\boldsymbol{x})$ of area $\boldsymbol{A}(\boldsymbol{x})$, and $\boldsymbol{A}$ is a continuous function of $\boldsymbol{x}$.

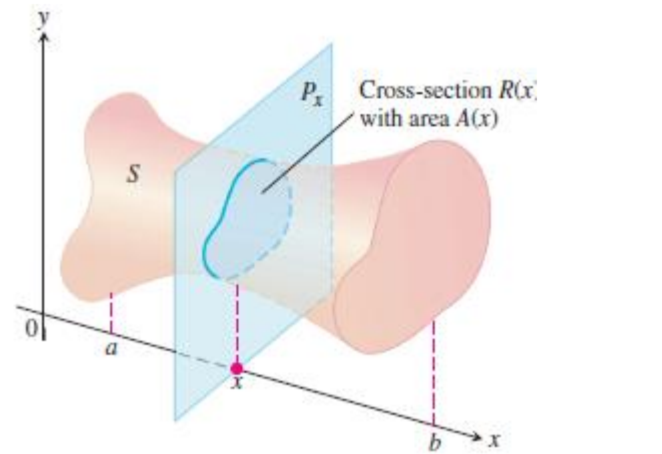


Figure 5.1

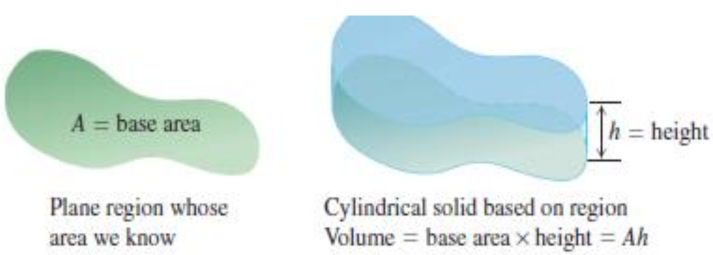


Figure 5.2

Chapter 5 Application of Definite Integral
5.2 Volume Using Cross-Section

Definition Volume

The volume of a solid of known integrable cross-sectional area $A(\mathrm{x})$ from $x = a$ to $x = b$ is the integral of A from a to b,

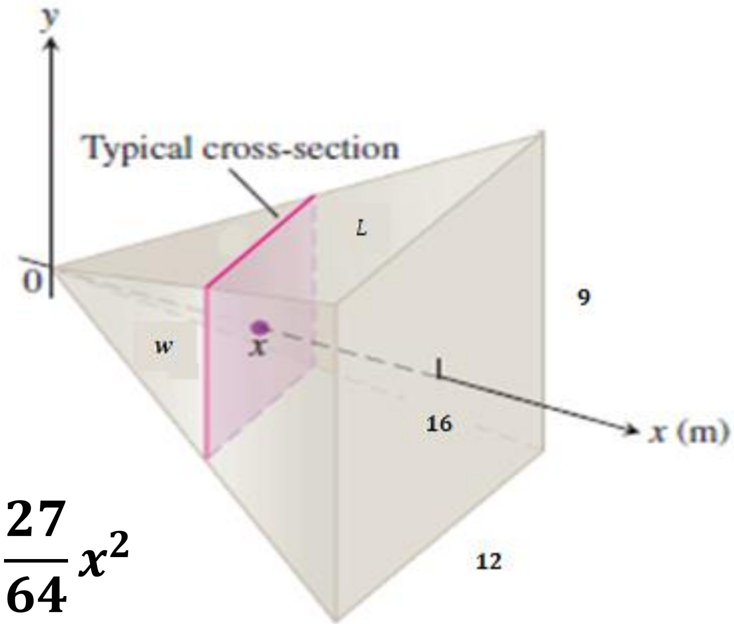
$$V = \int_a^b A(x)\, dx$$

Calculation the Volume of a Solid

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ using the Fundamental Theorem.

Chapter 5 Application of Definite Integral
5.2 Volume Using Cross-Section

Example: A pyramid $\boldsymbol{16\ m}$ high has a rectangular base that is $\boldsymbol{12\ m}$ length and $\boldsymbol{9\ m}$ width. The cross-section of the pyramid perpendicular to the altitude $\boldsymbol{x\ m}$ down from the vertex. Find the volume of the pyramid.



Solution:

1- Sketch the solid and typical cross-section

2- the cross-section at $\boldsymbol{x}$ is.

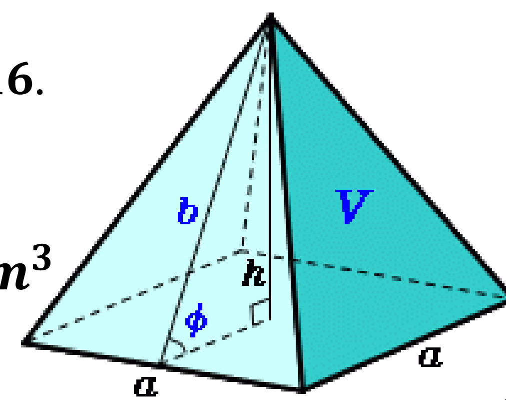
$$\boldsymbol{w = \frac{9}{16}\,x, \qquad L = \frac{12}{16}\,x}$$

$$\boldsymbol{A(x) = w.L = \left(\frac{9}{16}\,x\right).\left(\frac{12}{16}\,x\right) = \frac{27}{64}x^2}$$

3- From the figure, the limit of integration from $\boldsymbol{x = 0}$ to $\boldsymbol{x = 16}$.

4- The volume of pyramid

$$\boldsymbol{V = \int_0^{16} A(x)\,dx = \int_0^{16} \frac{27}{64}x^2\,dx = \left|\frac{9}{64}x^3\Big|_0^{16}\right| = 576\ m^3}$$



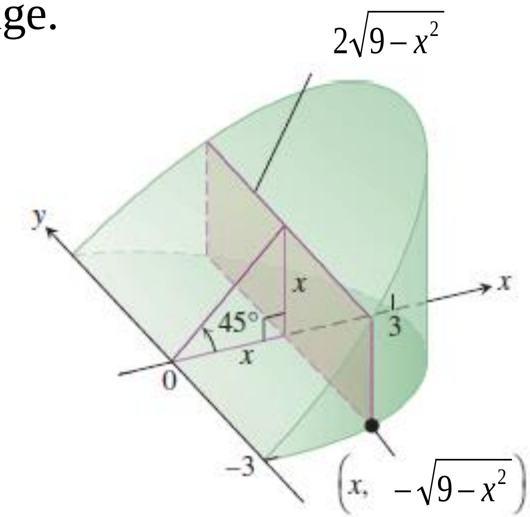
Chapter 5 Application of Definite Integral
5.2 Volume Using Cross-Section

Example: A curved wedge is cut from a cylinder of radius **3** by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a $\mathbf{45°}$ angle at the center of the cylinder. Find the volume of the wedge.

Solution:

1- Sketch the solid and typical cross-section

2- The cross-section at $\boldsymbol{x}$ is rectangular.



$$\boldsymbol{A(x) = 2x\sqrt{9-x^2}}$$

3- From the figure, the limit of integration from $\boldsymbol{x = 0}$ to $\boldsymbol{x = 3}$.

4- The volume of curved wedge

$$\boldsymbol{V = \int_0^3 A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx \Longrightarrow}$$

$$u = 9 - x^2$$
$$-du = 2x\,dx$$

$$\boldsymbol{= \int_0^3 -\sqrt{u}\,du = \left|-\frac{2}{3}u^{3/2}\Big|_0^3\right| = \left|-\frac{2}{3}(9-x^2)^{3/2}\Big|_0^3\right|}$$

$$\boldsymbol{= \left|-\frac{2}{3}(9)^{3/2} + 0\right| = 18\ m^3}$$