More than 80 questions in the subject of “Engineering Analysis”

1 In each of the following a periodic function $f(t)$ of period 2π is specified over one period. In each case sketch a graph of the function for $-4\pi \leqslant t \leqslant 4\pi$ and obtain a Fourier series representation of the function.

(a) $f(t) = \begin{cases} -\pi & (-\pi < t < 0) \\ t & (0 < t < \pi) \end{cases}$

(b) $f(t) = \begin{cases} t+\pi & (-\pi < t < 0) \\ 0 & (0 < t < \pi) \end{cases}$

(c) $f(t) = 1 - \dfrac{t}{\pi} \quad (0 \leqslant t \leqslant 2\pi)$

(d) $f(t) = \begin{cases} 0 & (-\pi \leqslant t \leqslant -\frac{1}{2}\pi) \\ 2\cos t & (-\frac{1}{2}\pi \leqslant t \leqslant \frac{1}{2}\pi) \\ 0 & (\frac{1}{2}\pi \leqslant t \leqslant \pi) \end{cases}$

(e) $f(t) = \cos \frac{1}{2}t \quad (-\pi < t < \pi)$

(f) $f(t) = |t| \quad (-\pi < t < \pi)$

(g) $f(t) = \begin{cases} 0 & (-\pi \leqslant t \leqslant 0) \\ 2t-\pi & (0 < t \leqslant \pi) \end{cases}$

(h) $f(t) = \begin{cases} -t+e^t & (-\pi \leqslant t < 0) \\ t+e^t & (0 \leqslant t < \pi) \end{cases}$

2 Obtain the Fourier series expansion of the periodic function $f(t)$ of period 2π defined over the period $0 \leqslant t \leqslant 2\pi$ by

$$f(t) = (\pi - t)^2 \quad (0 \leqslant t \leqslant 2\pi)$$

Use the Fourier series to show that

$$\frac{1}{12}\pi^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

3 Show that the Fourier series representing the periodic function $f(t)$, where

$$f(t) = \begin{cases} \pi^2 & (-\pi < t < 0) \\ (t-\pi)^2 & (0 < t < \pi) \end{cases}$$

$$f(t + 2\pi) = f(t)$$

is

$$f(t) = \tfrac{2}{3}\pi^2 + \sum_{n=1}^{\infty} \left[\frac{2}{n^2} \cos nt + \frac{(-1)^n}{n} \pi \sin nt \right]$$

$$- \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)^3}$$

Use this result to show that

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{12}\pi^2$

4 A periodic function $f(t)$ of period 2π is defined within the domain $0 \leqslant t \leqslant \pi$ by

$$f(t) = \begin{cases} t & (0 \leqslant t \leqslant \frac{1}{2}\pi) \\ \pi - t & (\frac{1}{2}\pi \leqslant t \leqslant \pi) \end{cases}$$

Sketch a graph of $f(t)$ for $-2\pi < t < 4\pi$ for the two cases where

(a) $f(t)$ is an even function
(b) $f(t)$ is an odd function

5 A periodic function $f(t)$ of period 2π is defined within the period $0 \leqslant t \leqslant 2\pi$ by

$$f(t) = \begin{cases} 2 - t/\pi & (0 \leqslant t \leqslant \pi) \\ t/\pi & (\pi \leqslant t \leqslant 2\pi) \end{cases}$$

Draw a graph of the function for $-4\pi \leqslant t \leqslant 4\pi$ and obtain its Fourier series expansion.

6 A periodic function of period 10 is defined within the period $-5 < t < 5$ by

$$f(t) = \begin{cases} 0 & (-5 < t < 0) \\ 3 & (0 < t < 5) \end{cases}$$

Determine its Fourier series expansion and illustrate graphically for $-12 < t < 12$.

7 Show that the half-range Fourier sine series expansion of the function $f(t) = 1$, valid for $0 < t < \pi$, is

$$f(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{2n-1} \quad (0 < t < \pi)$$

Sketch the graphs of both $f(t)$ and the periodic function represented by the series expansion for $-3\pi < t < 3\pi$.

8 Determine the half-range cosine series expansion of the function $f(t) = 2t - 1$, valid for $0 < t < 1$. Sketch the graphs of both $f(t)$ and the periodic function represented by the series expansion for $-2 < t < 2$.

9 The function $f(t) = 1 - t^2$ is to be represented by a Fourier series expansion over the finite interval $0 < t < 1$. Obtain a suitable

(a) full-range series expansion,
(b) half-range sine series expansion,
(c) half-range cosine series expansion.

Draw graphs of $f(t)$ and of the periodic functions represented by each of the three series for $-4 < t < 4$.

A function $f(t)$ is defined by

$$f(t) = \pi t - t^2 \quad (0 \leqslant t \leqslant \pi)$$

10 and is to be represented by either a half-range Fourier sine series or a half-range Fourier cosine series. Find both of these series and sketch the graphs of the functions represented by them for $-2\pi < t < 2\pi$.

11 A periodic function $f(t)$ is defined by

$$f(t) = \begin{cases} t^2 & (0 \leqslant t < \pi) \\ 0 & (\pi < t \leqslant 2\pi) \end{cases}$$

$$f(t + 2\pi) = f(t)$$

Obtain a Fourier series expansion of $f(t)$ and deduce that

$$\tfrac{1}{6}\pi^2 = \sum_{r=1}^{\infty} \frac{1}{r^2}$$

12 Determine the full-range Fourier series expansion of the even function $f(t)$ of period 2π defined by

$$f(t) = \begin{cases} \tfrac{2}{3}t & (0 \leqslant t \leqslant \tfrac{1}{3}\pi) \\ \tfrac{1}{3}(\pi - t) & (\tfrac{1}{3}\pi \leqslant t \leqslant \pi) \end{cases}$$

To what value does the series converge at $t = \tfrac{1}{3}\pi$?

13 A function $f(t)$ is defined for $0 \leqslant t \leqslant \tfrac{1}{2}T$ by

$$f(t) = \begin{cases} t & (0 \leqslant t \leqslant \tfrac{1}{4}T) \\ \tfrac{1}{2}T - t & (\tfrac{1}{4}T \leqslant t \leqslant \tfrac{1}{2}T) \end{cases}$$

Sketch odd and even functions that have a period T and are equal to $f(t)$ for $0 \leqslant t \leqslant \frac{1}{2}T$.

(a) Find the half-range Fourier sine series of $f(t)$.

14 Find a half-range Fourier sine and Fourier cosine series for $f(x)$ valid in the interval $0 < x < \pi$ when $f(x)$ is defined by

$$f(x) = \begin{cases} x & (0 \leqslant x \leqslant \frac{1}{2}\pi) \\ \pi - x & (\frac{1}{2}\pi \leqslant x \leqslant \pi) \end{cases}$$

Sketch the graph of the Fourier series obtained for $-2\pi < x \leqslant 2\pi$.

15 A function $f(t)$ is defined on $0 < t < \pi$ by

$$f(t) = \pi - t$$

Find

(a) a half-range Fourier sine series, and
(b) a half-range Fourier cosine series for $f(t)$ valid for $0 < t < \pi$.

Sketch the graphs of the functions represented by each series for $-2\pi < t < 2\pi$.

Laplace Transforms

1 Use the definition of the Laplace transform to obtain the transforms of $f(t)$ when $f(t)$ is given by

(a) $\cosh 2t$ (b) t^2 (c) $3 + t$ (d) $t\,e^{-t}$

stating the region of convergence in each case.

2 What are the abscissae of convergence for the following functions?

(a) e^{5t}
(b) e^{-3t}
(c) $\sin 2t$
(d) $\sinh 3t$
(e) $\cosh 2t$
(f) t^4
(g) $e^{-5t} + t^2$
(h) $3 \cos 2t - t^3$
(i) $3\,e^{2t} - 2\,e^{-2t} + \sin 2t$
(j) $\sinh 3t + \sin 3t$

3 Using the results shown in Figure 5.5, obtain the Laplace transforms of the following functions, stating the region of convergence:

(a) $5 - 3t$

(b) $7t^3 - 2\sin 3t$

(c) $3 - 2t + 4\cos 2t$

(d) $\cosh 3t$

(e) $\sinh 2t$

(f) $5e^{-2t} + 3 - 2\cos 2t$

(g) $4t\,e^{-2t}$

(h) $2e^{-3t}\sin 2t$

(i) $t^2 e^{-4t}$

(j) $6t^3 - 3t^2 + 4t - 2$

(k) $2\cos 3t + 5\sin 3t$

(l) $t\cos 2t$

(m) $t^2\sin 3t$

(n) $t^2 - 3\cos 4t$

(o) $t^2 e^{-2t} + e^{-t}\cos 2t + 3$

4 Find $\mathscr{L}^{-1}\{F(s)\}$ when $F(s)$ is given by

(a) $\dfrac{1}{(s+3)(s+7)}$ (b) $\dfrac{s+5}{(s+1)(s-3)}$

(c) $\dfrac{s-1}{s^2(s+3)}$ (d) $\dfrac{2s+6}{s^2+4}$

(e) $\dfrac{1}{s^2(s^2+16)}$ (f) $\dfrac{s+8}{s^2+4s+5}$

(g) $\dfrac{s+1}{s^2(s^2+4s+8)}$ (h) $\dfrac{4s}{(s-1)(s+1)^2}$

(i) $\dfrac{s+7}{s^2+2s+5}$

(j) $\dfrac{3s^2-7s+5}{(s-1)(s-2)(s-3)}$ (k) $\dfrac{5s-7}{(s+3)(s^2+2)}$

(l) $\dfrac{s}{(s-1)(s^2+2s+2)}$ (m) $\dfrac{s-1}{s^2+2s+5}$

(n) $\dfrac{s-1}{(s-2)(s-3)(s-4)}$ (o) $\dfrac{3s}{(s-1)(s^2-4)}$

(p) $\dfrac{36}{s(s^2+1)(s^2+9)}$

(q) $\dfrac{2s^2+4s+9}{(s+2)(s^2+3s+3)}$

(r) $\dfrac{1}{(s+1)(s+2)(s^2+2s+10)}$

5 Using Laplace transform methods, solve for $t \geqslant 0$ the following differential equations, subject to the specified initial conditions:

(a) $\dfrac{dx}{dt} + 3x = e^{-2t}$

subject to $x = 2$ at $t = 0$

(b) $3\dfrac{dx}{dt} - 4x = \sin 2t$

subject to $x = \frac{1}{3}$ at $t = 0$

(c) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 1$

subject to $x = 0$ and $\dfrac{dx}{dt} = 0$ at $t = 0$

(d) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 4\cos 2t$

subject to $y = 0$ and $\dfrac{dy}{dt} = 2$ at $t = 0$

(e) $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 2e^{-4t}$

subject to $x = 0$ and $\dfrac{dx}{dt} = 1$ at $t = 0$

(f) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 5x = 3e^{-2t}$

subject to $x = 4$ and $\dfrac{dx}{dt} = -7$ at $t = 0$

(g) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - 2x = 5e^{-t}\sin t$

subject to $x = 1$ and $\dfrac{dx}{dt} = 0$ at $t = 0$

(h) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = 3t$

subject to $y = 0$ and $\dfrac{dy}{dt} = 1$ at $t = 0$

(i) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 4x = t^2 + e^{-2t}$

subject to $x = \frac{1}{2}$ and $\dfrac{dx}{dt} = 0$ at $t = 0$

(j) $9\dfrac{d^2x}{dt^2} + 12\dfrac{dx}{dt} + 5x = 1$

subject to $x = 0$ and $\dfrac{dx}{dt} = 0$ at $t = 0$

(k) $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 16x = 16\sin 4t$

subject to $x = -\frac{1}{2}$ and $\dfrac{dx}{dt} = 1$ at $t = 0$

(l) $9\dfrac{d^2y}{dt^2} + 12\dfrac{dy}{dt} + 4y = e^{-t}$

subject to $y = 1$ and $\dfrac{dy}{dt} = 1$ at $t = 0$

6 Using Laplace transform methods, solve for $t \geqslant 0$ the following simultaneous differential equations subject to the given initial conditions:

(a) $2\dfrac{dx}{dt} - 2\dfrac{dy}{dt} - 9y = e^{-2t}$

$$2\frac{dx}{dt} + 4\frac{dy}{dt} + 4x - 37y = 0$$

subject to $x = 0$ and $y = \frac{1}{4}$ at $t = 0$

(b) $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} + x - y = 5 \sin t$

$$2\frac{dx}{dt} + 3\frac{dy}{dt} + x - y = e^{t}$$

subject to $x = 0$ and $y = 0$ at $t = 0$

(c) $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = e^{-3t}$

$$\frac{dy}{dt} + 5x + 3y = 5e^{-2t}$$

subject to $x = -1$ and $y = 4$ at $t = 0$

(d) $3\dfrac{dx}{dt} + 3\dfrac{dy}{dt} - 2x = e^{t}$

$$\frac{dx}{dt} + 2\frac{dy}{dt} - y = 1$$

subject to $x = 1$ and $y = 1$ at $t = 0$

(e) $3\frac{dx}{dt} + \frac{dy}{dt} - 2x = 3\sin t + 5\cos t$

$2\frac{dx}{dt} + \frac{dy}{dt} + y = \sin t + \cos t$

subject to $x = 0$ and $y = -1$ at $t = 0$

(f) $\frac{dx}{dt} + \frac{dy}{dt} + y = t$

$\frac{dx}{dt} + 4\frac{dy}{dt} + x = 1$

subject to $x = 1$ and $y = 0$ at $t = 0$

(g) $2\frac{dx}{dt} + 3\frac{dy}{dt} + 7x = 14t + 7$

$5\frac{dx}{dt} - 3\frac{dy}{dt} + 4x + 6y = 14t - 14$

subject to $x = y = 0$ at $t = 0$

(h) $\frac{d^2x}{dt^2} = y - 2x \qquad \frac{d^2y}{dt^2} = x - 2y$

subject to $x = 4$, $y = 2$, $dx/dt = 0$ and $dy/dt = 0$ at $t = 0$

(i) $$5\frac{d^2x}{dt^2} + 12\frac{d^2y}{dt^2} + 6x = 0$$

$$5\frac{d^2x}{dt^2} + 16\frac{d^2y}{dt^2} + 6y = 0$$

subject to $x = \frac{7}{4}$, $y = 1$, $dx/dt = 0$ and $dy/dt = 0$ at $t = 0$

(j) $$2\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} - \frac{dx}{dt} - \frac{dy}{dt} = 3y - 9x$$

$$2\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 5y - 7x$$

subject to $x = dx/dt = 1$ and $y = dy/dt = 0$ at $t = 0$

Solution of second order differential equations

<u>Example 1</u>: Solve the following diff. eqn.:

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

Note that:

$$D = \frac{d}{dx}$$

$$D^2 = \frac{d^2}{dx^2} \; ; \; D^3 = \frac{d^3}{dx^3}$$

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(2) Solve $(D^4 - 16)y = 0$

(3) <u>Solve</u>:

$$y''' - 3y'' + 3y' - y = 0$$

(4) Solve: $(D^6 - 64)y = 0$

5. Solve:

$$(D^4 - D)y = 0$$

6. Solve:

$$(D^4 - 2D^3 + 3D^2 + D)y = 0$$

7. Solve:

$$2y'' - y' - y = 0$$

Given: $y(0) = -1$; $y'(0) = 0$

8. Solve: $y'' + 4y = 0$; $y(0) = 0$; $y'(0) = 2$

Nonhomogeneous differential equations

① $y'' + 4y = \sec 2x$

② $x^2 y'' - 2x y' + 2y = x \ln x$

Given $y_1 = x$, $y_2 = x^2$

Find the P.I.