Calculus of Variations

Mathematics Department	Chapter 1	Fourth Stage
Second Semester	2023-2024	Lecturer: Dr. Andam

The Calculus of Variations owed its origin to the attempt to solve a very interesting and rather narrow class of problems in Maxima and Minima, in which it is required to find the form of a function such that the definite integral of an expression involving that function and its derivative shall be a maximum or a minimum.

1. **Maxima and Minima**

Let X and Y be two arbitrary sets and $f: X \to Y$ be a well-defined function having domain X and range Y. The function values $f(x)$ become comparable if Y is $\mathbb{R}$ or a subset of $\mathbb{R}$. Thus, optimization problem is valid for real valued functions. Let $f: X \to \mathbb{R}$ be a real valued function having X as its domain.

Now $x_0 \in X$ is said to be maximum point for the function f if $f(x_0) \geq f(x)\ \forall x \in X$. The value $f(x_0)$ is called the maximum value of f. Similarly, $x_0 \in X$ is said to be a minimum point for the function f if $f(x_0) \leq f(x) \forall x \in X$ and in this case $f(x_0)$ is the minimum value of f.

1.1 Sufficient condition for having maximum and minimum:

Theorem 1.1 (Weierstrass Theorem):

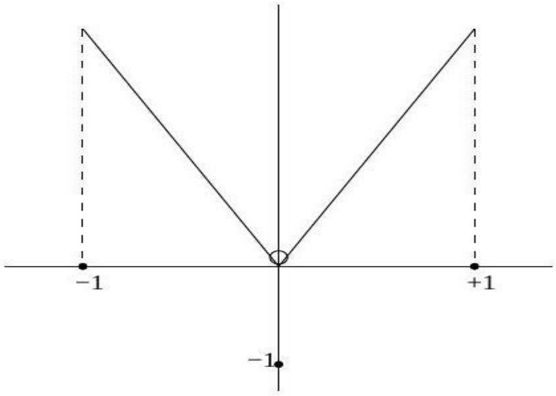
Let $S \subseteq \mathbb{R}$ and $f: S \to \mathbb{R}$ be a well-defined function. Then f will have a maximum/minimum under the following sufficient conditions.

1. $f: S \to \mathbb{R}$ is a continuous function.
2. $S \subset \mathbb{R}$ is a bound and closed (compact) subset of $\mathbb{R}$.

Note that the above conditions are just sufficient conditions but not necessary.

Example 1.1: Let $f: [-1,1] \to \mathbb{R}$ defined by

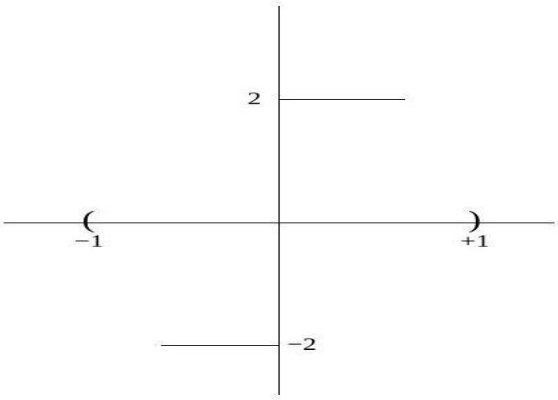
$$f(x) = \begin{cases} -1 & x = 0 \\ |x| & x \neq 0 \end{cases}$$



Obviously $f(x)$ is not continuous at $x = 0$. However, the $f(x)$ has a minimum point $x_0 = 0$ and maximum points at $x = -1, x = 1$. Continuity condition of the Weierstrass theorem is violated but still the function has maximum and minimum.

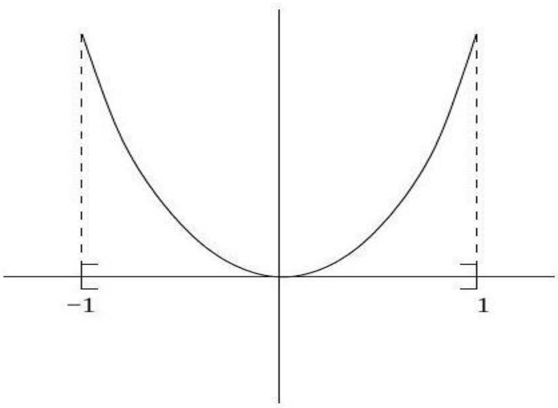
Example 1.2: Consider a function $f(-1,1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -2 & x < 0 \\ 2 & x \geq 0 \end{cases}$$



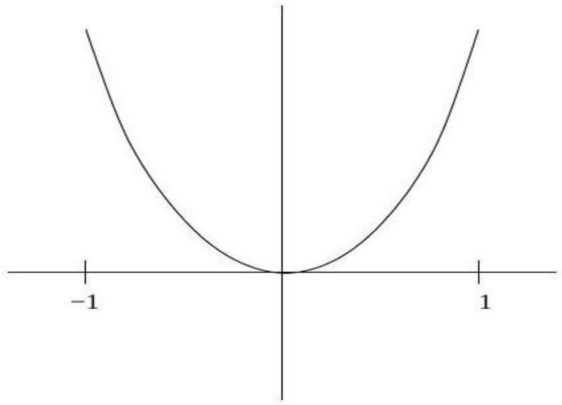
$f(x)$ has a maximum value $x = 2$ and a minimum value $x = -2$ even though both the conditions (a) and (b) of Weierstrass theorem are violated.

Example 1.3: Let $f: [-1,1] \to \mathbb{R}$ be defined by $f(x) = x^2$.



This function satisfies both the conditions of Weierstrass theorem. $f(x)$ has a minimum value 0 at $x = 0$ and maximum value 1 at $x = -1$ and $x = 1$.

Example 1.4: Let $f(-1,1) \to \mathbb{R}$ defined by $f(x) = x^2, f$ has a minimum at $x = 0$.



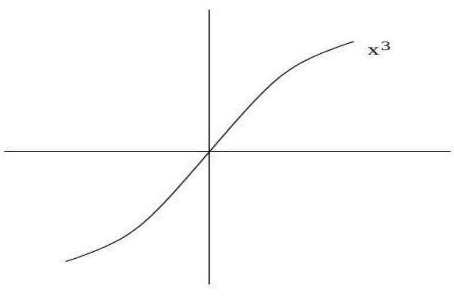
But f has no maximum point as $x = -1$ and $x = 1$ are outside the domain of the function. Here the condition (b) is violated.

1.2 Necessary condition for Maximum/Minimum when f is differentiable.

Theorem 1.2: Let $f: S \to \mathbb{R}$ be a differentiable function and let x_0 be an interior point of S and let x_0 is either a maximum point or minimum point of f. Then the first derivative of f vanishes at x_0. i.e. $f'(x_0) = 0$.

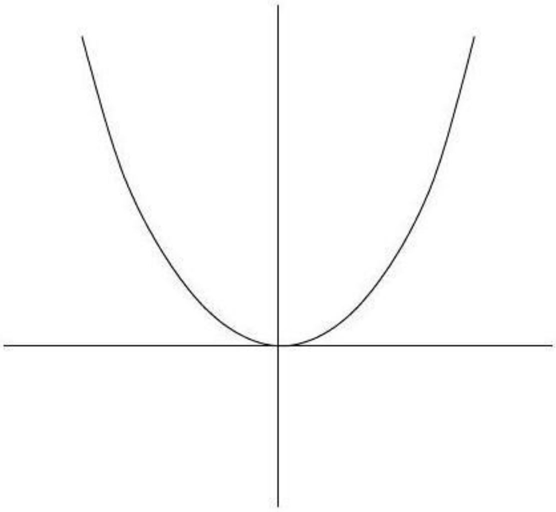
This condition is just a necessary condition but not sufficient condition. An interior point $x_0 \in D \subseteq \mathbb{R}$ is said to be a stationary point if $f'(x_0) = 0$. A stationary point x_0 need not be maximum point/minimum point. However, if $f''(x_0) > 0$ then x_0 is a minimum point and if $f''(x_0) < 0$ then x_0 is a maximum point.

Example 1.5: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. $f'(x) = 3x^2 = 0$ when $x = 0$



However, $x = 0$ is neither a maximum point nor a minimum point of $f(x) = x^3$.

Example 1.6: Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Hence, $f'(x_0) = 2x_0 = 0$ when $x_0 = 0$.



Obviously $x_0 = 0$ is a stationary point and this stationary point is minimum point of $f(x)$, as $f''(x_0) = 2 > 0$.

1.3 Necessary conditions for Maxima/Minima functions of several variables Multi-variable functions.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a real valued function of n - variables defined on $\mathbb{R}^n$. If f has partial derivatives at $x_0 \in \mathbb{R}^n$. If x_0 is a maximum point/minimum point of the function $f(x)$ then

$$\left.\frac{\partial f}{\partial x_1}\right|_{x=x_0} = 0,\ \left.\frac{\partial f}{\partial y_2}\right|_{x=x_0} = 0, \cdots \left.\frac{\partial f}{\partial x_{n-1}}\right|_{x=x_0} = 0$$

A stationary point x_0 is maximum point if the matrix

$$M = \left.\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}\right|_{x=x_0}$$

is negative definite and x_0 is minimum point if M is positive definite.

1.4 Functionals:

Let S be a set of functions. Let $f: S \to \mathbb{R}$ be a real valued function. Such functions are known as a functionals. In otherwords, a functional is a real valued function whose domain is a set of functions.

Example 1.7: Let $C[0,1]$ be the set of all continuous functions defined on $[0,1]$ Let $I: C[0,1] \to \mathbb{R}$ be a function defined by

$$I(y) = \int_0^1 y(x)dx$$

Obviously $I(y)$ is a functional on $C[0,1]$. The following table gives the values of $I(y)$ for different functions $y(x)$, listed in the table.

$y(x)$	$I(y)$
x	0.5
x^2	0.333
x^3	0.25
$\sin x$	0.4597
$\cos x$	0.8415
e^x	1.7183
1	1

We can find for which function y, the functional $I(y)$ has a maximum value or minimum value. In the above example, $I(y)$ will have minimum value for $y(x) = x^3$ and $I(y)$ will have maximum value for the function $y(x) = e^x$ out of the seven functions given here.

Let $C^1[x_1, x_2]$ denote the set of continuously differentiable functions defined on $[x_1, x_2]$. Now consider a functional $I: C^1[x_1, x_2] \to \mathbb{R}$ defined by $I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$ subject to the end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

In calculus of variations the basic problem is to find a function y for which the functional $I(y)$ is maximum or minimum. We call such functions as extremizing functions and the value of the functional at the extremizing function as extremum.

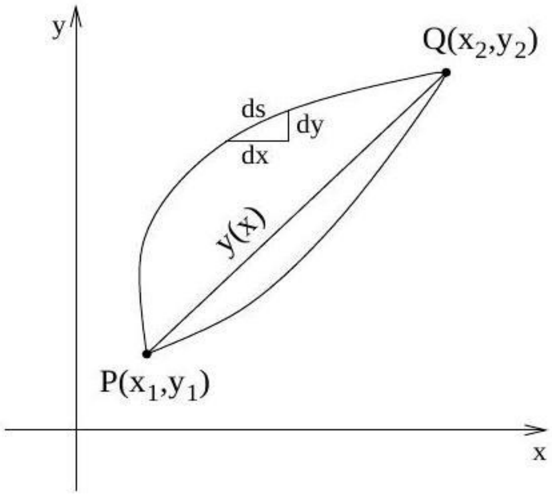
Consider the extremization problem

$$\underset{y}{\text{Extremize}} I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$$

subject to the end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, where F is a twice continuously differentiable function.

Example 1.8:

Find the shortest smooth plane curve joining two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$.



There are infinitely many functions y passing through the given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. We are looking for a function which will have minimum arc length. Let ds be a small strip on the curve then we have.

$$ds^2 = dx^2 + dy^2$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$\text{Total arc length } I(y) = \int_P^Q ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Thus, the problem is to minimize $I(y)$ subject to the end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.