Calculus of Variations

Mathematics Department	Lecture-2	Fourth Stage
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Lemma 2.1 (Fundamental Lemma of Calculus of Variations):

If $f(x)$ is a continuous function defined on $[a, b]$ and if $\int_a^b f(x)g(x)dx = 0$ for every function $g(x) \in C(a, b)$ such that $g(a) = g(b) = 0$ then $f(x) \equiv 0$ for all $x \in [a, b]$.

Proof:

Let $f(x) \neq 0$ for some $c \in (a, b)$. Without loss of generality let us assume that $f(c) > 0$. Now because of continuity of f we have $f(x) > 0$ for some interval $[x_1, x_2] \subset [a, b]$ that contains the point c.

$$\text{Let } g(x) = \begin{cases} (x - x_1)(x_2 - x) & \text{for } x \in [x_1, x_2] \\ 0 & \text{outside } [x_1, x_2] \end{cases}$$

Note that $(x - x_1)(x_2 - x)$ is positive for $x \in (x_1, x_2)$.

Now consider

$$\begin{aligned} \int_a^b f(x)g(x)dx &= \int_a^{x_1} f(x)g(x)dx + \int_{x_1}^{x_2} f(x)g(x)dx + \int_{x_2}^b f(x)g(x)dx \\ &= \int_{x_1}^{x_2} f(x)g(x)dx \\ &= \int_{x_1}^{x_2} f(x)(x - x_1)(x_2 - x)dx > 0 \end{aligned}$$

Thus, we get a contradiction to what is given in the Lemma. This implies that $f(x) \equiv 0$ on $[a, b]$.

2.1 Euler-Lagrange Equation (Necessary Condition for Extremum)

Theorem 2.1: If $y(x)$ is an extremizing function for the problem

$$\text{Minimize/Maximize } I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$$

with end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ then $y(x)$ satisfies the BVP.

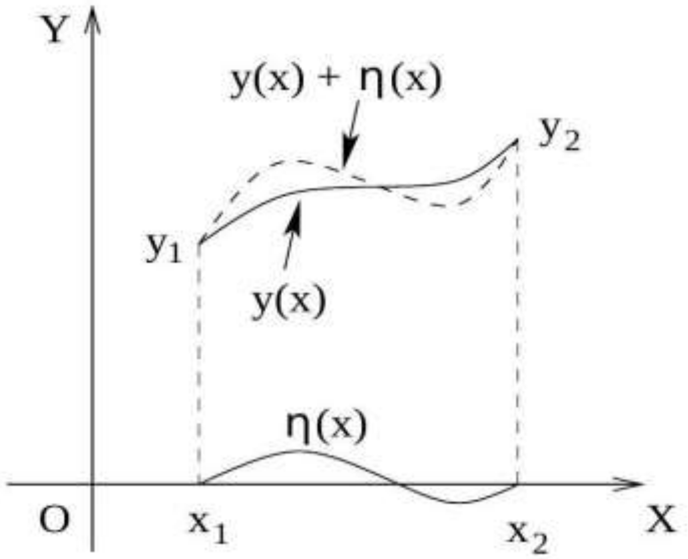
$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

$$y(x_1) = y_1 \qquad \text{and} \quad y(x_2) = y_2$$

Equation (2) is known as the Euler-Lagrange equation.

Proof:

Let $y(x)$ be an extremizing function for the functional $I(y)$ in (1).



Let $Y = y(x) + \epsilon\eta(x)$ be a variation of $y(x)$, where $\eta(x)$ is a continuously differentiable function with $\eta(x_1) = 0 = \eta(x_2)$ and ϵ is a small constant.

Hence I along the path $Y = y(x) + \epsilon\eta(x)$ is given by

$$I(\epsilon) = \int_{x_1}^{x_2} F\big(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\big)dx = \int_{x_1}^{x_2} F\big(x, Y(x), Y'(x)\big)dx$$

Since $y(x)$ is an extremizing function, $I(\epsilon)$ has extremum when $\epsilon = 0$.

Thus, by classical calculus,

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

$$\Rightarrow \frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial F}{\partial Y'}\frac{\partial Y'}{\partial \epsilon}\right]dx$$

But $\frac{\partial x}{\partial \epsilon} = 0$ as x is independent of ϵ.

$$\frac{\partial Y}{\partial \epsilon} = \eta(x)$$
$$\frac{\partial Y'}{\partial \epsilon} = \eta'(x)$$
$$\therefore \ \frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y}\eta(x) + \frac{\partial F}{\partial Y'}\eta'(x)\right] dx$$

Integration by parts we get

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial Y'}\eta'(x)dx = \frac{\partial F}{\partial Y'}\eta(x)\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\eta(x)dx$$
$$= -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\eta(x)dx$$
$$\frac{dI}{d\epsilon}\Big|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y} - \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\right]\Big|_{\epsilon=0} \eta(x)dx = 0$$
$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)dx = 0$$

as $Y(x)|_{\epsilon=0} = y(x)$ and $Y'(x)|_{\epsilon=0} = y'(x)$

Since $\eta(x)$ is arbitrary function, by applying the Fundamental Lemma of calculus of variations, we get

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

2.2 Different Forms of Euler Lagrange Equation

Suppose that $y(x)$ is an extremizer of $I(y)$.

Since $F = F(x, y, y')$

$$\frac{d}{dx}(F) = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx}$$
$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$$

Consider $\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}y''$

Subtracting (4) - (5) we get,

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y'$$

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]^0$$

As $y(x)$ is an extremizer and by using Euler-Lagrange equation we get

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$$

which is another form of Euler-Lagrange Equation.

2.3 Special Cases

$$\text{Extremize } I(y) = \int_{x_1}^{x_2} F(x, y, y')dx\ y(x_1) = y_1; y(x_2) = y_2$$

(i) When x does not appear in F explicitly

$$\frac{\partial F}{\partial x} = 0$$

Hence $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$ becomes $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$ or $F - y'\frac{\partial F}{\partial y'} =$ const. This is known as Beltrami Identity.

(ii) When y does not appear in F explicitly

$$\frac{\partial F}{\partial y} = 0$$

Hence, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ reduces to $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ or $\frac{\partial F}{\partial y'} =$ const.

(iii) When y' does not appear in F explicitly

$$\frac{\partial F}{\partial y'} = 0$$

Then $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ reduces to $\frac{\partial F}{\partial y} = 0$.

Example 2.1: Find the shortest smooth plane curve joining two distinct points (x_1, y_1) and (x_2, y_2). We are minimizing the arc length of the function y

$$\text{Minimize } I(y) = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\text{Here, } F(x, y, y') = \sqrt{1 + y'^2}$$

$$\frac{\partial F}{\partial y} = 0;\ \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

Euler-Lagrange equation

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0\, \frac{y'}{\sqrt{1 + y'^2}} = \text{ const } = \text{c}$$

$$y' = c\sqrt{1 + y'^2}$$

$$y'^2 = c^2(1 + y'^2)$$

$$y'^2 - c^2 y'^2 = c^2$$

$$(1 - c^2)y'^2 = c^2$$

$$\Rightarrow y'^2 = \frac{c^2}{1 - c^2}$$

$$\Rightarrow y' = \sqrt{\frac{c^2}{1 - c^2}} = m$$

$$y = mx + b$$

$$y(x_1) = y_1 \Rightarrow y_1 = mx_1 + b$$

$$y(x_2) = y_2 \Rightarrow y_2 = mx_2 + b$$

$$\frac{y_1 - y_2}{x_1 - x_2} = m\,;\, b = y_1 - \left(\frac{y_1 - y_2}{x_1 - x_2}\right) x_1$$

Thus, the curve having minimum arc length passing through the given two fixed point is a straight line.

Exercise 2.1: Show that another form of Euler - Lagrange equation is $F_y - F_{y'x} - F_{y'y}y' - F_{y'y'}y'' = 0$.

Example 2.2: Find the extremals of the functional $\int_{x_0}^{x_1} \frac{y'^2}{x^3} dx$

$$F(x, y, y') = \frac{y'^2}{x^3};\ \frac{\partial F}{\partial y} = 0;\ \frac{\partial F}{\partial y'} = \frac{2y'}{x^3}$$

Euler- Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = \frac{2x^3 y'' - 6y'x^2}{x^6} = \frac{2}{x^4}(xy'' - 3y') = 0$$

Thus, we have $xy'' - 3y' = 0$ or $\frac{y''}{y'} = \frac{3}{x}$

$$\int \frac{y''}{y'} = 3\int \frac{1}{x}dx + c$$
$$\ln y' = 3\ln x + c$$
$$\text{Integrating } \ln\left(\frac{y'}{x^3}\right) = c \text{ or } y' = Cx^3$$
$$y = \frac{Cx^4}{4} + C_2$$
$$y = Ax^4 + B$$

Example 2.3:

$$\text{Extremize } I(y) = \int_{x_1}^{x_2} 1 + y'^2 dx,\ y(x_1) = y(x_2) = 0$$
$$F = 1 + y'^2,\ \frac{\partial F}{\partial y'} = 2y'$$
$$\text{Euler Equation } \frac{d}{dx}(2y') = 0 \Longrightarrow 2y'' = 0$$
$$y'(x) = C,\ y(x) = Cx + D$$
$$C = \frac{y_1 - y_2}{x_1 - x_2};$$

Example 2.4: Find the curve y on which the functional $\int_0^1 y'^2 + 12xydx, y(0) = 0, y(1) = 1$ is exrtremum.

Solution:

$$\text{Here } F = y^2 + 12xy$$
$$\text{Euler Lagrange Equation: } 12x - 2y'' = 0 \text{ or } y'' = 6x$$
$$y' = 3x^2 + C$$
$$y = x^3 + cx + c'$$

Applying the conditions, we get

$$y = x^3$$