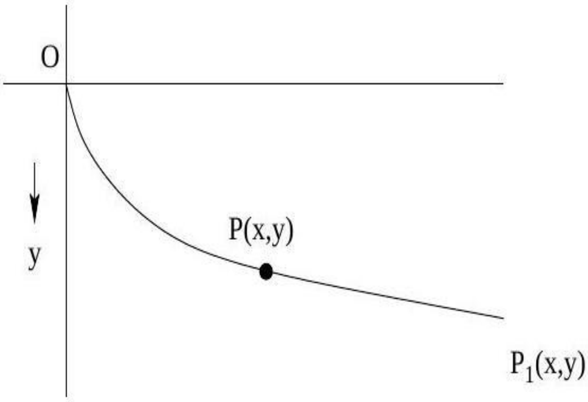
Calculus of Variations

Mathematics Department	Lecture-3	Fourth Stage
Second Semester	2023-2024	Lecturer: Dr. Andam

3.1 Brachistochrone Problem (Shortest Time of Descent Problem)

Find the shortest path on which a particle in the absence of friction will slide from one point to another point in the shortest time under the action of gravity.



Solution:

Let the particle slide from o along the path OP_1.

Let at time t, the particle be at $P(x, y)$. Let arc $OP = s$.

By the principle of work and energy, we have

KE at $P - KE$ at $O =$ Work done in moving the particle from O to P.

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 - 0 = mgy$$

That is

$$\frac{ds}{dt} = \sqrt{2gy}$$

⇒ Time taken by the particle to move from O to P_1 is given by

$$T = \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{2gy}} = \frac{1}{2g}\int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}}dx$$

$$T = \frac{1}{\sqrt{2g}}\int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}}dx$$

Brachistochrone Problem is to find y which minimizes the functional

$$I(y)=\frac{1}{\sqrt{2g}}\int_0^{x_1}\frac{\sqrt{1+y'^2}}{\sqrt{y}}dx \text{ and } F=\frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

Since x does not appear in F explicitly take the Beltrami Identity.

$$F-y'\frac{\partial F}{\partial y'}= \text{ const.}$$

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}}-y'\frac{\partial}{\partial y'}\left(\frac{\sqrt{1+y'^2}}{\sqrt{y}}\right)=c$$

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}}-y'\frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+y'^2}}=c$$

$$\frac{1+y'^2-y'^2}{\sqrt{y}\sqrt{1+y'^2}}=c$$

$$\frac{1}{\sqrt{y}\sqrt{1+y'^2}}=c\Rightarrow\sqrt{y(1+y'^2)}=\frac{1}{c}=\sqrt{a}(\text{ say })$$

$$y(1+y'^2)=a$$

$$1+y'^2=\frac{a}{y}$$

$$y'^2=\frac{a-y}{y}$$

$$y'=\sqrt{\frac{a-y}{y}}$$

$$\frac{dy}{dx}=\sqrt{\frac{a-y}{y}}$$

$$\sqrt{\frac{y}{a-y}}dy=dx$$

$$\int_0^x dx=\int_0^y\sqrt{\frac{y}{a-y}}dy$$

Since $(0,0)$ is point on the curve, we get $\mathrm{c}=0$.

Let $y=a\sin^2\theta\,; dy=2\,a\sin\theta\cos\theta d\theta$

$$x = \int_0^\theta \sqrt{\frac{a\sin^2\theta}{a - a\sin^2\theta}}\, 2\,\mathrm{asin}\,\theta\cos\theta d\theta$$

Thus, $$x = \int_0^\theta \frac{\sin\theta}{\cos\theta} 2\,\mathrm{asin}\,\theta\cos\theta d\theta = a\int_0^\theta 2\sin^2 d\theta = a\int_0^\theta (1-\cos 2\theta) d\theta$$

$$x = \frac{a}{2}[2\theta - \sin 2\theta]$$

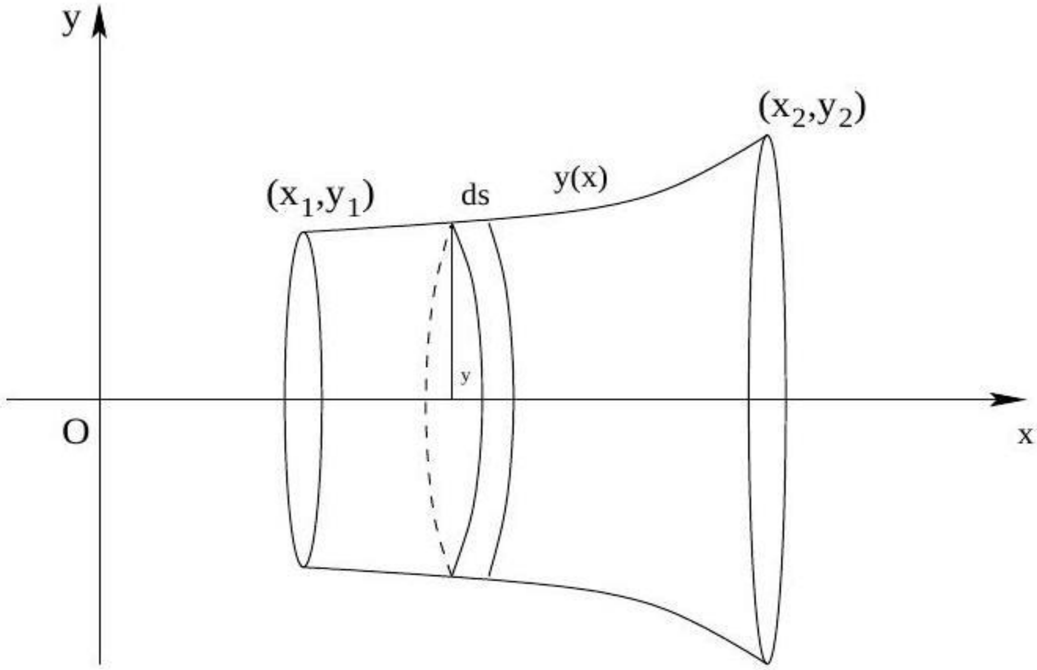
Let $\frac{a}{2} = b, 2\theta = \phi$

$$x = b(\phi - \sin\phi); y = b(1-\cos\phi)$$

which is a cycloid.

3.2 Minimum Surface Area of Rotation.

Find the curve passing through the points (x_1, y_1) and (x_2, y_2) which when rotated about the x-axis gives a minimum surface.



Let ds be a small strip on the curve y. Area of the surface generated by ds when revolved is $2\pi y ds$. In the figure the total surface area $= \int_{x_1}^{x_2} 2\pi y ds$

$$= 2\pi \int_{x_1}^{x_2} y\sqrt{(1 + y'^2)} dx$$

This has to be minimum.

Since $F = y\sqrt{(1+y'^2)}$ does not contain x explicitly, thus the Euler's equation reduces to

$$F - y'\frac{\partial F}{\partial y'} = c\text{: (say)}$$

$$y\sqrt{(1+y'^2)} - y'\frac{\partial}{\partial y'}y\sqrt{(1+y'^2)} = c$$

$$\text{i.e } y\sqrt{(1+y'^2)} - y'\frac{y}{2}(1+y'^2)^{-\frac{1}{2}} \cdot 2y' = c$$

$$\text{or } \frac{y}{\sqrt{(1+y'^2)}} = c$$

$$y^2 = c^2 + c^2y'^2$$

$$y' = \frac{dy}{dx} = \frac{\sqrt{(y^2-c^2)}}{c}$$

Separating the variables and integrating, we have

$$\int \frac{dy}{\sqrt{(y^2-c^2)}} = \int \frac{dx}{c} + c'$$

$$\cos h^{-1}\frac{y}{c} = \frac{x+a}{c}$$

$$\text{i.e } y = \text{c}\cosh\left(\frac{x+a}{c}\right)$$

which is catenary. The constants a and c are determined from the points (x_1, y_1) and (x_2, y_2).