Calculus of Variations

Mathematics Department	Lecture-4	Fourth Stage
Second Semester	2023-2024	Lecturer: Dr. Andam

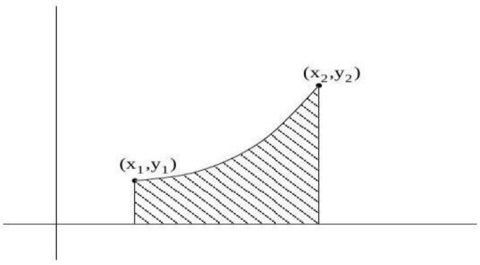
4.1 Constrained Extremization Problem

4.1.1 Isoperimetric Problems

In certain problems of calculus of variations, while extremizing a given functional $I(y)$, along with the end conditions $y(x_1) = y_1, y(x_2) = y_2$, we also need the extremizing function has to satisfy an additional integral constraint as we see in the following Dido's Problem.

4.1.2 Dido's Problem

Find the plane curve of fixed perimeter which has maximum area above x - axis.



The perimeter and the area under the curve are given by

Perimeter = Arc length = $\int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx$

Area under the curve $A = \int_{x_1}^{x_2} y(x)dx$.

Variational Problem: Maximize $I(y) = \int_{x_1}^{x_2} y(x)dx$

subject to the constraints

$$\int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx = L \text{ (given) and with end conditions}$$

$$y(x_1) = y_1 \text{ and } y(x_2) = y_2.$$

General Problem: Extremize $I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$

subject to the integral constraint $\int_{x_1}^{x_2} G(x, y, y')dx = L =$ constant

End conditions are $y(x_1) = y_1$ and $y(x_2) = y_2$.

4.2 Lagrange Multiplier Technique

Convert the constrained optimization problem into an unconstrained optimization problem by the Lagrange Multiplier Technique.

Define a new functional H by $H(x,y,y') = F(x,y,y') + \lambda G(x,y,y')$ and optimize $\int_{x_1}^{x_2} H(x,y,y')dx$ without constraints.

That is, Optimize $I(y) = \int_{x_1}^{x_2} F(x,y,y') + \lambda G(x,y,y')dx$

with end condition $y(x_1) = y_1$ and $y(x_2) = y_2$.

The problem is solved by solving the

Euler Lagrange Equation: $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

$$y(x_1) = y_1,\ y(x_2) = y_2$$

Solution of Dido's Problem:

$$\text{Maximize } I(y) = \int_{x_1}^{x_2} y(x)dx \ \text{ subject to}$$

$$\int_{x_1}^{x_2} \sqrt{1 + y'^2}dx = L \ \text{ and with end conditions}$$

$$y(x_1) = y_1,\ y(x_2) = y_2$$

$$\text{Here, } F(x,y,y') = y(x),\ G(x,y,y') = \sqrt{1 + y'^2}$$

$$H(x,y,y') = y + \lambda\sqrt{1 + y'^2}$$

$$\frac{\partial H}{\partial y} = 1,\ \frac{\partial H}{\partial y'} = \frac{\lambda y'}{\sqrt{1 + y'^2}}$$

Euler's Equation:

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1 + y'^2}}\right) = 1$$

$$\Rightarrow \frac{\lambda y'}{\sqrt{1 + y'^2}} = x + a$$

$$\Rightarrow \frac{\lambda y'}{x + a} = \sqrt{1 + y'^2}$$

$$\Rightarrow \lambda^2 y'^2 = (1 + y'^2)(x + a)^2$$

$$\Rightarrow y'^2(\lambda^2 - (x + a)^2) = (x + a)^2$$

$$y' = \frac{x + a}{\sqrt{\lambda^2 - (x + a)^2}}$$

$$\Rightarrow y = -\sqrt{\lambda^2 - (x + a)^2} + b$$

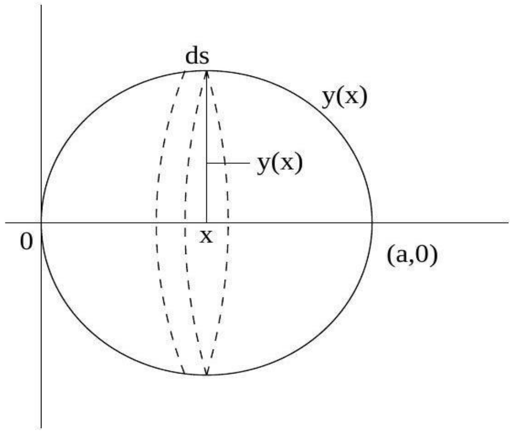
$$(y - b)^2 = \lambda^2 - (x + a)^2$$

$$\Rightarrow (x + a)^2 + (y - b)^2 = \lambda^2$$

which is a circle, where the constants a, b, λ can be obtained from 3 conditions namely, two end conditions and one constraint condition.

Problem 4.2.1:

Show that sphere is the solid figure of revolution which for a given surface area having maximum volume enclosed.



Consider a small circular strip having height ds and radius y. The surface area is $2\pi y ds$.

Thus the total surface area of revolution is $S = \int_0^a 2\pi y ds$

$$= \int_0^a 2\pi y\sqrt{1 + y'^2}dx$$

$$\text{Volume } = \int_0^a \pi y^2 dx$$

Variational Problem: Maximize $I(y) = \int_0^a \pi y^2 dx$

subject to the constraint

$$\int_0^a 2\pi y\sqrt{1 + y'^2}dx = S \text{ (constant)}$$

Define a function $H = F + \lambda G = \pi y^2 + \lambda 2\pi y\sqrt{1 + y^2}$.

As x is not appearing in H explicitly, we have Euler's Equation (Beltrami Identity)

$$H - y'\frac{\partial H}{\partial y'} = \text{ const.}$$

$$\pi y^2 + 2\pi\lambda y\sqrt{1 + y'^2} - y'\frac{\lambda 2\pi y y'}{\sqrt{1 + y'^2}} = c$$

$$\pi y^2 + \frac{2\pi\lambda y(1 + y'^2) - \lambda 2\pi y y'^2}{\sqrt{1 + y'^2}} = c$$

$$\pi y^2 + \frac{2\pi\lambda y}{\sqrt{1 + y'^2}} = c$$

Since the curve passes through (0,0), when $y = 0, c = 0$ $y^2 = \frac{-2\lambda y}{\sqrt{1+y'^2}}$.

$$y = \frac{-2\lambda}{\sqrt{1 + y'^2}}$$
$$\Rightarrow y^2(1 + y'^2) = 4\lambda^2$$
$$\Rightarrow y'^2 = \frac{4\lambda^2 - y^2}{y^2}$$
$$y' = \frac{\sqrt{4\lambda^2 - y^2}}{y}$$
$$\int \frac{ydy}{\sqrt{4\lambda^2 - y^2}} = \int dx + k$$
$$x = k - \sqrt{4\lambda^2 - y^2}$$
When $x = 0, y = 0 \Rightarrow x = 2\lambda$
$$x = 2\lambda - \sqrt{4\lambda^2 - y^2}$$
$$(x - 2\lambda) = -\sqrt{4\lambda^2 - y^2}$$
$$(x - 2\lambda)^2 + y^2 = 4\lambda^2$$

The curve is a circle, centred at $(2\lambda, 0)$ and radius 2λ. Hence the solid of revolution is a sphere.