Calculus of Variations

Mathematics Department	Lecture-5	Fourth Stage
Second Semester	2023-2024	Lecturer: Dr. Andam

5.1 Problem with Higher order Derivatives:

$$\text{Extremize } I(y) = \int_{x_1}^{x_2} F(x, y, y', y'')dx$$

$$y(x_1) = y_1,\ y(x_2) = y_2$$

$$y'(x_1) = y_1'\ y'(x_2) = y_1'$$

The Euler - Poisson Equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

Example 5.1: Extremize $I(y) = \int_{x_1}^{x_2} (y^2 - (y'')^2)dx$ with end conditions:

$$y(x_1) = y_1,\ y(x_2) = y_2$$

$$y'(x_1) = y_1',\ y'(x_2) = y_2'$$

$$\frac{\partial F}{\partial y} = 2y,\ \frac{\partial F}{\partial y'} = 0,\ \frac{\partial F}{\partial y''} = -2y''$$

Euler - Poisson Equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

$$2y - 0 + \frac{d^2}{dx^2}(-2y'') = 0$$

$$y^{(iv)} - y = 0$$

$$y(x_1) = y_1,\ y(x_2) = y_2$$

$$y'(x_1) = y_1',\ y'(x_2) = y_2'$$

5.2 Problems with several unknown functions:

Let u and v be the unknown functions which extremize the functional I.

$$\text{Extremize } I(u, v) = \int_{x_1}^{x_2} F(x, u, v, u', v')dx$$

$$u(x_1) = u_1,\ u(x_2) = u_2$$

$$v(x_1) = v_1,\ v(x_2) = v_2$$

Euler - Lagrange equations:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$$
$$\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) = 0$$

5.3 Problems with more than one independent variables:

Let z be the dependent variable and x and y be the independent variables. Extremize $I(z) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} F(x, y, z, z_x, z_y)dydx$

where z is prescribed on the boundary ∂D of the domain D where F is defined.

Euler - Lagrange Equation

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

Example 5.3: Find a function Φ whose mean square value of the magnitude of the gradient over a region D is minimum

The problem is

$$\text{Minimize } I(\Phi) = \iint (\Phi_x^2 + \Phi_y^2)dxdy$$

where Φ is prescribed on the boundary ∂D of D.

$$\text{Here } F = \Phi_x^2 + \Phi_y^2$$

Euler Lagrange Eq: $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \Phi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \Phi_y}\right) = 0$

$$\frac{\partial}{\partial x}(2\Phi_x) + \frac{\partial}{\partial y}(2\Phi_y) = 0$$
$$\Phi_{xx} + \Phi_{yy} = 0$$
$$\Delta\Phi = 0 \quad \text{(Laplace Equation)}$$
$$\Phi|_{\partial D} = \text{ prescribed.}$$