Calculus of Variations

Mathematics Department	Lecture-6	Fourth Stage
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6.1 The Variational Notation:

When a function changes its value from $y(x)$ to $y(x+\Delta x)$, the rate of change of this defines the derivative $y'(x)$. Whereas in variational calculus the function $y(x)$ is changed to a new function $y(x)+\epsilon\eta(x)$, where ϵ is a constant and $\eta(x)$ is a continuous differentiable function. The change $\epsilon\eta(x)$ in $y(x)$ as a function is called the variation of y and is denoted by δy.

That is $\delta y=\epsilon\eta(x)$. Similarly, we have $\delta y'=\epsilon\eta'(x)$. In $F=F(x,y,y')$ for a fixed x, change in y from y to $y+\epsilon\eta$ makes F to change to $F(x,y+\epsilon\eta,y'+\epsilon\eta')$. Thus, the change in F, denoted by ΔF is given by

$$\Delta F=F(x,y+\epsilon\eta,y'+\epsilon\eta')-F(x,y,y')$$

Expanding the first term on RHS in Taylors series

$$\Delta F=F(x,y,y')+\left(\frac{\partial F}{\partial y}\eta+\frac{\partial F}{\partial y'}\eta'\right)\epsilon+\left[\frac{\partial^2 F}{\partial y^2}\eta^2+2\frac{\partial^2 F}{\partial y\,\partial y'}\eta\eta'+\frac{\partial^2 F}{\partial y'^2}(\eta')^2\right]\frac{\epsilon^2}{2!}$$
$$+\text{ higher order terms of }(\epsilon)-F(x,y,y')$$
$$=\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'+\frac{1}{2!}\left[\frac{\partial^2 F}{\partial y^2}(\delta y)^2+2\frac{\partial^2 F}{\partial y\,\partial y'}\delta y\delta y'+\frac{\partial^2 F}{\partial y'^2}(\delta y')^2\right]^2$$
$$+\text{ higher order terms}$$

First variation $\delta F=\dfrac{\partial F}{\partial y}\delta y+\dfrac{\partial F}{\partial y'}\delta y'$

Second Variation $\delta^2 F=\dfrac{1}{2}\left[\dfrac{\partial^2 F}{\partial y^2}(\delta y)^2+2\dfrac{\partial^2 F}{\partial y\,\partial y'}\delta y\delta y'+\dfrac{\partial^2 F}{\partial y'^2}(\delta y')^2\right]$

6.2 Variation is analogous to derivative in calculus

Properties:

(1) $\delta(F_1\pm F_2)=\delta F_1\pm\delta F_2$

(2) $\delta(F_1F_2)=F_1\delta F_2+F_2\delta F_1$

(3) $\delta\left(\frac{F_1}{F_2}\right)=\frac{F_2\delta F_1-F_1\delta F_2}{F_2^2}$, $F_2\neq 0$

(4) $\delta(F^n) = nF^{n-1}\delta F$

Example 6.1:

(i) $\delta(y^2) = 2y\delta y$

(ii) $\delta(y'^2) = 2y'\delta y'$

(iii) $\delta(xy) = x\delta y$

(iv) $\delta(x^2) = 0$

Problem 6.1: If $I(y) = \int_{x_1}^{x_2} F(x,y,y')dx$, find the variation δI of I.

Solution:

$$\delta I = \delta\left(\int_{x_1}^{x_2} F(x,y,y')dx\right)$$
$$= \int_{x_1}^{x_2} \delta F(x,y,y')dx$$
$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]dx$$

In the classical calculus, if x_0 an optimizing point for a differentiable function $f(x)$ then $f'(x_0) = 0$. Analogous to this, we have the following result in calculus of variations.

Theorem 6.1: If $y(x)$ is an extremizing function for

$$\text{Extremize } I(y) = \int_{x_1}^{x_2} F(x,y,y')dx,\ y(x_1) = y_1,\ y(x_2) = y_2$$

Then the first variations $\delta I(y) = 0$

Proof:

The first variation of I is given by

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]dx$$
$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)\right]dx$$

Integrating by points on the

$$\int \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)dx = \frac{\partial F}{\partial y'}\delta y\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\delta x$$

second term,
$$\delta I(y) = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y dx$$

=0 because of Euler-Lagrange Equation for an extremizer.

Hence $\delta I(y) = 0$ if y is an extremizing function.

6.3 Hamilton's Principle

Let T be the kinetic energy and V be the potential energy of a particle in motion. Let $L = T - V$ be the kinetic potential or the Lagrangian function.

$$\text{Let } A = \int_{t_1}^{t_2} Ldt \quad \text{(Action integral)}$$

$$\delta A = 0 \text{ (Principle of Least Action)}$$

$$\delta\left(\int_{t_1}^{t_2} Ldt\right) = \delta\left(\int_{t_1}^{t_2} T - Vdt\right) = 0$$

Hamilton Principle states that the motion is such that the integral of the difference between kinetic and potential energies is stationary for the true path. Over a sufficiently small-time interval the integral is a minimum. That is, nature tends to equalize the kinetic and potential energies over motion. Hence $\delta A = 0$ along truth path.

6.4 Second Order Conditions for Extremum.

As in the classical calculus, if f is differentiable and x_0 is a stationary point of f then $f''(x_0) > 0$ implies x_0 is minimum & $f''(x_0) < 0$ implies x_0 is maximum point, we have the following second order conditions for testing extremum in Calculus of variations.

6.5 Legendre Test for Extremum

Let I be a functional and y be an extremizer of I then

(i) $\delta I(y) = 0$. (Euler-Lagrange Equation)

(ii) $\delta^2 I(y) > 0 \Rightarrow y$ is a minimizing function.

(iii) $\delta^2 I(y) < 0 \Rightarrow y$ is a maximizing function.