Calculus of Variations

Mathematics Department	Lecture-7	Fourth Stage
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7.1 Reduction of BVP into Variational Problems

If a variational problem is given, the corresponding Euler-Lagrange equation is a BVP. Now, we ask the question, if a BVP is given, can we find its corresponding variational problem. The answer is yes for a class of BVP. We demonstrate it with the following

Example 7.1: Reduce the BVP

$$y'' - y + x = 0$$
$$y(0) = y(1) = 0$$

into a variational problem.

Solution: Multiply both sides of (6) by δy and integrate over $(0,1)$.

$$\int_0^1 y''\delta y dx - \int \ y\delta y dx + \int \ x\delta y dx = 0$$

Integration by parts,

$$y'\delta y']_0^1 - \int_0^1 y'\delta y' dx - \int \ y\delta y dx + \int \ x\delta y dx = 0$$

But $\delta(y'^2) = 2y'\delta y', \delta y^2 = 2y\delta y \delta(xy) = x\delta y$

$$-\int_0^1 \frac{1}{2}\delta y'^2 dx - \int_0^1 \frac{1}{2}\delta y^2 dx + \int \ \delta xy dx = 0$$

$$\int_0^1 \delta\left(-\frac{1}{2}y'^2 - \frac{1}{2}y^2 + xy\right) dx = 0$$

$$\delta\left(\int_0^1 y'^2 + y^2 - 2xy dx\right) = 0$$

It is of the form $\delta I(y) = 0$

Thus, the corresponding variational problem is

$$\left.\begin{array}{c}\text{Extremize } \ I(y) = \int_0^1 y'^2 + y^2 - 2xy dx \\ y(0) = 0, \ y(1) = 0\end{array}\right\} \text{V.P}$$

If we find the Euler - Lagrange equation of the above V.P, we have

$$F = y'^2 - y^2 - 2xy$$
$$\frac{\partial F}{\partial y} = 2y - 2x$$
$$\frac{\partial F}{\partial y'} = 2y'$$

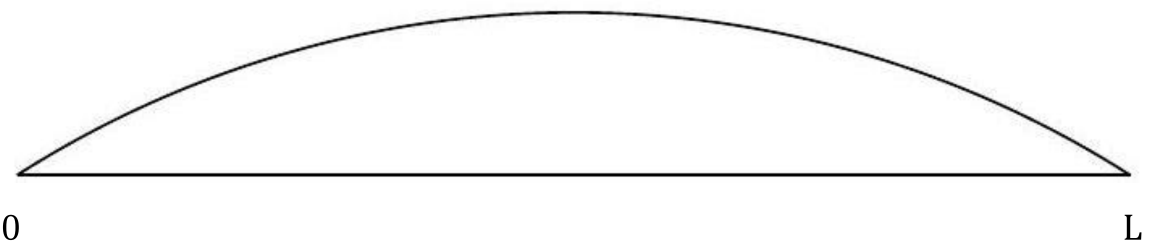
Euler-Lagrange eqn. is given by

$$\left.\begin{array}{c} 2y - 2x - \frac{d}{dx}(2y') = 0 \\ y'' - y + x = 0 \\ y(0) = y(1) = 0 \end{array}\right\} \text{ which is same as the original BVP}$$

Example 7.2: Deflection of a rotating string of Length L.

Consider the boundary value problem

$$\begin{array}{r} \frac{d}{dx}\left(F(x)\frac{dy}{dx}\right) + \rho\omega^2 y + p(x) = 0 \\ y(0) = 0 \; y(L) = 0 \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$



where

$y(x) -$ displacement of a point from the axis of rotation.
$F(x) -$ tension.
$\rho(x) -$ linear mass density.
$\omega -$ angular velocity of rotation.
$p(x) -$ intensity of distributed radial load.

We now reduce this BVP into a variational problem as follows: Multiply (7) by a variation δy and integrate over $(0, L)$ to obtain

$$\int_0^L \frac{d}{dx}\left(F\frac{dy}{dx}\right)\delta y dx + \int_0^L \rho\omega^2 y\delta y dx + \int_0^L p\delta y dx = 0$$

Consider the first term and integration by parts gives

$$\int_0^L \frac{d}{dx}\left(F\frac{dy}{dx}\right)\delta y dx = \left(F\frac{dy}{dx}\delta y\right]_0^{E-0} - \int_0^L F\frac{dy}{dx}\delta\frac{dy}{dx}dx$$

But $\delta(y'^2) = 2y'\delta y',\ \delta(y^2) = 2y\delta y$ reduce

$$\int_0^L \delta\left(-\frac{1}{2}Fy'^2\right) + \rho\omega^2\delta\left(\frac{1}{2}y^2\right) + \delta py dx = 0$$

$$\int_0^L \delta\left(-\frac{1}{2}Fy'^2 + \frac{1}{2}\rho\omega^2y^2 + py\right)dx = 0$$

That is, $\delta I = 0$

Thus, the variational problem is:

$$\text{Extremize } I(y) = \int_0^L (-Fy'^2 + \rho\omega^2y^2 + 2py)dx$$

$$y(0) = 0,\ y(L) = 0$$

Example 7.3: Reduce the BVP

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + y = \mathrm{x}$$

$$y(0) = 0, y(1) = 1$$

into a variational problem.

Solution: Multiplying by δy and integrating over $(0,1)$

$$\int_0^1 \left(x\frac{dy}{dx}\right)'\delta y dx + \int_0^1 y\delta y - \int_0^1 x\delta y = 0$$

$$\left(x\frac{dy}{dx}\right)\delta y\Big]_0^{x^0} - \int_0^1 x\frac{dy}{dx}\delta\frac{dy}{dx}dx + \int \delta\left(\frac{1}{2}y^2\right)dx - \int \delta xy dx = 0$$

$$\int_0^1 \delta\left(\frac{-xy'^2}{2} + \frac{1}{2}y^2 - xy\right)dx = 0$$

$$\delta\int_0^1 (-xy'^2 + y^2 - 2xy)dx = 0$$

$$\delta I = 0$$

$$\text{where } I(y) = \int_0^1 -xy'^2 + y^2 - 2xydx$$

$$y(0) = 0, y(1) = 1$$