Mathematics Department	Lecture-8	Fourth Stage
Second Semester	2023-2024	Lecturer: Dr. Andam

8.1 Direct Method to Solve Variational Problems (Rayleigh Ritz Method to find approximate solution): Let $C^1[x_1, x_2]$ be the set of all continuously differentiable functions defined on $[x_1, x_2]$. Consider the variational problem:

$$\text{Min } I(y) = \int_{x_1}^{x_2} F(x, y, y')dx, \ y(x_1) = y_1 \,\&\, y(x_2) = y_2$$

Let $y(x) \in C^1[x_1, x_2]$ be the solution to the V.P. Let $B = \{\phi_0(x), \phi_1(x), \dots, \phi_n(x), \dots\}$ be basis for the infinite dimensional vectorspace $C^1[x_1, x_2]$. Let $\bar{y}(x)$ be an approximation of y given by

$$\bar{y}(x) = \sum_{i=0}^{n} c_i \phi_i(x)$$

The basis functions are taken such that the boundary condition $\bar{y}(x_1) = y_1$ and $\bar{y}(x_2) = y_2$ are satisfied.

The problem becomes

$$\text{Minimize } I(y) = \int_{x_1}^{x_2} F\left(x, \sum_{i=0}^{\infty} c_i \phi_i(x), \sum_{i=0}^{\infty} c_i \phi_i'(x)\right) dx$$

$$y(x_1) = y_1 \,\&\, y(x_2) = y_2$$

The problem to find an approximate solution $\bar{y}$ in

$$\text{Minimize } I(\bar{y}) = \int_{x_1}^{x_2} F\left(x, \sum_{i=0}^{n} c_i \phi_i(x), \sum_{i=0}^{n} c_i \phi_i'(x)\right) dx$$

$$\bar{y}(x_1) = y_1 \,\&\, \bar{y}(x_2) = y_2$$

Since $\phi_0, \phi_1, \dots$ are known basic functions, the only unknown are $c_0, c_1, \dots, c_n$, we have

$$\min \quad I(\bar{y}) = \min_{c_0, c_1, \cdots, c_n} I(c_0, c_1, \cdots, c_n)$$

Using the classical calculus, we have

$$\frac{\partial I}{\partial c_i} = 0,\ i = 0,1,2, \cdots, n.$$

If we simplify this $n+1$ equation, we need to solve $n+1$ linear equation in $n+1$ unknowing to set $c_0, c_1, \cdots, c_n$.

Example Find approximate solution to the BVP

$$y'' - y + x = 0,\ y(0) = y(1) = 0$$

by using Rayleigh-Ritz Method.

Solution:

$$I(y) = \int_0^1 2xy - y^2 - y'^2 dx$$

Let $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$ be an approximate solution.

Applying the both condition

$$\bar{y}(0) = 0 \Longrightarrow c_0 = 0$$

$$\bar{y}(1) = 0 \Longrightarrow c_1 + c_2 = 0\ c_2 = -c_1$$

Thus $\bar{y}(x) = c_1 x(1-x)$, where c_1 has to be determined

$$I(c_1) = \int_0^1 2x\bar{y} - \bar{y}^2 - \bar{y'}^2 dx$$

$$= \int_0^1 (2c_1(x^2 - x^3) - c_1^2(x - x^2)^2 - c_1(1-x)^2 dx$$

$$= \frac{1}{6} c_1 - \frac{11}{30} c_1^2$$

$$\Rightarrow \frac{dI(c_1)}{dc_1} = 0$$

$$\Rightarrow \frac{1}{6} - \frac{22}{30} c_1 = 0$$

$$\Rightarrow \frac{22}{30} c_1 = \frac{1}{6}$$

$$\Rightarrow c_1 = \frac{5}{22}$$

$$\bar{y}(x) = \frac{5}{22}x(1-x)$$ is the approximate solution.

Exact Solution: $y(x) = x - \frac{e^x - e^{-x}}{e - e^{-1}}$

x	Approximate solution	Exact solution
0.25	0.043	0.035
0.50	0.057	0.057
0.75	0.43	0.05