

# Calculus of Variations

<b>Mathematics Department</b>	<b>Assignment</b>	<b>Fourth Stage</b>
<b>Second Semester</b>	<b>2023-2024</b>	<b>Lecturer: Dr. Andam</b>

1. Solve the Euler-Lagrange equation for the functional

$$\int_{\frac{1}{10}}^1 y'(1 + x^2 y') dx$$

subject to the end conditions  $y\left(\frac{1}{10}\right) = 19, y(1) = 1$ .

2. Derive Euler-Lagrange equation for the variational problem

$$\text{Extremize } I(y) = \int_{x_1}^{x_2} F(x, y, y') dx, y(x_1) = y_1 \text{ and } y(x_2) = y_2.$$

Deduce Beltrami identity from it.

\*3. Find the curve on which the functional

$$\int_0^1 (y'^2 + 12xy) dx \text{ with } y(0) = 0, y(1) = 1$$

has extremum value.

4. Find an extremal for the functional  $I(y) = \int_0^{\frac{\pi}{2}} [y'^2 - y^2] dx$  which satisfies the boundary conditions  $y(0) = 0$  and  $y\left(\frac{\pi}{2}\right) = 1$ .

5. Show that the Euler-Lagrange equation can also be written in the form

$$F_y - F_{y'x} - F_{y'y} y' - F_{y'y'} y'' = 0$$

\*6. It is required to determine the continuously differentiable function  $y(x)$  which minimizes the integral  $I(y) = \int_0^1 (1 + y'^2) dx$ , and satisfies the end conditions  $y(0) = 0, y(1) = 1$ .

(a) Obtain the relevant Euler equation, and show that the stationary function is  $y = x$ .

(b) With  $y(x) = x$  and the special choice  $\eta(x) = x(1 - x)$  and with the notation  $I(\epsilon) = \int_0^1 F(x, y + \epsilon\eta(x), y' + \epsilon\eta'(x)) dx$ , calculate  $I(\epsilon)$  and verify directly that  $\frac{dI(\epsilon)}{d\epsilon} = 0$  when  $\epsilon = 0$ .

7. Find the extremal of the following functionals

(a)  $I(y) = \int_{x_1}^{x_2} [y^2 - (y')^2 - 2y \cos hx] dx, y(x_1) = y_1 \& y(x_2) = y_2$

$$(b) I(y) = \int_{x_1}^{x_2} \frac{1+y^2}{y'^2} dx$$

$$*(c) I(y) = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x} dx$$

$$(d) I(y) = \int_0^1 (xy + y^2 - 2y^2 y') dx, y(0) = 1, y(1) = 2$$

$$(e) I(y) = \int_{x_1}^{x_2} (y^2 + y'^2 - 2y \sin x) dx$$

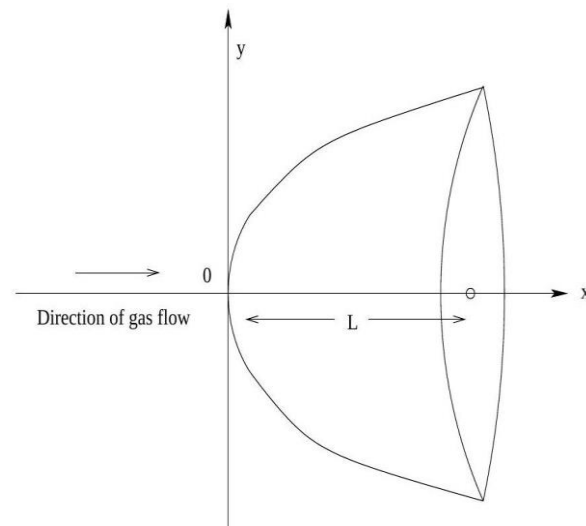
$$(f) \int_0^{\frac{\pi}{2}} (y'^2 - y^2 + 2xy) dx, y(0) = 0, y\left(\frac{\pi}{2}\right) = 0$$

$$(g) \int_{x_1}^{x_2} (y^2 + 2xyy') dx; y(x_1) = y_1, y(x_2) = y_2$$

$$*(h) \int_0^{\pi} (4y \cos x - y^2 + y'^2) dx; y(0) = 0, y(\pi) = 0$$

$$*(i) I(y) = \int_{x_0}^{x_1} (y^2 + y'^2 + 2ye^x) dx$$

8. Determine the shape of solid of revolution moving in a flow of gas with least resistance.



(Hint : The total resistance experienced by the body is  $I(y) = 4\pi\rho v^2 \int_0^L yy'^3 dx$  where  $\rho$  is the density,  $v$  is the velocity of gas relative to the solid).

9. Prove the following facts by using COV:

(a) The shortest distance between two points in a plane is a straight line.

(b) The curve passing through two points on  $xy$  plane which when rotated about  $x$ -axis giving a minimum surface area is a Catenary.

(c) The path on which a particle in absence of friction slides from one point to another in the shortest time under the action of gravity is a Cycloid(Brachistochrone Problem).

\*10. Find the extremal of the functional

$$I(y) = \int_0^{\pi} (y'^2 - y^2) dx, \quad y(0) = 0, y(\pi) = 1$$

and subject to the constraint  $\int_0^{\pi} y dx = 1$ .

11. Find the extremal of the isoperimetric problem

$$\text{Extremize } I(y) = \int_1^4 y'^2 dx, \quad y(1) = 3, y(4) = 24$$

subject to  $\int_1^4 y dx = 36$ .

\*12. Determine  $y(x)$  for which  $\int_0^1 x^2 + y'^2 dx$  is stationary subject to  $\int_0^1 y^2 dx = 2$ ,  $y(0) = 0, y(1) = 0$ .

13. Find the extremal of  $I = \int_0^{\pi} y'^2 dx$  subject to  $\int_0^{\pi} y^2 dx = 1$  and satisfying  $y(0) = y(\pi) = 0$ .

\*14. Given  $F(x, y, y') = (y')^2 + xy$ . Compute  $\Delta F$  and  $\delta F$  for  $x = x_0, y = x^2$  and  $\delta y = \epsilon x^n$ .

15. Find the extremals of the isoperimetric problem

$$I(y) = \int_{x_0}^{x_1} y'^2 dx$$

given that  $\int_{x_0}^{x_1} y dx = \text{constant}$ .

16. Prove the following facts by using COV:

(a) The geodesics on a sphere of radius  $a$  are its great circles.

(b) The sphere is the solid figure of revolution which, for a given surface area has maximum volume.

17. If  $y$  is an extremizing function for

$$I(y) = \int_{x_1}^{x_2} F(x, y, y'), \quad y(x_1) = y_1, \text{ and } y(x_2) = y_2$$

then show that  $\delta I = 0$  for the function  $y$ .

\*18. Find  $y(x)$  for which

$$\delta \left\{ \int_{x_0}^{x_1} \left( \frac{y'^2}{x^3} \right) dx \right\} = 0$$

and  $y(x_1) = y_1$  and  $y(x_2) = y_2$ .

19. Write down the Euler-Lagrange equation for the following extremization problems

(i) Extremize  $I(u, v) = \int_D F(x, y, u, v, u_x, u_y, v_x, v_y) dx dy$  where  $x, y$  are independent variables and  $u, v$  are dependent variables.  $D$  is a domain in  $xy$  plane and  $u$  and  $v$  are prescribed on the boundary of  $D$ .

(ii) Extremize  $I(y) = \int_{x_0}^{x_1} F(x, y, y^{(1)}, y^{(2)}, \dots, y^{(m)}) dx$

$$y(x_0) = y_0, y(x_1) = y_1$$

$$y'(x_0) = y'_0, y'(x_1) = y'_1$$

.....

$$y^{(m-1)}(x_0) = y_0^{(m-1)}, y^{(m-1)}(x_1) = y_1^{(m-1)}$$

(iii) Max or Min  $I(y) = \int_{x_1}^{x_2} F(x, y, y') dx$  where  $y$  is prescribed at the end points  $y(x_1) = y_1$ ,  $y(x_2) = y_2$ , and  $y$  is also to satisfy the integral constraint condition  $J(y) = \int_{x_1}^{x_2} G(x, y, y') dx = k$ , where  $k$  is a prescribed constant.

\*20. Show that the extremals of the problem

Extremize  $I(y) = \int_{x_1}^{x_2} [p(x)y'^2 - q(x)y^2] dx$

where  $y(x_1)$  and  $y(x_2)$  are prescribed and  $y$  satisfies a constraint  $\int_{x_1}^{x_2} r(x)y^2 dx = 1$ , are

solutions of the differential equation  $\frac{d}{dx} \left( p \frac{dy}{dx} \right) + (q + \lambda r)y = 0$

where  $\lambda$  is a constant.

\*21. Reduce the BVP

$$\frac{d}{dx} \left( x \frac{dy}{dx} \right) + y = x, y(0) = 0, y(1) = 1$$

into a variational problem and use Rayleigh-Ritz method to obtain an approximate solution in the form

$$y(x) \approx x + x(1-x)(c_1 + c_2x)$$

22. (Principle of least Action) A particle under the influence of a gravitational field moves on a path along which the kinetic energy is minimal. Using calculus of variation prove that the trajectory is parabolic.

(Hint: Minimize  $I = \int \frac{1}{2}mv^2 dt = \int \frac{1}{2}mvd s = \int \sqrt{u^2 - 2gy} \sqrt{1 + y'^2} dx$  ) where  $u$  is the initial speed.

23. Show that the curve which extremizes the functional  $I(y) = \int_0^{\frac{\pi}{4}} (y''^2 - y^2 + x^2) dx$

under the conditions  $y(0) = 0, y'(0) = 1, y\left(\frac{\pi}{4}\right) = y'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$  is  $y = \sin x$ .

24. Find a function  $y(x)$  such that  $\int_0^{\pi} y^2 dx = 1$  which makes  $\int_0^{\pi} (y'')^2 dx$  a minimum

if  $y(0) = 0 = y(\pi), y''(0) = 0 = y''(\pi)$ .

\* 25. Find the extremals of the following functional

$$I(y) = \int_{x_1}^{x_2} 2xy + (y''')^2 dx$$

26. Find the extremals of the functional

$$I(u, v) = \int_{x_0}^{x_1} 2uv - 2u^2 + u'^2 - v'^2 dx$$

where  $u$  and  $v$  are prescribed at the end points.

27. Find a function  $y(x)$  such that  $\int_0^{\pi} y^2 dx = 1$  which makes  $\int_0^{\pi} y''^2 dx$  a minimum if

$y(0) = 0 = y(\pi), y''(0) = 0 = y''(\pi)$

\*28. Show that the functional  $\int_0^{\frac{\pi}{2}} 2xy \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 dt$  such that  $x(0) = 0, x\left(\frac{\pi}{2}\right) =$

$-1, y(0) = 0, y\left(\frac{\pi}{2}\right) = 1$  is stationary for  $x = -\sin t, y = \sin t$ .

\*29. Explain Rayleigh - Ritz method to find an approximate solution of the variational problem

$$\text{Extremize } I(y) = \int_{t_0}^{t_1} F(x, y, y') dx$$

with prescribed end conditions  $y(x_1) = y_1$  &  $y(x_2) = y_2$ .

30. Solve the BVP  $y'' + y + x = 0, y(0) = y(1) = 0$  by Rayleigh - Ritz method.

31. Use Rayleigh - Ritz method to find an approximate solution of the problem  $y'' - y +$

$4xe^x = 0, y'(0) - y(0) = 1, y'(1) + y(1) = -e$ .