Mathematics Department	**Exercises** Chapter one	Fourth Stage
First Semester	$2023 - 2024$	Lecturer: Dr Herish & Dr. Andam

Exercises 1

1) Prove that a number is triangular if and only if it is of the form $n(n+1)/2, n \geq 1$.

We need to prove two parts:

Part 1(→) : If a number is triangular, then it is of the form $n(n+1)/2$ for $n \geq 1$.

Suppose we have a triangular number T. By definition, T is a number that can be represented by a triangular arrangement of objects. Let n be the number of rows in this arrangement.

The nth triangular number T can be expressed as the sum of the first n natural numbers:

$$T = 1 + 2 + 3 + \cdots + n$$

Using the formula for the sum of an arithmetic series:

$$T = (n/2)(1 + n)$$

This simplifies to:

$$T = n(n+1)/2$$

So, any triangular number T can be represented in the form $n(n+1)/2$ for $n \geq 1$.

Part 2(←) : If a number is of the form $n(n+1)/2$ for $n \geq 1$, then it is triangular.

Suppose we have a number in the form $n(n+1)/2$ for some integer $n \geq 1$. We want to show that it is triangular.

Let $T = n(n+1)/2$. We can think of T as a sum of consecutive integers starting from 1 up to n :

$$T = 1 + 2 + 3 + \cdots + n$$

This arrangement forms a triangle with n rows, and T is indeed a triangular number.

Therefore, we have shown both directions, and a number is triangular if and only if it is of the form $n(n+1)/2$ for $n \geq 1$.

2) If P_n denotes the n^{th} pentagonal number, where $P_1 = 1$ and $P_n = P_{n-1} + (3n - 2)$.

We will prove this statement using mathematical induction.

Base Case $(n = 1)$;

- P_1=1 (Given)
- $n(3n-1)/2 = 1(3(1)-1)/2 = 1(2)/2 = 1$

The base case holds.

Inductive Hypothesis:

Assume that for some positive integer $k \geq 1$, the statement holds: $P\_k = k(3k-1)/2$.

Inductive Step:

We need to show that the statement also holds for $k+1$, i.e., $P\_(k+1) = (k+1)(3(k+1) - 1)/2$.

Using the recurrence relation for P_n, we have:

$$P\_(k+1) = P\_k + (3(k+1) - 2)$$

By the inductive hypothesis, $P\_k = k(3k-1)/2$.

Substituting this into the equation:

$$P\_(k+1) = (k(3k-1)/2) + (3(k+1) - 2)$$

Simplify the expression:

$$P\_(k+1) = (3k\^2 - k)/2 + (3k+1)$$

Now, let's simplify the right-hand side of the equation:

$$P\_(k+1) = (3k\^2 - k)/2 + (6k+2)/2$$
$$P\_(k+1) = (3k\^2 - k + 6k + 2)/2$$
$$P\_(k+1) = (3k\^2 + 5k + 2)/2$$

Now, let's simplify the expression on the right further:

$$P\_(k+1) = [(k+1)(3k+2)]/2$$

We have shown that $P\_(k+1) = [(k+1)(3k+2)]/2$, which is the same as $(k+1)(3(k+1)-1)/2$.

Therefore, the statement is true for k + 1.

By mathematical induction, the statement holds for all positive integers $n \geq 1$, and we have proved that $P\_n = n(3n-1)/2$ for $n \geq 1$.

3) for $n \geq 2$, then prove that $P_n = \frac{n(3n-1)}{2}, n \geq 1$.

Base Case $(n = 2)$:

- For $n = 2$, we have:

$$P\_2 = P\_(2-1) + (3(2)-2) = P\_1 + (6-2) = 1 + 4 = 5$$
$$n(3n-1)/2 = 2(3{*}2-1)/2 = 2(6-1)/2 = 2(5)/2 = 5$$

The base case holds.

Inductive Hypothesis:

Assume that for some positive integer $k \geq 2$, the statement holds: $P\_k = k(3k-1)/2$.

Inductive Step:

We need to show that the statement also holds for $k+1$, i.e., $P\_(k+1) = (k+1)(3(k+1)-1)/2$.

Using the recurrence relation for P_n :

$$P\_(k+1) = P\_k + (3(k+1)-2)$$

By the inductive hypothesis, $P\_k = k(3k-1)/2$:

$$P\_(k+1) = (k(3k-1)/2) + (3(k+1)-2)$$

Simplify the expression:

$$P\_(k+1) = (3k\^2 - k)/2 + (3k+3-2)$$

Now, let's simplify the right-hand side of the equation:

$$P\_(k+1) = (3k\^2 - k)/2 + (3k+1)$$

Now, combine the terms with common denominators:

$$P\_(k+1) = (3k\^2 - k + 6k + 2)/2$$
$$P\_(k+1) = (3k\^2 + 5k + 2)/2$$

Now, factor the right-hand side:

$$P\_(k+1) = [(k+1)(3k+2)]/2$$

We have shown that $P\_(k+1) = [(k+1)(3k+2)]/2$, which is the same as $(k+1)(3(k+1)-1)/2$.

Therefore, the statement is true for $\mathrm{k}+1$.

By mathematical induction, the statement holds for all positive integers $n \geq 2$, and we have proved that $P\_n = n(3n-1)/2$ for $n \geq 1$.

Exercises 2

1. **Show that $5|25, 19|38$ and $2 \mid 98$.**

- $5 \mid 25$ because $25 = 5 \times 5$.
- $19 \mid 38$ because $38 = 19 \times 2$.
- $2 \mid 98$ because $98 = 2 \times 49$.

2. **Use the division algorithm to find the quotient and the remainder when 76 is divided by 13.**

Quotient q and remainder r are found such that $76 = 13q + r$ where $0 \leq r < 13$.

$76 = 13 \times 5 + 11$, so the quotient is 5 and the remainder is 11 .

3. **Use the division algorithm to find the quotient and the remainder when -100 is divided by 13.**

$-100 = 13 \times (-8) + 4$, so the quotient is -8 and the remainder is 4 .

4. **Show that if a, b, c and d are integers with a and c nonzero, such that $a \mid b$ and $c \mid d$, then $ac \mid bd$.**

If $a \mid b$, then $b = ak$ for some integer k.

If $c \mid d$, then $d = cl$ for some integer l.

Thus, $bd = (ak)(cl) = ac(kl)$, which means ac divides bd.

5. **Show that if a and b are positive integers and $a \mid b$, then $a \leq b$.**

$a \mid b$ implies $b = ak$ for some integer k.

Since k cannot be 0 and b is positive, k must be at least 1 , hence $a \leq b$.

6. Prove that the sum of two even integers is even, the sum of two odd integers is even and the sum of an even integer and an odd integer is odd.

Two even integers: $2n + 2m = 2(n + m)$, which is divisible by 2 , hence even.

Two odd integers: $(2n + 1) + (2m + 1) = 2(n + m + 1)$, which is divisible by 2 , hence even.

An even and an odd integer: $2n + (2m + 1) = 2n + 2m + 1$, which is not divisible by 2 , hence odd.

7. Show that the product of two even integers is even, the product of two odd integers is odd and the product of an even integer and an odd integer is even.

Two even integers: $(2n)(2m) = 4nm = 2(2nm)$, which is divisible by 2 , hence even.

Two odd integers: $(2n + 1)(2m + 1) = 4nm + 2n + 2m + 1$, which is not divisible by 2 , hence odd.

An even and an odd integer: $2n(2m + 1) = 4nm + 2n$, which is divisible by 2 , hence even.

8. Show that if m is an integer then 3 divides $m^3 - m$.

$m^3 - m = m(m^2 - 1) = m(m - 1)(m + 1)$.

Among $m, m - 1$, and $m + 1$, at least one is divisible by 3 , hence $3 \mid m^3 - m$.

9. Show that the square of every odd integer is of the form $8m + 1$.

Let an odd integer be $2n + 1$, then $(2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1$.

Since either n or $n + 1$ is even, $4n(n + 1)$ is divisible by 8 , hence the form $8m + 1$.

10.Show that the square of any integer is of the form $3m$ or $3m + 1$ but not of the form $3m + 2$.

Any integer is of the form $3n, 3n + 1$, or $3n + 2$.

Squaring these gives $9n^2, 9n^2 + 6n + 1$, and $9n^2 + 12n + 4$, respectively.

The first is $3m$, the second is $3m + 1$, and the third reduces to $3(3n^2 + 4n + 1) + 1$, not $3m + 2$.

11.Show that if $ac \mid bc$, then $a \mid b$.

If $ac \mid bc$, then $bc = ack$ for some integer k.

If $c \neq 0$, then $b = ak$, thus $a \mid b$.

12.Show that if $a \mid b$ and $b \mid a$ then $a = \pm b$.

$a \mid b$ means $b = ak$ for some integer k.

$b \mid a$ means $a = bl$ for some integer l.

Combining gives $a = akl$, so $kl = 1$.

Since k and l are integers, the only possibility is $k = l = \pm 1$, so $a = \pm b$.

Exercises 3

1. Convert $(\mathbf{7482})_{\mathbf{10}}$ to base 6 notation.

Divide 7482 by 6 , which results in a quotient of 1247 and a remainder of 0 .

Next, divide 1247 by 6 , which results in a quotient of 207 and a remainder of 5 .

Continue this process until the quotient becomes 0 .

Reading the remainders in reverse order, the base 6 representation of 7482 is 50210.

2. Convert $(\mathbf{98156})_{\mathbf{10}}$ to base 8 notation.

To convert the decimal number 98156 to base 8 :

- Divide 98156 by 8 , which results in a quotient of 12269 and a remainder of 4 .
- Next, divide 12269 by 8 , which results in a quotient of 1533 and a remainder of 5 .
- Continue this process until the quotient becomes 0 .
  Reading the remainders in reverse order, the base 8 representation of 98156 is 25454.

3. Convert (101011101) $_{2}$ to decimal notation.

To convert the binary number 101011101 to decimal:

- Start from the rightmost digit and assign powers of 2 to each bit, starting from 2^0 and increasing by 1 for each bit to the left.
- Multiply each bit by the corresponding power of 2 and sum the results.
  Calculation:

$$1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4 + 1 \cdot 2^5 + 0 \cdot 2^6 + 1 \cdot 2^7 + 1 \cdot 2^8 = 173$$

So, the decimal representation of $(101011101)_2$ is 173_{10}.

4. **Convert $(AB6C7D)_{16}$ to decimal notation.**

To convert the hexadecimal number AB6C7D to decimal:

- Assign values to the hexadecimal digits: $A = 10, B = 11, C = 12, D = 13$.
- Start from the rightmost digit and assign powers of 16 to each digit, starting from 16^0 and increasing by 1 for each digit to the left.
- Multiply each digit by the corresponding power of 16 and sum the results. Calculation:

$$D \cdot 16^0 + 7 \cdot 16^1 + C \cdot 16^2 + 6 \cdot 16^3 + B \cdot 16^4 + A \cdot 16^5 = 4393453$$

So, the decimal representation of (AB6C7D) $)_{16}$ is 4393453_{10}.

5. **Convert $(9A0B)_{16}$ to binary notation.**

To convert the hexadecimal number 9AOB to binary:

- Convert each hexadecimal digit to its 4-bit binary representation.
- Concatenate the binary representations of the digits.
  Conversion:
- 9 (hex) = 1001 (binary)
- A(hex) = 1010 (binary)
- O (hex) = 0000 (binary)
- B(hex) = 1011 (binary)
  Concatenating the binary representations: 1001101000001011_{12}
  So, the binary representation of $(9AOB)_{16}$ is 1001101000001011_2.

Exercises 4

1. **Find the greatest common divisor of 15 and 35.**

The greatest common divisor (GCD) of 15 and 35 is 5.

2. **Find the greatest common divisor of 100 and 104.**

The GCD of 100 and 104 is 4.

3. **Find the greatest common divisor of -30 and 95.**

The GCD of -30 and 95 is 5.

4. **Let m be a positive integer. Find the greatest common divisor of m and $m + 1$.**

When m is a positive integer, the GCD of m and m+1 is always 1. This is because consecutive positive integers are always coprime, meaning they share no common factors other than 1.

5. **Let m be a positive integer, find the greatest common divisor of m and $m + 2$.**

When m is a positive integer, the GCD of m and m+2 can be found as follows:

If m is even, then GCD(m, m+2) = 2, because both numbers are divisible by 2.

If m is odd, then GCD(m, m+2) = 1, because consecutive odd integers are always coprime, sharing no common factors other than 1.

So, the GCD of m and m+2 depends on whether m is even or odd:

If m is even, GCD(m, m+2) = 2.

If m is odd, GCD(m, m+2) = 1.

6. **Show that if m and n are integers such that $(m, n) = 1$, then $(\mathrm{m + n, m - n}) = 1$ or 2 .**

Proof:

Case 1: $(m + n, m - n) = 1$
This means that $(m + n)$ and $(m - n)$ have no common factors other than 1 , which implies that they are relatively prime.

Case 2: $(m + n, m - n) = 2$
This means that $(m + n)$ and $(m - n)$ are both even. Since their sum is even, it implies that m is even as well (because the sum of two odd or two even integers is even). Therefore, both

$(m+n)$ and $(m-n)$ are divisible by 2 , and their GCD is 2 .

Hence, either $(m+n, m-n)=1$ or $(m+n, m-n)=2$.

7. **Show that if m is a positive integer, then $3m+2$ and $5m+3$ are relatively prime.**

8. **Show that if a and b are relatively prime integers, then $(a+2b, 2a+b)=1$ or 3.**

Proof:

Let $d=$ GCD (a, b). Since a and b are relatively prime, $d=1$.

Now, consider the GCD of $a+2b$ and $2a+b$, denoted as d'. Using the Euclidean algorithm:

$$d' = \text{GCD } (a+2b, 2a+b) = \text{GCD } (a+2b, (2a+b)-2(a+2b)) =$$
$$\text{GCD } (a+2b, 2a+b-2(2a+2b)) = \text{GCD } (a+2b, -3b)$$

Since a and b are relatively prime, $d=1$. Therefore, d' is the GCD of 1 and $-3b$.

The GCD of 1 and any integer is either 1 or the absolute value of that integer. Thus, $d'=1$ or 3 (depending on the sign of $-3b$).

Therefore, GCD $(a+2b, 2a+b)=1$ or 3.

9. **Show that if $a_1, a_2, \ldots, a_n$ are integers that are not all 0 and c is a positive integer, then $(ca_1, ca_2, \ldots, ca_n)=c(a_1, a_2, \ldots a_n)$.**

Proof:

Let $d=$ GCD $(a_1, a_2, \ldots, a_n)$. This means that d is the greatest common divisor of all the integers $a_1, a_2, \ldots, a_n$.

Consider the GCD of the numbers $ca_1, ca_2, \ldots, ca_n$. Let's call it d'. By definition, d' is the greatest common divisor of $ca_1, ca_2, \ldots, ca_n$.

Since d is a common factor of all a_i, we can write each a_i as $a_i=dx_i$, where x_i are integers.

Now, consider $ca_i=c(dx_i)=(cd)x_i$.

Since d is a common factor of all ca_i, it is also a factor of $(cd)x_i$ for each i.

Therefore, d is a common factor of $ca_1, ca_2, \dots, ca_n$, and it follows that $d' \leq d$.

On the other hand, since c is a positive integer, it is clear that c is a common factor of $ca_1, ca_2, \dots, ca_n$, and d' is the greatest common divisor of these numbers. Thus, $d' \geq c$.

Therefore, $d' = c$.

Hence, GCD $(ca_1, ca_2, \dots, ca_n) = c$GCD $(a_1, a_2, \dots, a_n)$, as desired.

Exercises 5

1. **Use the Euclidean algorithm to find the greatest common divisor of 412 and 32 and express it in terms of the two integers.**

$$412 = 12 \times 32 + 28$$
$$32 = 28 + 4$$
$$4 = 7 \times 4$$

Therefore, the greatest common divisor of 412 and 32 is 4 , and it can be expressed as follows:

$$4 = -1 \times 412 + 13 \times 32$$

2. **Use the Euclidean algorithm to find the greatest common divisor of 780 and 150 and express it in terms of the two integers.**

Greatest common divisor of 780 and 150

$$780 = 5 \times 150 + 30$$
$$150 = 5 \times 30$$
$$30 = 3 \times 10$$
$$10 = 2 \times 5$$

Therefore, the greatest common divisor of 780 and 150 is 30 , and it can be expressed as follows:

$$30 = -1 \times 780 + 2 \times 150$$

3. **Greatest common divisor of 2457 and 1343**

$$2457 = 1 \times 1343 + 1114$$
$$1343 = 1114 + 229$$
$$1114 = 4 \times 229 + 170$$
$$229 = 229 + 0$$

Therefore, the greatest common divisor of 2457 and 1343 is 229 , and it can be expressed as follows:

$$229 = -5 \times 2457 + 17 \times 1343$$

3. Let a and b be two positive even integers. Prove that $(a, b) = 2(a/2, b/2)$.

Let $a = 2x$ and $b = 2y$, where x and y are positive integers.
Using the Euclidean algorithm:

$(a, b) = (2x, 2y) = 2(x, y)$.
Therefore, $(a, b) = 2(a/2, b/2)$.

4. Show that if a and b are positive integers where a is even and b is odd, then $(a, b) = (a/2, b)$.

Let a be even ($a = 2x$) and b be odd.
Using the Euclidean algorithm:

$(a, b) = (2x, b) = (2x, b - 2x^*k)$, where k is a positive integer.
Since b is odd, $b - 2x^*k$ is also odd. Therefore, the GCD of an even number $(2x)$ and an odd number $(b - 2x\ ^*k)$ must be 1 , as even and odd numbers are always coprime.
So, $(\mathrm{a}, \mathrm{b}) = 1$.
Now, consider $(\mathrm{a}/2, \mathrm{b})$:

$(\mathrm{a}/2, \mathrm{b}) = (2\mathrm{x}/2, \mathrm{b}) = (\mathrm{x}, \mathrm{b})$.
Since x is a positive integer and b is an odd positive integer, (x, b) can only be 1 because an even number (x) and an odd number (b) are always relatively prime.
So, $(\mathrm{a}/2, \mathrm{b}) = 1$.
Since $(a, b) = 1$ and $(a/2, b) = 1$, it follows that $(a, b) = (a/2, b) = 1$.
