CHAPTER 3

Functions of a Complex Variable

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((Chapter - Three))

((Functions of a Complex Variable)) ~~4016~~ 4030

$z = x + jy \longrightarrow$ is Complex Variable

$f(z) = u + jv$, where $u = u(x,y)$ & $v = v(x,y)$

$\hookrightarrow$ is function of a complex Variable.

* The necessary & sufficient Conditions for $f(z) = u(x,y) + jv(x,y)$, to be analytic, are

① $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are Continous functions of x & y in a given region R.

② $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$ $\Rightarrow$ Cauchy Riemann equation for Rect-Form

if $u = u(r,\theta)$ & $v = v(r,\theta)$ then

$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$ $\Rightarrow$ Cauchy Riemann equation for Polar Form

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EXP/ Determine a, b, c, d such that $f(z) = (x^2 + axy + by^2) + j(cx^2 + dxy + y^2)$ is analytic.

Sol/ let $f(z) = u + jv$

$\therefore u = x^2 + axy + by^2 \quad ; \quad v = cx^2 + dxy + y^2$

Given $f(z)$ is analytic, there for C.R. eq^n must be Satisfied :-

Now

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ---①$$

$$2x + ay = dx + 2y$$

$$\boxed{\therefore d = 2 \quad \& \quad a = 2}$$

Again $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \quad ---②$

$$ax + 2by = -2cx - dy$$

$$2x + 2by = -2cx - 2y$$

$$\boxed{\therefore c = -1 \quad \& \quad b = -1}$$

Exp// Determine which of the following functions are analytic !-

① e^{z} ② $\frac{1}{2} \ln(x^2+y^2) + j \tan^{-1}\frac{y}{x}$

③ $\frac{1}{z}$

Sol/ ① $f(z) = e^{z}$

$f(z) = e^{x+jy}$

$= e^{x} \cdot e^{jy}$

$= e^{x}(\cos y + j\sin y)$

$f(z) = e^{x}\cos y + j e^{x}\sin y$

$\therefore f(z) = u + j v \Rightarrow$ function of a complex variable.

$\because$ Eular formula

$e^{jy} = \cos y + j\sin y$

$u = e^x \cos y \quad ; \quad v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

↳ Hence the C. R. eqn are satisfied

$\therefore f(z) = e^z \rightarrow$ is analytic.

② $f(z) = \underbrace{\frac{1}{2} \ln (x^2+y^2)}_{u} + j \underbrace{\tan^{-1} \frac{y}{x}}_{v}$

$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{x^2+y^2} = \frac{x}{x^2+y^2} \quad ; \quad \frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} * \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}} * \frac{\cancel{2}y}{x^2+y^2} = \frac{y}{x^2+y^2} \quad ; \quad \frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} * \frac{1}{x} = \frac{x}{x^2+y^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ C. R. eqn are satisfied. Also the first partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are all continous.

$\therefore f(z)$ is analytic.

③ $f(z) = \frac{1}{z} = \frac{1}{x+jy} * \frac{x-jy}{x-jy}$

$= \frac{x-jy}{x^2+y^2}$

$$f(z) = \underbrace{\frac{x}{x^2+y^2}}_{u} - j\underbrace{\frac{y}{x^2+y^2}}_{V}$$

Now

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \quad ; \quad \frac{\partial v}{\partial y} = \frac{-[(x^2+y^2) - y \cdot 2y]}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

↳ ① C.R-eq^n are satisfied

② $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ ⇒ are all continuous.

$\therefore f(z)$ is analytic.

EXP// Show that $f(z) = xy + jy$ is everywhere continuous but is not analytic.

Sol/ $u = xy$; $V = y$

$\frac{\partial u}{\partial x} = y$ $\quad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = x$ $\quad$ $\frac{\partial v}{\partial y} = 1$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$$

↳ C.R. eq^n are not satisfied.

$\therefore f(z)$ is not analytic.

* Milne - Thomson's Method determine an analytic function, when $u(x,y)$ or $v(x,y)$ is given.-

let $f(z) = u + jv$, $z = x + jy$, $\bar{z} = x - jy$

$\frac{z+\bar{z}}{2} = x$, $\frac{z-\bar{z}}{2j} = y$, then

$$f(z) = u(x,y) + j\,v(x,y) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2j}\right) + j\,v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2j}\right)$$

if we take this as an identity in two variables & take $z = \bar{z}$, we have

$$f(z) = u(z,0) + j\,v(z,0)$$

which is same as we replace $\boxed{x \text{ by } z}$, $\boxed{y \text{ by } 0}$ in $f(z)$.

EXP/ Determine the analytic function $f(z)$ in terms of z whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Sol/

Here $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

let $f(z) = u + jv$, where $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ be an analytic function we apply Milne thomson Method:-

Now $f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y}$

the function is analytic
C.R. Eq
$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$= 3x^2 - 3y^2 + 6x - j(-6xy - 6y)$

Replacing x by z, y by 0, we get

$f'(z) = 3z^2 + 6z - j(0) = 3z^2 + 6z$

integrating,

$$\int f'(z) = \int (3z^2 + 6z)\,dz$$

$$f(z) = z^3 + 3z^2 + c \quad , \text{ where } \underline{c = a + jb}$$

$$= z^3 + 3z^2 + a + jb \qquad \text{Complex No.}$$

Now u contains 1 as constant $\therefore a = 1$

$$\therefore f(z) = z^3 + 3z^2 + jb$$

Exp / find the analytic function $f(z)$ in terms of z whose imaginary part is $\frac{x-y}{x^2+y^2}$

Sol / let $V = \frac{x-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2) - (x-y)2x}{(x^2+y^2)^2} = \frac{y^2 - x^2 + 2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{-x^2 + y^2 - 2xy}{(x^2+y^2)^2}$$

the $f(z)$ is analytic $\therefore$ C.R. eq $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

let $f(z) = u + jv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$

$$\therefore f'(z) = \frac{\partial v}{\partial y} + j\frac{\partial v}{\partial x}$$

$$= \frac{-x^2 + y^2 - 2xy}{(x^2+y^2)^2} + j\,\frac{y^2 - x^2 + 2xy}{(x^2+y^2)^2}$$

assuming this to be an identity & Repalcing

x by z & y by 0, we get (Milne thomson Method)

$$f'(z) = \frac{-z^2}{z^4} + j\,\frac{-z^2}{z^4}$$

$$f'(z) = -\frac{(1+j)}{z^2}$$

$$\int f'(z) = \int -\frac{(1+j)}{z^2}\,dz \Rightarrow f(z) = \frac{1+j}{z} + C$$

$$f(z) = \frac{1+j}{z} + a + jb$$

Now V is free from consant. ∴ $b = 0$

$$\therefore f(z) = \frac{1+j}{z} + a$$

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* Conjugate Functions :-

let $f(z) = u + jv$ be an analytic function, then $u(x,y)$ & $v(x,y)$ are called Conjugate functions.

* Harmonic Functions :-

let $\phi(x,y)$ be a function of x & y. then $\phi(x,y)$ is Called harmonic function if

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

∴, if $\phi(x,y)$ is the solution of Laplace eqn.

Exp/ prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find v such that $f(z) = u + jv$ is analytic. Also express $f(z)$ in terms of z.

Sol/ $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \quad ; \quad \frac{\partial u}{\partial y} = -2y - 2x + 3$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Now u is harmonic if u satisfies Laplace eqn

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

here $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0$, which is true.

Hence the first part.

let $f(z) = u + jv$

C.R.E

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}}$$

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y}$$

$$= 2x - 2y - 2 - j(-2y - 2x + 3)$$

Milne-Thomson Method :- Replacing x by z & y by 0

$$f'(z) = 2z - 2 - j(-2z + 3) = 2z - 2 + 2jz - 3j$$

$$\int f'(z) = \int [2(1+j)z - (2+3j)]\, dz$$

$$f(z) = (1+j)z^2 - (2+3j)z + C$$

$$= (1+j)z^2 - (2+3j)z + a + jb$$

As u is free from constants $\therefore a = 0$

$$f(z) = (1+j)z^2 - (2+3j)z + jb$$

$f(z) = (1+j)(x^2-y^2+2xyj) - (2+3j)(x+jy) + jb$

equating imaginary part, we get

$$V = 2xy + (x^2-y^2) - 2y - 3x + b$$

H.w① / if $u(x,y) = x^2-y^2$ & $v(x,y) = \frac{y}{x^2+y^2}$, prove that both u & v satisfy Laplace's equation but are not harmonic conjugates.

H.w② / Find the values of c_1 & c_2 such that the function is analytic.

$$f(z) = x^2 + c_1 y^2 - 2xy + j(c_2 x^2 - y^2 + 2xy)$$

Exp/ Show that the following function are analytic or not:-

① $Z \cdot \bar{Z}$ ② $Z \cdot |Z|$ ③ $Z e^{Z}$

① $f(z) = Z \cdot \bar{Z} = |Z|^2$

$\because |Z| = \sqrt{x^2+y^2}$

$|Z| = \sqrt{Z \cdot \bar{Z}}$

$= (x+jy) \cdot (x-jy) = \left(\sqrt{x^2+y^2}\right)^2$

$f(z) = x^2+y^2 = x^2+y^2$; $f(z) = u + jv$

$\therefore u = (x^2+y^2)$ & $v = 0$

$\frac{\partial u}{\partial x} = 2x$ $\qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 2y$ $\qquad \frac{\partial v}{\partial y} = 0$

$\therefore$ C.R. eqⁿ are

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$\therefore f(z) = Z\bar{Z} = |Z|^2 \Rightarrow$ is not analytic

② $f(z) = Z|Z|$

$\boxed{\because |Z| = \sqrt{x^2+y^2} = \sqrt{Z\bar{Z}}}$

$= (x+jy)\sqrt{x^2+y^2}$

$= \underbrace{x\sqrt{x^2+y^2}}_{u(x,y)} + j\underbrace{y\sqrt{x^2+y^2}}_{v(x,y)}$

$\frac{\partial u}{\partial x} = x \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x + \sqrt{x^2+y^2} = \frac{x^2}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}$

$\frac{\partial u}{\partial y} = x * \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y = \frac{xy}{\sqrt{x^2+y^2}}$

$\frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}$; $\frac{\partial v}{\partial y} = \frac{y^2}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}$

$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \Rightarrow$ $\therefore$ C.R. eqⁿ are not satisfied $f(z)$ is not analytic.

③ $f(z) = z e^{z}$; $\because z = x + jy$

$$= (x+jy)\, e^{x+jy} = (x+jy)\, e^{x} \cdot e^{jy}$$

$$= (x+jy)\, e^{x} (\cos y + j \sin y) = (x+jy)\left(e^{x}\cos y + j e^{x} \sin y\right)$$

$$= e^{x} x \cos y + j y e^{x} \cos y + j x e^{x} \sin y - y e^{x} \sin y$$

$$= \underbrace{e^{x} x \cos y - y e^{x} \sin y}_{u(x,y)} + j\,(\underbrace{y e^{x} \cos y + x e^{x} \sin y}_{v(x,y)})$$

$$\frac{\partial u}{\partial x} = e^{x} \cos y + e^{x} x \cos y - e^{x} y \sin y$$

$$\frac{\partial u}{\partial y} = - e^{x} x \sin y - y e^{x} \cos y - e^{x} \sin y$$

$$\frac{\partial v}{\partial x} = e^{x} y \cos y + e^{x} \sin y + x e^{x} \sin y$$

$$\frac{\partial v}{\partial y} = e^{x} \cos y - y e^{x} \sin y + x e^{x} \cos y$$

C.R. eq^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ the $f(z) = z e^{z}$ is analytic

∵ Exp / Determine which of the following function are analytic :-

① $\sin z$ ② $\cosh z$ ③

Sol /

① $f(z) = \sin z$

$= \sin(x + jy)$

$= \sin x \cos jy + \sin jy \cos x$

∵ $\sin(A+B) = \sin A \cos B + \sin B \cos A$

∵ $\cos jy = \cosh y$

$\sin jy = j \sinh y$

$f(z) = \sin x \cosh y + j \sinh y \cos x$

∵ $f(z) = u(x,y) + j v(x,y)$

$u(x,y) = \sin x \cosh y$, $v = \sinh y \cos x$

$\frac{\partial u}{\partial x} = \cos x \cosh y$ $\qquad$ $\frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y$ $\qquad$ $\frac{\partial v}{\partial y} = \cosh y \cos x$

∵ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

∴ $f(z) = \sin z \Rightarrow$ is analytic

(2) $f(z) = \cosh z$ $\qquad \because z = x + jy$

$= \cos jz$ $\qquad \cosh z = \cos jz$

$= \cos j(x+jy)$

$= \cos(jx - y)$ $\qquad \because \cos(A-B) = \cos A \cos B + \sin A \sin B$

$= \cos jx \cos y + \sin jx \sin y$ $\qquad \because \sin jx = j \sinh x$

$$f(z) = \underbrace{\cos y \cosh x}_{u(x,y)} + \underbrace{j \sinh x \sin y}_{v(x,y)}$$

$\frac{\partial u}{\partial x} = \cos y \sinh x$ $\qquad \frac{\partial v}{\partial x} = \cosh x \sin y$

$\frac{\partial u}{\partial y} = -\sin y \cosh x$ $\qquad \frac{\partial v}{\partial y} = \sinh x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore f(z) = \cosh z \Rightarrow$ is analytic

Exp / If $u(x,y) = x^2 - y^2$ & $v(x,y) = \frac{y}{x^2+y^2}$, prove that both u & v satisfy Laplace's equation but are not harmonic conjugates.

analytic function

Sol / * $u = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = -2y$

$\frac{\partial^2 u}{\partial x^2} = 2$; $\frac{\partial^2 u}{\partial y^2} = -2$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow 2 - 2 = 0 \rightarrow u(x,y)$ satisfy Laplace's eqn

$\therefore u(x,y)$ is harmonic

* $v(x,y) = \frac{y}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$; $\frac{\partial v}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)}$

$\frac{\partial^2 v}{\partial x^2} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$; $\frac{\partial^2 v}{\partial y^2} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \rightarrow \frac{\cancel{6x^2y - 2y^3 - 6x^2y + 2y^3}}{\cancel{(x^2+y^2)^3}} = 0$

$\therefore v(x,y)$ satisfies Laplace's eqn

$\therefore v(x,y)$ is harmonic.

but $\left.\begin{array}{l} \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \end{array}\right\}$ $\therefore$ C.R-eq are not satisfies $f(z) = u + jv$

$\therefore u(x,y)$ and $v(x,y)$ are not harmonic Conjugates

meanings that are not analytic

EXP/ show that the function $u = \frac{1}{2} \ln(x^2+y^2)$ is harmonic and find the harmonic conjugate function. analytic function $f(z)$

Sol/

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - 2x\cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0 \rightarrow \therefore u(x,y) \text{ is harmonic.}$$

$f(z) = u + jv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$

C.R. eq^n

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$f'(z) = \frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{x}{x^2+y^2} - j\frac{y}{x^2+y^2}$$ $\rightarrow$ Replacing $x = z$ & $y = 0$

$$f'(z) = \frac{z}{z^2} - \cancel{j\frac{0}{\ }}0 \Rightarrow f'(z) = \frac{1}{z} \Rightarrow \int f'(z)\,dz = \int \frac{1}{z}\,dz$$

$$f(z) = \ln z + a + jb$$; $a + jb$ = Complex No

let = $a = 1$

$$\therefore f(z) = \ln z + 1 + jb$$

EXP/ Find the analytic function $f(z)$ in termes of z whose real part is $u = \frac{x}{x^2+y^2}$. also express $f(z)$ in terms of z.

Sol/ $u = \frac{x}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$f(z) = u + jv$ $\Rightarrow$ we apply Milne thomson Method

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y}$$

Cauchy Riemann eq^n are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{y^2-x^2}{(x^2+y^2)^2} + j\frac{2xy}{(x^2+y^2)^2}$$

Replacing x by z & y by 0, we get

$$f'(z) = \frac{0-z^2}{(z^2+0)^2} + \cancel{j\frac{2z\times 0}{(z^2+0)^2}} \Rightarrow f'(z) = \frac{-z^2}{z^4}$$

$$f'(z) = -\frac{1}{z^2} \Rightarrow \int f'(z)\,dz = \int -\frac{1}{z^2}\,dz \Rightarrow f(z) = \frac{1}{z} + c$$

$$\therefore f(z) = \frac{1}{z} \Rightarrow f(z) = \frac{1}{x+jy}$$

$$f(z) = \frac{1}{(x+jy)} * \frac{x-jy}{(x-jy)}$$

$$f(z) = \frac{x-jy}{x^2+y^2}$$

$\therefore f(z) = \dfrac{x}{x^2+y^2} - j\dfrac{y}{x^2+y^2}$

$\therefore u = \dfrac{x}{x^2+y^2}$ & $v = \dfrac{-y}{x^2+y^2}$

* and also show that the function $f(z) = \dfrac{x}{x^2+y^2} - j\dfrac{y}{x^2+y^2}$ is analytic

$\dfrac{\partial v}{\partial x} = \dfrac{2xy}{(x^2+y^2)^2}$ & $\dfrac{\partial v}{\partial y} = \dfrac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

we apply Cauchy Riemann eqn

$\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{y^2-x^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\boxed{\dfrac{-2xy}{(x^2+y^2)^2} = \dfrac{2xy}{(x^2+y^2)^2}}$

$\therefore f(z) \rightarrow$ is analytic.

Exp/ Determine the analytic function $f(z)$ in termes of z whose real part $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$

Sol/ $\dfrac{\partial u}{\partial x} = \dfrac{2\cos 2x(\cosh 2y + \cos 2x) + 2\sin 2x * \sin 2x}{(\cosh 2y + \cos 2x)^2}$

$= \dfrac{2\cos 2x \cosh 2y + 2\cos^2 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{2(\cos 2x \cosh 2y + 1)}{(\cosh 2y + \cos 2x)^2}$ $\qquad \because \sin^2 2x + \cos^2 2x = 1$

$\frac{\partial u}{\partial y} = \frac{\cancel{0 * (\cosh 2y + \cos 2x)} - \sin 2x * 2\sinh 2y}{(\cosh 2y + \cos 2x)^2}$

$= \frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$

$f(z) = u + jv$ $\Rightarrow$ we apply M.T.M

$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$ $\qquad \Rightarrow$ C.R. eqn $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y}$

$= \frac{2(\cos 2x \cosh 2y + 1)}{(\cosh 2y + \cos 2x)^2} - j\frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$

Raplacing x by z & y by 0, we get

$f'(z) = \frac{2\cancel{(\cos 2z + 1)}}{(1 + \cos 2z)^{\cancel{2}}} - \cancel{j * 0}$

$\boxed{\begin{aligned} &\because \cos^2 z = \tfrac{1}{2}(1 + \cos 2z) \\ &\therefore 2\cos^2 z = 1 + \cos 2z \end{aligned}}$

$f'(z) = \frac{2}{1 + \cos 2z}$

$f'(z) = \frac{1}{\cos^2 z} = \sec^2 z$ $\qquad \because \sec^2 z = \frac{1}{\cos^2 z}$

$\int f'(z)dz = \int \sec^2 z \, dz$

$\therefore f(z) = \tan z + C$

$f(z) = \tan z = \tan(x+jy)$, $\because z = x+jy$

$$= \frac{\sin(x+jy)}{\cos(x+jy)}$$

$$= \frac{\sin(x+jy)\cos(x-jy)}{\cos(x+jy)\cos(x-jy)}$$

$\circ$ let $A = x+jy$ & $B = x-jy$

$A \neq B$

$\because \sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\sin A \sin B = \frac{1}{2}[-\cos(A+B) + \cos(A-B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$= \frac{\sin(x+jy+x-jy) + \sin(x+jy-x+jy)}{\cos(x+jy+x-jy) + \cos(x+jy-x+jy)}$$

$$= \frac{\sin 2x + \sin 2jy}{\cos 2x + \cos 2jy}$$

$\because \sin j\alpha = j\sinh\alpha$

$\cos j\alpha = \cosh\alpha$

$$= \frac{\sin 2x + j\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\therefore f(z) = \underbrace{\frac{\sin 2x}{\cos 2x + \cosh 2y}}_{u(x,y)} + j\underbrace{\frac{\sinh 2y}{\cos 2x + \cosh 2y}}_{v(x,y)}$$

* Also show that the fuction $f(z) = \frac{\sin 2x}{\cos 2x + \cosh 2y} + j\frac{\sinh 2y}{\cos 2x + \cosh 2y}$ is analytic or not.