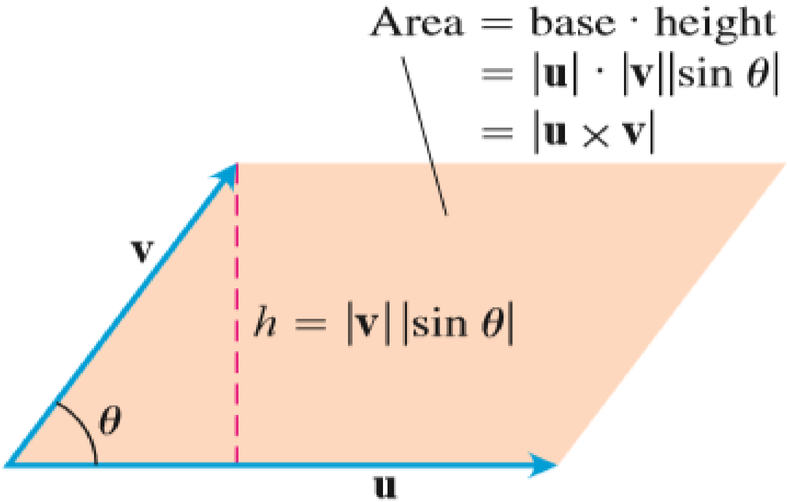
Geometric Properties Of The Cross Product

Let u and v be nonzero vectors in space then:

1. The vector $u \times v$ is orthogonal to both u and v.
2. $u \times v = 0$ if and only if u and v are scalar multiples of each other (u and v are parallel).
3. $|u \times v| =$ area of the parallelogram determined by u and v .



Proposition (Properties Of The Cross Product)

If u, v, and w are any vectors and r, s are scalars, then

1. $v \times u = -(u \times v)$
2. $(ru) \times (sv) = rs(u \times v)$
3. $(ru) \times v = r(u \times v) = u \times (rv)$
4. $u \times (v + w) = u \times v + u \times w$
5. $(v + w) \times u = v \times u + w \times u$
6. $u.(v \times w) = (u \times v).w$
7. $u \times (v \times w) = (u.w)v - (u.v)w$
8. $0 \times u = 0$

Examples:

1. Find a vector perpendicular to the plane of P(1,-1,0), Q(2,1,-1), and R(-1,1,2)

2. Find the area of the triangle with vertices P(1,-1,0), Q(2,1,-1), and R(-1,1,2).

3. If $\vec{a} = \langle 2,1,-1 \rangle$ and $\vec{b} = \langle -3,4,1 \rangle$ compute each of the following.

   a) $\vec{a} \times \vec{b}$ b) $\vec{b} \times \vec{a}$

Note :

1. The dot product of two vectors is a scalar; the cross product is another vector (perpendicular to each of the original).

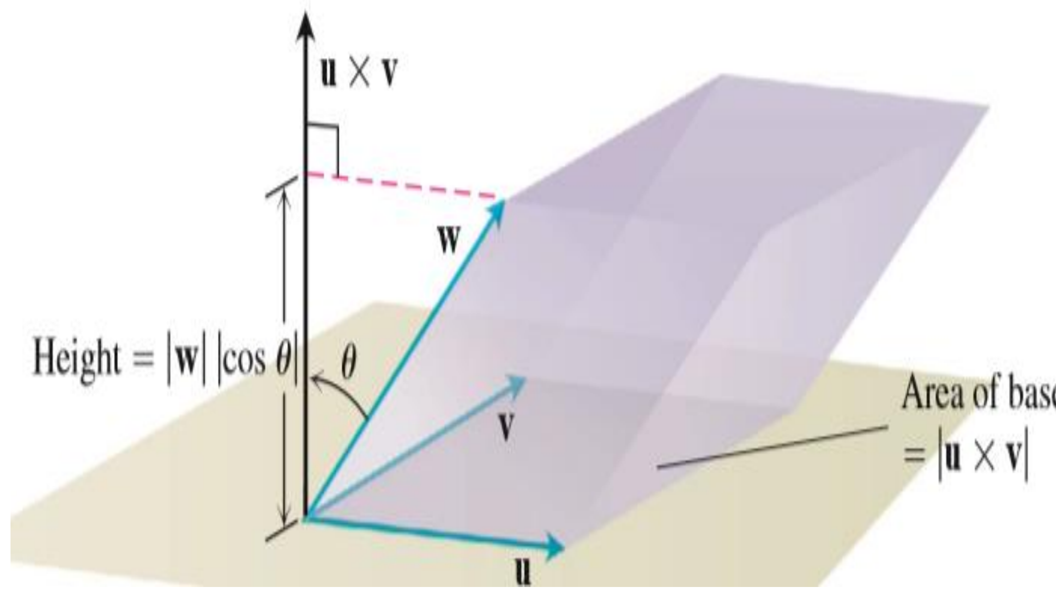
2. A dot product is commutative; a cross product is not. In fact,

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$$

Volume of the parallelepiped

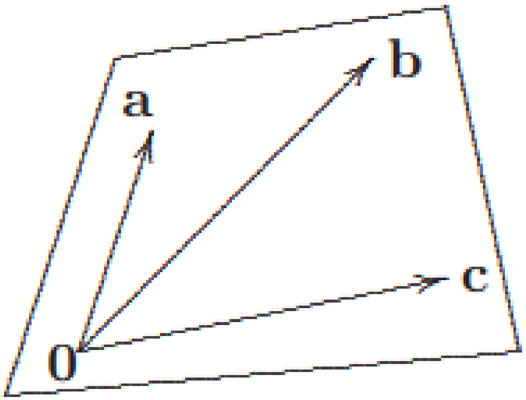
The volume of the parallelepiped determined by the vectors u,v, and w is the magnitude of their scalar triple product: $v = |(u \times v).w|$

$$\begin{aligned} \text{Volume} &= \text{area of base} \cdot \text{height} \\ &= |\mathbf{u} \times \mathbf{v}|\,|\mathbf{w}|\,|\cos\theta| \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \end{aligned}$$



Corollary:

The vectors a, b, c are coplanar (i.e. they all lie on a plane) if and only if $a \cdot (b \times c) = 0$.



a,b,c are coplanar

Examples:

1-Find the volume of the box (parallelepiped) determined By u=i+2j-k, v=-2i+3k, and w=7j-4k.

2- Show that the vectors a = <1, 4, −7>, b = <2, −1, 4>, c = <0, −9, 18> are coplanar.

Homework :

1- Show that $a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$.

2- Show that

$$(a \times b) \times (c \times d) = (a \cdot (c \times d))\, b - (b \cdot (c \times d))\, a$$
$$= (a \cdot (b \times d))\, c - (b \cdot (b \times c))\, d.$$

3- Suppose the vectors a,b,c,d are coplanar. Show that $(a \times b) \times (c \times d) = 0$.