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Limit of a function

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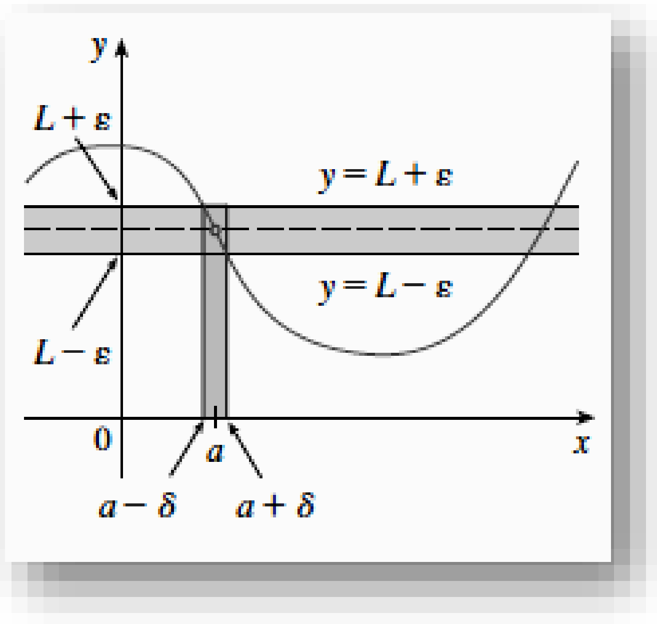
Introduction

The concept of limit is one of the ideas that distinguish calculus from algebra and trigonometry. In this report, we show how to define and calculate limits of function values. The calculation rules are straightforward and most of the limits we need can be found by substitution, graphical investigation, numerical approximation, algebra, or some combination of these. One of the uses of limits is to test functions for continuity. Continuous functions arise frequently in scientific work because they model such an enormous range of natural behavior. They also have special mathematical properties, not otherwise guaranteed. Also differential and integral calculus are built on the foundation concept of a limit

Definition of Limits:

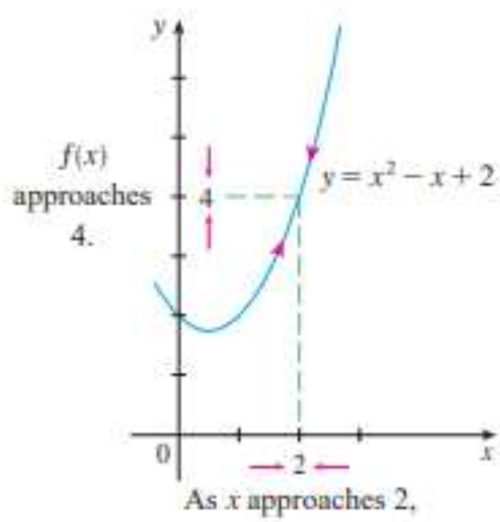
Conceptually, the limit of a function $f(x)$at some point a simply means that if your value of x is very close to the value a, then the function $f(x)$ stays very close to some particular value.

Definition: The limit of a function $f(x)$ at some point a exists and is equal to L if and only if every "small" interval about the limit L, say the interval $(L-\varepsilon, L+\varepsilon)$, means you can find a "small" interval about a , say the interval $(a-\delta, a+\delta)$, which has all values of $f(x)$ existing in the former "small" interval about the limit L, except possibly at a itself



Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001



Remark If we have $\lim_{x \to a} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

Example Let $f: \mathbb{R} \to \mathbb{R}$. Prove that $\lim_{x\to 3}(4x - 5) = 7$.

Solution: Let $\varepsilon > 0$ be given we have to find $\delta > 0$ (δ depends only on ε)

$$\therefore |f(x) - l| < \varepsilon \implies |(4x - 5) - 7| < \varepsilon$$

$$\implies |4x - 12| = 4|x - 3| < \varepsilon$$

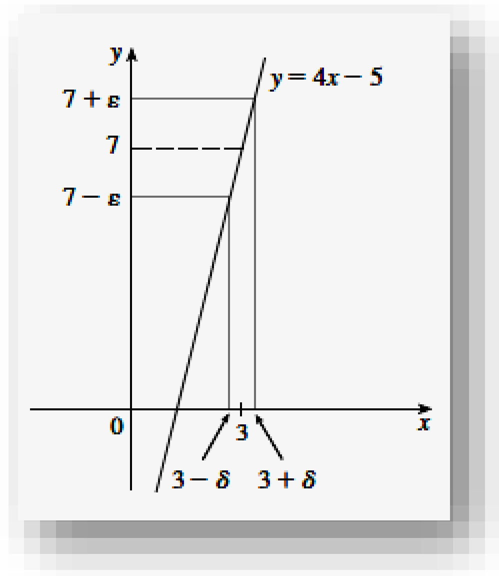
$$\implies |x - 3| < \frac{\varepsilon}{4}.$$

Choose $\delta = \frac{\varepsilon}{4}$ such that

$$x \in D_f,\ 0 < |x - 3| < \delta \implies |x - 3| < \frac{\varepsilon}{4}$$
$$\implies 4|x - 3| < \varepsilon$$
$$\implies |4x - 12| < \varepsilon$$
$$\implies |(4x - 5) - 7| < \varepsilon$$
$$\implies |f(x) - l| < \varepsilon.$$

Therefore, by the definition of a limit,

$$\lim_{x\to 3}(4x - 5) = 7$$



Theorem *Suppose that c is a constant and the limits* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. Then*

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)\times g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ *where n is a positive integer.*
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ *where n is a positive integer.*
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ *where n is a positive integer, and if n is even, we assume that* $a > 0$.

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *where n is a positive integer, and if n is even, we assume that* $\lim_{x\to a} f(x) > 0$.

Find: $\lim_{x \to -4} (x + 9)$.

Solution The two functions in this limit problem are $f(x) = x$ and $g(x) = 9$. We seek the limit of the sum of these functions.

$$\lim_{x \to -4} (x + 9) = \lim_{x \to -4} x + \lim_{x \to -4} 9 \quad \text{The limit of a sum is the sum of the limits.}$$

$$= -4 + 9 \quad \lim_{x \to a} x = a \text{ and } \lim_{x \to a} c = c.$$

$$= 5$$

Find: $\lim_{x \to 5} (12 - x)$.

Solution The two functions in this limit problem are $f(x) = 12$ and $g(x) = x$. We seek the limit of the difference of these functions.

$$\lim_{x \to 5} (12 - x) = \lim_{x \to 5} 12 - \lim_{x \to 5} x \quad \text{The limit of a difference is the difference of the limits.}$$

$$= 12 - 5 \quad \lim_{x \to a} c = c \text{ and } \lim_{x \to a} x = a.$$

$$= 7$$

Find: $\lim_{x \to 5} (-6x)$.

Solution The two functions in this limit problem are $f(x) = -6$ and $g(x) = x$. We seek the limit of the product of these functions.

$$\lim_{x \to 5} (-6x) = \lim_{x \to 5} (-6) \cdot \lim_{x \to 5} x \quad \text{The limit of a product is the product of the limits.}$$

$$= -6 \cdot 5 \quad \lim_{x \to a} c = c \text{ and } \lim_{x \to a} x = a.$$

$$= -30$$

Find: $\lim_{x \to 1} \frac{x^3 - 3x^2 + 7}{2x - 5}$.

$$\lim_{x \to 1} \frac{x^3 - 3x^2 + 7}{2x - 5} = \frac{\lim_{x \to 1}(x^3 - 3x^2 + 7)}{\lim_{x \to 1}(2x - 5)} = \frac{1^3 - 3 \cdot 1^2 + 7}{2 \cdot 1 - 5} = \frac{5}{-3} = -\frac{5}{3}$$

Find: $\lim_{x \to 5}(2x - 7)^3$.

$$\lim_{x \to 5}(2x - 7)^3 = \left[\lim_{x \to 5}(2x - 7)\right]^3 = (2 \cdot 5 - 7)^3 = 3^3 = 27$$

Find the following limits:

a. $\lim_{x \to 4} 7$ **b.** $\lim_{x \to 0}(-5)$.

a. $\lim_{x \to 4} 7 = 7$ **b.** $\lim_{x \to 0}(-5) = -5$

Find the following limits:

a. $\lim_{x \to 7} x$ **b.** $\lim_{x \to -\pi} x$.

a. $\lim_{x \to 7} x = 7$ **b.** $\lim_{x \to -\pi} x = -\pi$

Find: $\lim_{x \to -2} \sqrt{4x^2 + 5}$.

$$\lim_{x \to -2} \sqrt{4x^2 + 5} = \sqrt{\lim_{x \to -2}(4x^2 + 5)} = \sqrt{4(-2)^2 + 5} = \sqrt{16 + 5} = \sqrt{21}$$

One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^+} f(x) = L \quad \text{and} \quad \lim_{x\to c^-} f(x) = L.$$

Example If

$$f(x) = \frac{|x-2|}{x^2+x-6}$$

find: $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $\lim_{x\to 2} f(x)$

Solution Observe that

$$|x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$

Therefore,

$$\begin{aligned}
\lim_{x\to 2^+} \frac{|x-2|}{x^2+x-6} &= \lim_{x\to 2^+} \frac{x-2}{(x+3)(x-2)} \\
&= \lim_{x\to 2^+} \frac{1}{x+3} \\
&= \frac{1}{5} \\
\lim_{x\to 2^-} \frac{|x-2|}{x^2+x-6} &= \lim_{x\to 2^-} \frac{-(x-2)}{(x+3)(x-2)} \\
&= \lim_{x\to 2^-} \frac{-1}{x+3} \\
&= -\frac{1}{5}
\end{aligned}$$

Since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, then the limit $\lim_{x\to 2} f(x)$ does not exist.

Multiplying by the Conjugate.

Example Find

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$$

Solution We can not apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} \\
&= \lim_{t\to 0} \frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)} \\
&= \lim_{t\to 0} \frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)} \\
&= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} \\
&= \frac{1}{6}
\end{aligned}$$

Formal Definition: L'Hôpital's Rule

If the limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ results in one of the following forms:

$$\frac{0}{0}, \quad \pm\frac{\infty}{\infty}, \quad 0 * \pm\infty, \quad \infty-\infty, \quad 0^0, \quad 1^\infty, \quad \infty^0$$

And $\lim_{x\to c} \frac{f\prime(x)}{g\prime(x)}$ exits and $g'(x) \neq 0$, then:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Example 1: indeterminate form of $\frac{0}{0}$

Find the limit $\lim_{x\to 0}\frac{e^x-1}{x}$

$$\lim_{x\to 0}\frac{e^x-1}{x}=\frac{0}{0}=\lim_{x\to 0}\frac{e^x}{1}$$

Using L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{e^x}{1}=\frac{1}{1}=1$$

Example 2: indeterminate form of $\frac{\infty}{\infty}$

Find the limit $\lim_{x\to\infty}\frac{x^2}{2^x}$

$$\lim_{x\to\infty}\frac{x^2}{2^x}=\frac{\infty}{\infty}$$

Using L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{x^2}{2^x}=\lim_{x\to\infty}\frac{2x}{2^x ln2}=\frac{2}{ln2}\lim_{x\to\infty}\frac{x}{2^x}=\frac{\infty}{\infty}$$

Using L'Hôpital's Rule again:

$$\frac{2}{ln2}\lim_{x\to\infty}\frac{x}{2^x}=\frac{2}{ln2}\lim_{x\to\infty}\frac{1}{2^x ln2}=\frac{2}{(ln2)^2}\lim_{x\to\infty}\frac{1}{2^x}=\frac{2}{(ln2)^2}*0=0$$

Example 3: indeterminate form of $\infty - \infty$

Find the limit $\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)$

$$\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right) = \infty - \infty$$

Using L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right) = \lim_{x\to 2}\frac{4-(x+2)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{2-x}{x^2-4} = \lim_{x\to 2}\frac{-1}{2x} = -\frac{1}{4}$$

Example 4: indeterminate form of $\mathbf{1}^{\infty}$

Find the limit $\lim_{x\to 1} x^{\frac{1}{1-x}}$

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = 1^{\infty}$$

Let $y = x^{\frac{1}{1-x}}$. Then $\ln y = \ln x^{\frac{1}{1-x}} = \frac{\ln x}{1-x}$

Using L'Hôpital's Rule:

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{\ln x}{1-x} = \lim_{x\to 1}\frac{\frac{1}{x}}{-1} = -\lim_{x\to 1}\frac{1}{x} = -1$$

Therefore

$$\lim_{x\to 1}\ln y = -1$$

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \lim_{x\to 1} y = e^{-1} = \frac{1}{e}$$

Calculation of Limits of Rational Functions as x approaches to $+\infty$ or $-\infty$

Consider the function $f(x) = \frac{p(x)}{q(x)}$, in order to find the limit of $f(x)$ as x approaches to (∞) or $(-\infty)$**,** we devide both numerator and denominator by highest power of x that occur in the denominator.

If $\frac{P_m(x)}{Q_n(x)}$, in which $P_m(x)$ and $Q_n(x)$ are polinomials of degrees m and n respectively, then

$$P_m(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

$$Q_n(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

$$\lim_{x\to\mp\infty} f(x) = \begin{cases} \frac{a_m}{b_n} & \text{if} \quad m = n \\ +\infty \quad or \quad -\infty & \text{if} \quad n < m \\ 0 & \text{if} \quad n > m \end{cases}$$

Example Evaluate

1. $\lim_{x\to\infty} \frac{2x^{10}-6x^9+14x+1}{25x^6+14x+2}$

Solution: $\lim_{x\to\infty} \frac{\frac{2x^{10}-6x^9+14x+1}{x^6}}{\frac{25x^6+14x+2}{x^6}} = \lim_{x\to\infty} \frac{2x^4-6x^3+\frac{14}{x^5}+\frac{1}{x^6}}{25+\frac{14}{x^5}+\frac{2}{x^6}} = \infty$.

2. $\lim_{x\to\infty} \frac{1}{x^5} = \lim_{x\to\infty} (\frac{1}{x})^5 = (\lim_{x\to\infty} \frac{1}{x})^5 = 0$

3. $\lim_{x\to\infty} \frac{5x+6}{12x-20} = \frac{5}{12}$

Application of limit

EXAMPLE: Finding an Average Speed A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

SOLUTION: Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2$$

feet in the first t seconds. The average speed of the rock over any given time interval is the distance traveled, Δy, divided by the length of the interval Δt. For the first 2 seconds of fall, from $t = 0$ to $t = 2$, we have

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\frac{\text{ft}}{\text{sec}}.$$

EXAMPLE 2 Finding an Instantaneous Speed

Find the speed of the rock in Example 1 at the instant $t = 2$.

SOLUTION

Solve Numerically We can calculate the average speed of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

$$\frac{\Delta y}{\Delta t} = \frac{16(2 + h)^2 - 16(2)^2}{h}. \tag{1}$$

We cannot use this formula to calculate the speed at the exact instant $t = 2$ because that would require taking $h = 0$, and $0/0$ is undefined. However, we can get a good idea of what is happening at $t = 2$ by evaluating the formula at values of h *close* to 0. When we do, we see a clear pattern (Table 2.1 on the next page). As h approaches 0, the average speed approaches the limiting value 64 ft/sec.

Confirm Algebraically If we expand the numerator of Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2 + h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h.$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $64 + 16h$ ft/sec. We can now see why the average speed has the limiting value $64 + 16(0) = 64\ ft/\sec\ as\ h\ approaches\ 0$

References

[1] **Robert T. Smith and Roland B. Minton, (2007)**, Calculus: Early Transcendental Functions, Third Edition, Publishing by McGraw-Hill, a business unit of the McGraw-Hill companies, Inc.

[2] Calculus, Schaum's outline series