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Integrability Analysis of the Smallest 3D Biochemical Reaction Model

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Abstract

In this paper the complex dynamics of a smallest biochemical system model in three-dimensional systems with the reaction scheme. This model is described by a system of three nonlinear ordinary differential equations with five positive real parameters, are analyzed and studied. We present a thorough analysis of their invariant algebraic surfaces and exponential factors and investigate the integrability and nonintegrabilty of this model. Particularly, we show the non–existence of polynomial, rational, Darboux and local analytic first integrals in a neighborhood of the equilibrium. Moreover, we prove that, the model is not integrable in the sense of Bogoyavlensky in the class of rational functions.

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1 Introduction and the main result

Nonlinear systems of ordinary differential equations is appeared in many branches of physics, chemistry, biology, mechanics and economics. Exact solutions to those equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. However, even if there exists a solution, only for a few nonlinear system of ordinary differential equations it is possible to determine this exact solution. There is not a general analytical approach to find analytical solutions, see for instance [2, 11, 15]. The integrability theory of dynamical systems plays a quite important role in studying complex dynamics of many differential systems. Since the differential systems in general cannot be solved explicitly, the qualitative information provided by the theory of dynamical systems is the best that one can expect to obtain in general. One of the more classical problems in the qualitative theory of polynomial differential systems depending on parameters is to characterize the existence or not of first integrals.

In this paper we consider the smallest biochemical chemical reaction system introduced by Wilhelm et al in [14] in three-dimensional systems with the reaction scheme

$$\begin{gathered} A + X \xrightarrow{\tilde{k}} 2\ X \\ X + Y \xrightarrow{\tilde{k}_2} A + Y \\ Y \xrightarrow{\tilde{k}_3} A \\ X \xrightarrow{\tilde{k}_4} Z \\ Z \xrightarrow{\tilde{k}_5} Y \end{gathered}$$

where A denotes outer reactants representing at least two different substances for thermos dynamical reasons, and X, Y and Z are the autocatalytic reactants, k, k_2, k_3, k_4 and k_5 values are the reaction rate coefficients for each component reaction. By considering the mass action law, the system dynamics which is given by the following ordinary differential equations

$$\begin{aligned} \dot{x} &= kx - k_2xy, \\ \dot{y} &= -k_3y + k_5z, \\ \dot{z} &= k_4x - k_5z. \end{aligned} \quad (1)$$

where k , k_2 , k_3 , k_4 , $k_5 > 0$. Also in [13] on the basis of a sufficient condition for a Hopf bifurcation of system (1) for finding one reaction system which is according to the given characterization, the smallest one. The stability and bifurcations of this system have been investigated in detail in [14]. In 2009, Wilhelm in [12] discussed the roles of the reactions concerning the necessary conditions for the bistability of system (1) and proved that near the bifurcation point a stable limit cycle appears and transcritical bifurcation by using the methods of local bifurcation theory, especially the center manifold and the normal form Theorem. Smith presented the global behavior of solutions for system (1) that exhibits a Hopf bifurcation, a competitive system with a monotone cyclic feedback, with the help of the Poincar$\acute{e}$ - Bendixson theory by ruling out periodic orbits with a Bendixson criterion in 2012 [10]. In [4], it was demonstrated that supercritical Hopf bifurcation occurs for the flow of system (1) restricted to the center manifold. This was done by obtaining a center manifold up to third degree for the study of system (1). This study analyzes the invariant algebraic surfaces and exponential factors of system (1) to determine the Darboux integrability of the system. Moreover, we study local analytic and Bogoyavlensky first integrals. The system (1) which have no analytical solutions, allow the investigations to be carried on with this computational algebraic method, but no analytical solution for this system is known. According to our knowledge, the integrability and nonintegrabilty problems for the system have not studied. This study focus for studying some types of first integrals of system (1) in the analysis we use Darboux Theorem and some preliminary results. All mathematical analysis, particularly, solving partial differential equations are verified with the help of Maple.

In the following, we summarize the main results related to the Darboux and analytic first integrals of system (1).

Proposition 1. *System* (1) *does not admit polynomial first integrals.*

Proposition 2. *The only irreducible invariant algebraic surfaces of system* (1) *with non-zero cofactor is* $x = 0$ *with cofactor* $k - k_2 y$.

Proposition 3. *The only exponential factors of system* (1) *are* e^y *and* e^z *with cofactors* $-k_3 y + k_5 z$ *and* $k_4 x - k_5 z$ *respectively and also exponential of linear combinations of* y *and* z.

Theorem 4. *System* (1) *does not admit Darboux first integrals.*

Corollary 1. *System* (1) *does not admit rational first integrals.*

Theorem 5. *The system* (1) *has no a local analytic first integral in a neighborhood of the origin if* $\frac{k_3}{k}$ *and* $\frac{k_5}{k}$ *are not positive integer numbers.*

Theorem 6. *The system* (1) *has no a local analytic first integral in a neighborhood of the equilibrium point* $(\frac{k}{k_2}\frac{k_3}{k_4}, \frac{k}{k_2}, \frac{k}{k_2}\frac{k_3}{k_5})$ *if one of the following conditions holds:*

1. $k = \frac{-\mu(a^2+b^2)}{a^2+2\ a\mu+b^2}$, $k_3 = -a - \frac{\mu+\sqrt{-4a\mu-4b^2+\mu^2}}{2}$, $k_5 = -a - \frac{\mu-\sqrt{-4a\mu-4b^2+?^2}}{2}$, $k \neq \frac{-2an(4na^2+2ank_5+2ak_5+k_5{}^2)}{k_5(2an+2a+k_5)}$, $k_3 \neq -2an - 2a - k_5$, $\mu, a, b \in \mathbb{R}\backslash\{0\}, n \in \mathbb{Z}$.

2. $k_5 = k - k_3$.

3. $k_5 + k_3 - k > 0$.

Theorem 7. *System* (1) *is not completely integrable with two functionally independent rational first integrals. Moreover, system* (1) *is not B-integrable in the class of rational functions and does not possess any rational first integral.*

2 Elementary results

We recall some definitions given in [5, 7]. Some well-known results on the Darboux theory of integrability and analytic first integrals may be found in [1, 2, 5, 6]. We characterize here integrability and non-integrability of system (1). Thus to prove the main results, we use Darboux theorem of integrability in order to find invariant algebraic surfaces, exponential factors and characterize its local analytic first integrals of system (1). By χ we denote the corresponding vector field of system (1)

$$\chi = (kx - k_2xy)\frac{\partial}{\partial x} + (-k_3y + k_5z))\frac{\partial}{\partial y} + (k_4x - k_5z)\frac{\partial}{\partial z}.$$

A continuously differentiable function $H(x, y, z)$ in a neighborhood $U \in \mathbb{R}^3$ is said to be a first integral of the vector field (1) if $H(x, y, z)$ is a constant on the trajectories of system (1), that is $\chi(H) = 0$.

We call H a polynomial (respectively analytic) first integral if H is polynomial (respectively analytic) see [6]. The existence of Draboux first integral depends on the exponential factors and on the invariant algebraic surfaces. Hence we recall definitions of Darboux polynomial and exponential factor. Let $f \in \mathbb{C}[x, y, z]$ be a non constant polynomial and f satisfies the partial differential equation

$$(kx - k_2xy)\frac{\partial f}{\partial x} + (-k_3y + k_5z)\frac{\partial f}{\partial y} + (k_4x - k_5z))\frac{\partial f}{\partial z} = Cf,$$

for some polynomial $C \in \mathbb{C}[x, y, z]$. We call $f = 0$ an invariant algebraic surface (and f a Darboux polynomial) of system (1) and C is the cofactor of f of degree one.
Let $f, g \in \mathbb{C}[x, y, z]$ be relatively coprime. A non-constant function $e^{\frac{f}{g}}$ is called an exponential factor of system (1) if it satisfies the partial differential equation

$$(kx - k_2xy)\frac{\partial e^{\frac{f}{g}}}{\partial x} + (-k_3y + k_5z)\frac{\partial e^{\frac{f}{g}}}{\partial y} + (k_4x - k_5z)\frac{\partial e^{\frac{f}{g}}}{\partial z} = Le^{\frac{f}{g}},$$

for some polynomial $L \in \mathbb{C}[x, y, z]$ of degree one. We call L the cofactor of $e^{\frac{f}{g}}$. For more details see [1, 7].
We recall that a Darboux first integral is a product of complex powers of invariant algebraic surfaces and exponential factors.
The following result restricted to the invariant algebraic surfaces goes back to Darboux which concerning the existence of Darboux first integrals, see [5, 7].

Theorem 8. *Suppose that a polynomial system* (1) *admits* p *invariant algebraic surfaces* $f_i = 0$ *with cofactors* C_i *for* $i = 1, ..., p$ *and* q *exponential factors* $exp\left(\frac{g_j}{h_j}\right)$ *with cofactors* L_j *for* $j = 1, ..., q$. *Then there exist* λ_i *and* $\mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = 0,$$

if and only if the function

$$f_1{}^{\lambda_1} ... f_p{}^{\lambda_p} \left(exp\left(\frac{g_1}{h_1}\right) \right)^{\mu_1} ... \left(exp\left(\frac{g_q}{h_q}\right) \right)^{\mu_q},$$

is a Darboux first integral of system (1).

Theorem 9. *[2] The following statements hold.*

(a) If $exp\left(\frac{g}{h}\right)$ *is an exponential factor for the polynomial differential system* (1) *and* h *is not a constant polynomial, then* $h = 0$ *is an invariant algebraic surface.*

(b) Eventually $exp\,(g)$ *can be an exponential factor, coming from the multiplicity of the invariant plane at infinity.*

We now introduce some results that will be used through the paper. The following result is due Falconi and Llibre and its proof can be found in [3], see also [9].

Proposition 10. *The real linear differential system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

with $abc \neq 0$ *has two independent first integrals of the form* $H_1 = \frac{(x^2+y^2)^c}{z^{2a}}$ *and* $H_2 = (x^2 + y^2)^b e^{-2a\ arctan(\frac{y}{x})}$ *and if* $a = 0$, *such a system has two independent first integrals of the form*

$$H_1 = x^2 + y^2, H_2 = z^b\ e^{-c\ arctan(\frac{y}{x})}.$$

Proposition 11. *[7] If the linear part of system* (1) *has no polynomial first integrals in a neighborhood of equilibrium point q, then the whole system has no analytic first integrals in a neighborhood of q.*

An equilibrium point q is said to be an attractor (repeller) provided solutions starting nearby limit to the point as $t \to \infty$ $(t \to -\infty)$.

Proposition 12. *[8] If system* (1) *has an isolated equilibrium point q which is either attractor or repeller, then it has no C^1-first integrals defined in a neighborhood of q.*

There is a natural concept of non-Hamiltonian integrability for dynamical systems, which was probably originally studied by Bogoyavlenskij, see [11] but was also independently investigated by other persons from various perspectives.
System (1) is B-integrable if it possesses k functionally independent first integrals H_1, ..., H_k, where $0 \leq k < 3$ and an abelian $(3-k)$ dimensional Lie algebra S_a of symmetries which preserve first integrals H_j and which are linearly independent.
We also need the following result concerning complete integrability of non-linear three dimensional differential systems [11].

Theorem 13. *Consider system*

$$\begin{aligned}
\dot{x} &= P(x,y) + Q\,(x,y)\,z, \\
\dot{y} &= R(x,y) + S\,(x,y)\,z, \\
\dot{z} &= T(x,y) + G\,(x,y)\,z,
\end{aligned}$$

where P, Q, R, S, T and G are meromorphic functions.

1. *If there exists a point $c = (x_1, y_1)$ such that $P\,(c) = Q\,(c) = R\,(c) = S\,(c) = 0$, $\frac{\partial Q(c)}{\partial y} = 0$ and at least $T(c) \neq 0$ or $G\,(c) \neq 0$. Then the system is not completely integrable with two functionally independent rational first integrals in variables x, y, z.*

2. *If $G(c) \neq 0, \triangle \neq 0$ and $\triangle_1(c) \notin \mathbb{N}$, then the above system is not B-integrable in the class of rational functions and does not possess*

any rational first integral, where

$$\triangle(x,y)=\frac{\partial S}{\partial y}-\frac{\partial Q}{\partial x} \text{ and}$$
$$\triangle_1(x,y)=\frac{\frac{\partial P}{\partial y}}{\triangle^2 F^2}[\triangle(\frac{\partial P}{\partial x}-\frac{\partial R}{\partial y})\frac{\partial S}{\partial x}-\frac{\partial P}{\partial y}(\frac{\partial S}{\partial x})^2+\triangle^2\frac{\partial R}{\partial x}] .$$

3 Proofs of the main results

In the study of the first integrals of Darboux type of system (1), one has to find polynomial first integrals, all Darboux polynomials and exponential factors of system (1) and this is due to the fact that the Darboux first integrals can be constructed using these kind of functions.

Proof of Proposition 1. Let $H(x,y,z)=\sum_{i=1}^{n} h_i(x,y,z)$ be a polynomial first integral of degree n of system (1), where each h_i is a homogeneous polynomial of degree i in the variables x, y, z and $h_n\neq 0$, $n\geq 1$. Then H satisfies

$$(k\ x-k_2x\ y)\frac{\partial H}{\partial x}+(-k_3y+k_5z)\frac{\partial H}{\partial y}+(k_4x-k_5z)\frac{\partial H}{\partial z}=0. \qquad (2)$$

We distinguish the following two cases.

I. If h_n is a function of variables of x, y and z. The terms of degree $n+1$ in (2) satisfy

$$-k_2x\ y\frac{\partial h_n}{\partial x}=0.$$

Since $k_2>0$, so the solution of this linear partial differential equation is

$$h_n=F_n(y,z),$$

where F_n is a polynomial function. Since h_n is a homogenous polynomial of degree n, then

$$h_n(y,z)=(a\ z+b\ y)^n,$$

where a and b are constants such that at least a or b is not zero. Also tacking the homogeneous part of degree n in equation (2) we obtain

$$kx\frac{\partial h_n(y,z)}{\partial x} - k_2xy\,\frac{\partial h_{n-1}(x,y,z)}{\partial x} + (-k_3y+k_5z)\,\frac{\partial h_n(y,z)}{\partial y} + (k_4x-k_5z)\,\frac{\partial h_n(y,z)}{\partial z} = 0.$$

We get that the general solution is

$$h_{n-1}(x,y,z) = -\frac{n(az+by)^{n-1}\left(bk_3y + (a-b)\,k_5z\right)\ln(x) - a\,k_4x)}{k_2y} + F_{n-1}(x,y),$$

where F_{n-1} is a polynomial function in the variables x and y. Since h_{n-1} is a polynomial of degree $n-1$ and $k_3,\ k_5 > 0,\ n \geq 1$ then must be $a = b = 0$. We conclude that $h_n = 0$. This is a contradiction.

II. If h_n is a function only of variables of y and z. Since h_n is a homogenous polynomial of degree n, then $h_n(y,z) = (az+by)^n$, where a and b are not zero simultaneously. Proceeding as in the proof of case I we obtain that $h_n = 0$. This is contradiction. Hence the result follows. ■

Proof of Proposition 2. Since system (1) is quadratic then the cofactor must be of the form $C = c_0 + c_1\,x + c_2y + c_3\,z$, where $c_i \in \mathbb{C}$, $i = 0,1,2,3$. Let $f(x,y,z) = \sum_{i=1}^{n} f_i(x,y,z)$ be an invariant algebraic surface of system (1) of degree $n \geq 1$, where each $f_i(x,y,z)$ is a homogeneous polynomial of degree i with $f_n \neq 0$. Then it satisfies the partial differential equation

$$(kx - k_2xy)\,\frac{\partial f}{\partial x} + (-k_3y + k_5z)\,\frac{\partial f}{\partial y} + (k_4x - k_5z))\,\frac{\partial f}{\partial z} = (c_0 + c_1x + c_2y + c_3z)\,f. \tag{3}$$

Computing the terms of degree $n+1$ in equation (3) we obtain

$$-k_2xy\frac{\partial f_n(x,y,z)}{\partial x} = (c_1x + c_2y + c_3z)\,f_n(x,y,z).$$

First if $\frac{\partial f_n}{\partial x} = 0$, since $f_n \neq 0$ then $c_1 = c_2 = c_3 = 0$ and so $C = c_0$.

We taking $f_n = (az+by)^n$, where at least a or b is non zero. Taking the homogeneous part of degree n of (3) we get the equation

$$-k_2xy\frac{\partial f_{n-1}(x,y,z)}{\partial x}+(-k_3y+k_5z)\frac{\partial f_n(y,z)}{\partial y}+(k_4x-k_5z)\frac{\partial f_n(y,z)}{\partial y}=c_0f_n(y,z),$$

then

$$f_{n-1}(x,y,z)=\frac{(-bk_3y-nk_5(a-b)z-c_0(az+by))\,ln(x)}{k_2y}+\frac{nak_4x(az+by)^{n-1}}{k_2y}+F_{n-1}(y,z),$$

where F_{n-1} is a polynomial function in its variables y and z. Since this is to be a polynomial of degree $n-1$, we must take $a=0$. Moreover, the logarithm must be eliminated, we have

$$-b(nk_3+c_0)=bnk_5=0.$$

Since $b\neq 0$ and $k_3,\ k_5>0$ then $n=c_0=0$. This is contradiction with the fact $x=0$ be an invariant algebraic surface of system (1). Then this case $\frac{\partial f_n}{\partial x}=0$ does not hold.
Now we assume that $\frac{\partial f_n}{\partial x}\neq 0$. Solving the following partial linear differential equation with respect to f_n

$$-k_2xy\frac{\partial f_n(x,y,z)}{\partial x}=(c_1x+c_2y+c_3z)f_n(x,y,z),$$

we get

$$f_n\left(x,y,z\right)=x^{\frac{-c_2y-c_3\ z}{k_2y}}\ e^{-\ \frac{c_1}{k_2}\frac{x}{y}}F_n\left(y,z\right),$$

where F_n is a function. Since f_n is a homogenous polynomial of degree n then must be $c_1=c_3=0$ and $c_2=rk_2,\ -r\in\mathbb{N}\cup\{0\}$. Hence

$$f_n\left(x,y,z\right)=(az+b\ y)^{n+r}x^{-r},$$

where a and b are constants . Since $f_n\neq 0$ then at least a or b is not zero. Also the differential equation corresponding to the terms of degree n in equation (3) we obtain

$$kx\frac{\partial f_n(x,y,z)}{\partial x} - k_2xy\ \frac{\partial f_{n-1}(x,y,z)}{\partial x} + (-k_3y+k_5z)\ \frac{\partial f_n(x,y,z)}{\partial y}$$
$$+ (k_4x-k_5z)\ \frac{\partial f_n(x,y,z)}{\partial z} = rk_2yf_{n-1}(x,y,z) + c_0f_n(x,y,z)\,.$$

It is general solution is

$$f_{n-1}(x,y,z) = F_{n-1}(y,z)\ \ x^{-r} + \frac{ak_4(n+r)(az+by)^{n+r-1}x^{-r+1}}{k_2y} - \frac{Q\ ln(x)}{k_2y},$$

where

$$Q = (az+by)^{n+r-1}\,x^{-r}\,((k_3by + k_5(a-b)z)\,(n+r) + (kr+c_0)(az+by))\,,$$

and F_{n-1} is a polynomial function. Since the logarithm in the polynomail f_{n-1} must be removed, we have

$$b\,(kr + (n+r)\,k_3 + c_0) = 0,$$
$$(akr + a\ \ (n+r)\,k_5 - b\,(n+r)\,k_5 + ac_0) = 0.$$

Solving the above equations and using a and b are not zero simultaneously, we have three conditions either

1. $b = 0,\ c_0 = -kr - (n+r)\,k_5$, or
2. $n + r = 0,\ c_0 = -kr$, or
3. $b = \frac{-a(k_3-k_5)}{k_5},\ c_0 = -k\ r - (n+r)\ k_3$.

For the first case we get $f_n = a^{n+r}z^{n+r}x^{-r}$ and $f_{n-1} = F_{n-1}(y,z)\ x^{-r} + \frac{ak_4(n+r)(az)^{n+r-1}x^{-r+1}}{k_2y}$. Since $a \neq 0$ and f_{n-1} is a polynomial, we must have $r = -n$, then $f_n = x^n$ and $f_{n-1} = F_{n-1}(y,z)\ x^n$, but f_{n-1} is a homogenous polynomial of degree $n-1$, then must be $F_{n-1}(y,z) = 0$, and hence $f_{n-1}(x,y,z) = 0$.
For the second case $r = -n, c_0 = -k\ r$ we obtain $f_n = x^n$ and $f_{n-1} = F_{n-1}(y,z)\ x^n$. By the same argument of first case we get $f_{n-1}(x,y,z) = 0$.

Finally for the third case $b = \frac{-a(k_3-k_5)}{k_5}$, $c_0 = -kr - (n+r)\, k_3$, then $f_n = \left(a(z+(1-\frac{k_3}{k_5})y)\right)^{n+r} x^{-r}$ and $f_{n-1}(x,y,z) = F_{n-1}(y,z)\, x^{-r} - \frac{a^{n+r} k_4 (n+r)(k_3-k_5)y - k_5 z)^{n+r-1} x^{-r+1}}{{k_5}^{n+r-1} k_2 y}$. Also since f_{n-1} is a homogenous polynomial of degree $n-1$, we have $r = -n$, then we obtain $f_n = x^n$ and $f_{n-1} = F_{n-1}(y,z)\, x^n$. By the same argument of first case we get $f_{n-1}(x,y,z) = 0$.

So for all three cases we obtain $f_n = x^n$ and $f_{n-1}(x,y,z) = 0$ with $c_0 = -k\ r, r = -n$. Now computing the terms of degree $n-1$ in equation (3) we obtain

$$-k_2 x\ y \frac{\partial f_{n-2}(x,y,z)}{\partial x} = -n\ k_2 y f_{n-2}(x,y,z),$$

First if f_{n-2} is a function of variables of x, y and z, solving it we obtain

$$f_{n-2}(x,y,z) = F_{n-2}(y,z)\, x^n,$$

where F_{n-2} is a polynomial function in the variables y and z. But f_{n-2} is a homogenous polynomial of degree $n-2$, then must be $F_{n-2}(y,z) = 0$, hence $f_{n-2}(x,y,z) = 0$. Second if $\frac{\partial f_{n-2}}{\partial x} = 0$ we obtain directly $f_{n-2} = 0$. Now we will prove by induction that $f_i = 0$ for $i = 1, ..., n-1$. Now we assume that the above equation is true for $i = 2, ..., n-1$.

For the terms of degree 2 in (3) we have

$$-k_2 xy \frac{\partial f_1(x,y,z)}{\partial x} = -k_2 y f_1(x,y,z),$$

then

$$f_1(x,y,z) = F_1(y,z)x,$$

where F_1 is an arbitrary function in the variable y and z. Since f_1 is a homogeneous polynomial of degree 1 then $F_1(y,z) = 0$, so $f_1(x,y,z) = 0$. So, $f_1(x,y,z) = 0$. But if $\frac{\partial f_1}{\partial x} = 0$, also we obtain directly $f_1 = 0$. Hence $f(x,y,z) = x^n = 0$ is an invariant algebraic surface of system (1) with cofactor $n(-k + k_2 y)$. This concludes the proof of the proposition. ∎

Proof of Proposition 3. Since system (1) has only the irreducible invariant algebraic surface $x = 0$, then in view of Theorem 9 system (1) can have an exponential factor of the form: either $E = e^h$ with $h \in \mathbb{C}[x, y, z]$ or $E = e^{\frac{h}{x^n}}$ with $n \geq 1$ and such that h is coprime with x and the degree of h is at most n.

First we show that system (1) has no exponential factors of the form $E = e^{\frac{h}{x^n}}$. Clearly E satisfies the partial differently equation

$$(kx-k_2x\ y)\,\frac{\partial E}{\partial x}+(-k_3y+k_5z)\,\frac{\partial E}{\partial y}+(k_4x-k_5z))\,\frac{\partial E}{\partial z} = LE,$$

where the cofactor $L = a_0+a_1x+a_2y+a_3z$, where $a_i \in \mathbb{C}$, $i = 0, 1, 2, 3$. After simplifying the above equation becomes

$$(kx-k_2xy)\,\frac{\partial h}{\partial x} - nh\,(k-k_2y)+(-k_3y+k_5z)\,\frac{\partial h}{\partial y}+(k_4x-k_5z))\,\frac{\partial h}{\partial z}$$
$$= (a_0+a_1x+a_2y+a_3z)x^n.$$

We denote the restriction of h to $x = 0$, by $\widetilde{h}$ in the above equation, then $\widetilde{h} \neq 0$ and $\widetilde{h}$ satisfies

$$-n\widetilde{h}\,(k\ -k_2\ y)+(-k_3y+k_5z)\,\frac{\partial \widetilde{h}}{\partial y}-k_5z\frac{\partial \widetilde{h}}{\partial z}= 0,$$

and whose solution is

$$\widetilde{h}\,(y, z)= F_1((k_3y-k_5y-k_5z)z^{\frac{-k_3}{k_5}})z^{\frac{-nk}{k_5}}e^{\frac{n\ k_2(y+z)}{k_3}},$$

where F_1 is a function. Since $\widetilde{h}$ is a polynomial and $n \geq 1$, $k_2 > 0$ then we have $\widetilde{h} = 0$, which is a contradiction and this case is not possible. Now if a system (1) has an exponential factor it must be of the form $E = e^h$ with $h \in \mathbb{C}[x, y, z]$. Then h must be satisfies

$$(kx-k_2x\ y)\,\frac{\partial h}{\partial x}+(-k_3y+k_5z)\,\frac{\partial h}{\partial y}+(k_4x-k_5z)\,\frac{\partial h}{\partial z} = L. \tag{4}$$

We now show that h is a polynomial of degree at most one. We can write $h = \sum_{i=0}^{n} h_i\,(x, y, z)$, where each h_i is a homogeneous polynomial in the variables x, y and z of degree i and $n > 0$. If there are terms of degree $n+1$ with $n \geq 2$ in equation (4) satisfy

$$k_2 xy \frac{\partial h_n(x,y,z)}{\partial x} = 0.$$

Since $k_2 > 0$, then we obtain

$$h_n(x,y,z) = F_n(y,z).$$

where F_n is a polynomial function. Since f_n is a homogenous polynomial of degree n, then

$$h_n(x,y,z) = (a\ z + b\ y)^n,$$

where a and b are not zero simultaneously. Also computing the terms of degree n in equation (4) we obtain

$$kx\frac{\partial h_n(x,y,z)}{\partial x} - k_2 xy \frac{\partial h_{n-1}(x,y,z)}{\partial x} + (-k_3 y + k_5 z)\frac{\partial h_n(x,y,z)}{\partial y} + (k_4 x - k_5 z)\frac{\partial h_n(x,y,z)}{\partial z} = 0.$$

It is general solution is

$$h_{n-1}(x,y,z) = -\frac{n(az+by)^{n-1}((bk_3 y + (a-b)k_5 z)\ln(x) - ak_4 x)}{k_2 y} + F_{n-1}(x,y),$$

where F_{n-1} is a polynomial function. Also must be the logarithm in the polynomail h_{n-1} removed, we have $a = b = 0$. Then $h_n = 0$ in contradiction with the fact that $n \geq 2$. Then, we must h is a polynomial of degree one in its variables that we write it as

$$h(x,y,z) = d_0 + d_1 x + d_2 y + d_3 z.$$

Imposing that h satisfies (4) with $L = a_0 + a_1\ x + a_2 y + a_3\ z$. We finally obtain that $h = d_0 + d_2 y + d_3 z$ and $L = d_2(-k_3 y + k_5 z) + d_3(k_4 x - k_5 z)$. This concludes the proof of the proposition. ■

Proof of Theorem 4. Suppose that H is a Darboux first integral of system (1). From Propositions 2, 3 and Theorem 8 then $H= x^{\lambda}\ e^{\mu_1 y}\ \ e^{\mu_2 z}$ where $\lambda,\ \mu_1,\ \mu_2 \in \mathbb{C}$. So H satisfies

$$(kx-k_2xy)\,\frac{\partial H}{\partial x}+(-k_3y+k_5z)\,\frac{\partial H}{\partial y}+(k_4x-k_5z)\,\frac{\partial H}{\partial z}=0.$$

Then $(\lambda\,(k\ x-k_2x\ y)+\mu_1\,(-k_3y+k_5z)+\mu_2\,(k_4x-k_5z)\,))H=0$, since the parameters k,k_2, k_3, k_4 and k_5 are positive numbers, then must be $\lambda = \mu_1 = \mu_2 = 0$. Therefore H is a constant, this is contradiction. ■

Proof of Corollary 1. From Propositions 1 and 2 system (1) has no polynomial first integrals and has no two different invariant algebraic surfaces with the same cofactor, then system (1) has no rational first integrals. ■

We now show that the system has no analytic first integrals at neighbourhood of equilibrium points. We compute the equilibrium points of system (1) which are

$$E_0 = (0,0,0) \quad \text{and } E_1 = \left(\frac{k\ k_3}{k_2\ k_4}, \frac{k}{k_2}, \frac{k\ k_3}{k_2\ k_5}\right).$$

Proof of Theorem 5. The linear part of the system (1) at the origin is

$$\begin{aligned} \dot{x} &= kx, \\ \dot{y} &= -k_3y+k_5z, \\ \dot{z} &= k_4x-k_5z. \end{aligned} \tag{5}$$

Direct calculations shows

$$kx\frac{\partial H_i}{\partial x}+(-k_3y+k_5z)\,\frac{\partial H_i}{\partial y}+(k_4x-k_5z)\,\frac{\partial H_i}{\partial z}=0,$$

where $H_1 = x^{\frac{k_5}{k}}(kz-k_4x+k_5z)$ and

$$\begin{aligned} H_2 =&(k+k_3)x^{\frac{k_3}{k}}(k_5\,(k\ z-k_4\ x+k_5z)-(k_3-k_5)\,(k+k_5)\,y)+ \\ &k_4k_5\,(k_3-k_5)\,x^{\frac{k+k_3}{k}} \end{aligned}$$

Therefore, the function H_i is a constant over the solutions of the system (5) for $i = 1, 2$.
If $\frac{k_3}{k}$ and $\frac{k_5}{k}$ are not positive integer numbers, then the linear part of system (1) has no polynomial first integrals in a neighborhood of the origin, hence the result follows by Theorem 11. ■

The change of variables $(X, Y, Z) \to (x - \frac{k\ k_3}{k_2\ k_4}, y - \frac{k}{k_2}, z - \frac{k\ k_3}{k_2\ k_5})$ move to equilibrium point E_1 to an equilibrium point at the origin and system (1) becomes

$$\begin{aligned} \dot{X} &= -\frac{kk_3}{k_4}Y - k_2XY, \\ \dot{Y} &= -k_3Y + k_5Z, \\ \dot{Z} &= k_4X - k_5Z. \end{aligned} \tag{6}$$

The Jacobian matrix of system (6) evaluated at $(0, 0, 0)$ is

$$J = \begin{pmatrix} 0 & \frac{-k\ k_3}{k_4} & 0 \\ 0 & -k_3 & k_5 \\ k_4 & 0 & -k_5 \end{pmatrix}.$$

The characteristic equation of the matrix J is given by

$$\lambda^3 + (k_5 + k_3)\,\lambda^2 + k_3k_5\lambda + kk_3k_5 = 0. \tag{7}$$

Before proving Theorem 6 we need to characterize all the polynomial first integrals of the linear part of system (6).

Lemma 1. *The characteristic equation* (7) *has one simple real root* μ *and two complex roots* $a \pm i\ b$ *with* $a, b \in \mathbb{R}$, *if satisfying*

$$\begin{aligned} -2a - \mu &= k_3 + k_5, \\ a^2 + 2\ a\mu + b^2 &= k_3k_5, \\ -\ \mu(a^2 + b^2) &= kk_3k_5. \end{aligned} \tag{8}$$

Proof. The characteristic equation (7) can be rewritten

$$\lambda^3+(k_5+k_3)\,\lambda^2+k_3k_5\lambda+kk_3k_5=(\lambda-\mu)(\lambda-a-ib)(\lambda-a+ib).$$

By comparing the above equation of λ we obtain the conditions (8). ■

Lemma 2. *If $k_5=k-k_3$, then linear part of system* (6) *has only one polynomial first integral.*

Proof. Let $f_2=X^2-\frac{2k_3}{k_4}XY+\frac{kk_3}{{k_4}^2}Y^2+\frac{2(k-k_3)k_3}{{k_4}^2}YZ+\frac{(k-k_3)k_3}{{k_4}^2}Z^2$. It is easy to check that

$$-\frac{kk_3}{k_4}Y\frac{\partial f_2}{\partial X}+(-(Y+Z)\,k_3+kZ)\,\frac{\partial f_2}{\partial Y}+(k_4X+(k_3-k)\,Z)\,\frac{\partial f_2}{\partial Z}=0.$$

So f_2 is a polynomial first integral of linear part of system (6). The Jacobian matrix J has two complex conjugate eigenvalues $\pm i\sqrt{kk_3-{k_3}^2}$ and one real eigenvalue $-k$. So, the real Jordan matrix of the linear differential system (6) is

$$\begin{pmatrix} 0 & -\sqrt{kk_3-{k_3}^2} & 0 \\ \sqrt{kk_3-{k_3}^2} & 0 & 0 \\ 0 & 0 & -k \end{pmatrix}.$$

Then from Theorem 10 our linear differential system has at most one polynomial first integrals because the eigenvalues of J are pure imaginary complex numbers ■

Proof of Theorem 6. **1.** If $k=\frac{-\mu(a^2+b^2)}{a^2+2a\mu+b^2}$, $k_3=-a-\frac{\mu+\sqrt{-4a\mu-4b^2+\mu^2}}{2}$, $k_5=-a-\frac{\mu-\sqrt{-4a\mu-4b^2+\mu^2}}{2}$, $\mu,a,b\in\mathbb{R}\backslash\{0\}$, from Lemma 1, the characteristic equation (6) has one simple real root μ and two complex roots $a\pm ib$. By Theorem 10, the real Jordan matrix of the linear part of differential system (6) is

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix}=\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & \mu \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

and the above linear system has two first integrals $H_1{=}\frac{(X^2+Y^2)^{\mu}}{Z^{2a}}$ and $H_2{=}e^{-2a\ arctan(\frac{Y}{X})}\left(X^2{+}Y^2\right)^b$ with $\mu, a, b \in \mathbb{R}\backslash\{0\}$. Then only H_1 is a polynomial first integral if and only if we have that condition $\mu = 2an$, where a is positive integer and n is negative integer (or n is positive integer and a is negative integer because $\frac{1}{H_1}$ is also the first integral), in this case a, b, μ satisfies equation (8), so substitution $\mu{=}\ 2an$, in equation (8), we obtain the solution

$$k = \frac{-2an(4na^2 + 2a\ nk_5 + 2ak_5 + {k_5}^2)}{k_5\,(2an + 2a + k_5)},\ k_3 = -2an - 2a - k_5. \qquad (9)$$

By the hypothesis none of them are possible. Therefore, the linear part of system (6) has no polynomial first integrals. Then, directly using Theorem 11 we can say that system (1) has no local analytic first integral at the neighborhood of the equilibrium point E_1.

2. Now we prove that system (6) has no local analytic first integrals at the origin when $k_5 = k - k_3$. We assume that $F = F(X, Y, Z)$ is a local analytic first integral at the origin of system (6) and we write it as $F{=}\sum_{i=1} F_i(X, Y, Z)$ where F_i is a homogeneous polynomial of degree i for $i{=}\ 1, 2, 3, \ldots.$ We will show by induction that $F_i{=}\ 0$ for all $i \geq 1$. Since F is a first integral of system (6) it must satisfy

$$-(\frac{kk_3}{k_4}Y{+}k_2XY)\frac{\partial F}{\partial X}{+}\left(-\left(Y{+}Z\right)k_3{+}kZ\right)\frac{\partial F}{\partial Y}{+}\left(k_4X{+}\left(k_3{-}k\right)Z\right)\frac{\partial F}{\partial Z} = 0. \qquad (10)$$

First, computing the terms of degree one in equation (10) satisfy the differential equation

$$-\frac{kk_3}{k_4}Y\frac{\partial F_1}{\partial X}{+}\left(-\left(Y{+}Z\right)k_3{+}kZ\right)\frac{\partial F_1}{\partial Y}{+}\left(k_4X{+}\left(k_3{-}k\right)Z\right)\frac{\partial F_1}{\partial Z} = 0,$$

we obtain that by Lemma 2 the linear part of system (6) has no polynomial first integral of degree one, this gives us $F_1 = 0$. Now, computing the homogeneous terms of degree two in equation (10), satisfy

$$-\frac{kk_3}{k_4}Y\frac{\partial F_2}{\partial X}{+}\left(-\left(Y{+}Z\right)k_3{+}kZ\right)\frac{\partial F_2}{\partial Y}{+}\left(k_4X{+}\left(k_3{-}k\right)Z\right)\frac{\partial F_2}{\partial Z}{=}\ 0.$$

By Lemma 2 we have that $F_2 = c_2 f_2$, where f_2 is a polynomial first integral in the statement of the Lemma 2, and c_2 is a constant. Computing the polynomial homogeneous terms of degree three in equation (10), we have

$$-\frac{kk_3}{k_4}Y\frac{\partial F_3}{\partial X}+(-(Y+Z)k_3+kZ)\frac{\partial F_3}{\partial Y}+(k_4X+(k_3-k)Z)\frac{\partial F_3}{\partial Z}=k_2XY\frac{\partial F_2}{\partial Z}.$$

taking an arbitrary homogeneous polynomial of degree three for F_3 and substituting it in the equation above. Using Maple for computing and after some easy computations we have that

$$F_3=\frac{-2k_2c_2f_3}{3kk_3k_4^2\left(4k^2+kk_3-{k_3}^2\right)\left(k^2+4kk_3-4{k_3}^2\right)},$$

where

$$\begin{aligned}f_3=&4k^5{k_3}^2y^3+12\,k^5{k_3}^2zy^2+12k^5{k_3}^2yz^2+4k^5{k_3}^2z^3+4k^4{k_4}^3x^3-12k_4k^4{k_3}^2xy^2\\&+6\,k_4k^4{k_3}^2xz^2+9\,k^4{k_3}^3y^3+3k^4{k_3}^3zy^2-6k^4{k_3}^3yz^2-2k^4{k_3}^3z^3+17k_3k^3{k_4}^3x^3\\&-12k^3{k_3}^2{k_4}^2yx^2-27k_4k^3{k_3}^3xy^2-42k_4k^3{k_3}^3xyz-18\,k_4k^3{k_3}^3xz^2-19k^3{k_3}^4y^3\\&-45\,k^3{k_3}^4zy^2-42k^3{k_3}^4yz^2-14k^3{k_3}^4z^3-23k^2{k_3}^2x^3{k_4}^3+24\,k^2{k_3}^3x^2y{k_4}^2\\&+57k_4k^2{k_3}^4xy^2+84k_4k^2{k_3}^4xyz+24k_4k^2{k_3}^4z^2x+8k^2{k_3}^5y^3+42k^2{k_3}^5zy^2\\&+54k^2{k_3}^5yz^2+18k^2{k_3}^5z^3+12kk_4^3{k_3}^3x^3-24kk_4^2{k_3}^4yx^2-24k_4k{k_3}^5xy^2\\&-54k{k_3}^5k_4xyz-18k{k_3}^5k_4xz^2-12k{k_3}^6zy^2-18k{k_3}^6yz^2-6k{k_3}^6z^3-6{k_3}^4{k_4}^3x^3\\&+12{k_3}^5{k_4}^2yx^2+12k_4{k_3}^6xyz+6\,k_4{k_3}^6xz^2.\end{aligned}$$

Then, computing the terms of degree four in (10) we get

$$-\frac{kk_3}{k_4}Y\frac{\partial F_4}{\partial X}+(-(Y+Z)k_3+kZ)\frac{\partial F_4}{\partial Y}+(k_4X+(k_4-k)Z)\frac{\partial F_4}{\partial z}=k_2XY\frac{\partial F_3}{\partial Z}.$$

Taking an arbitrary homogeneous polynomial of degree 4 for F_4 and substituting it in the above equation, using Maple and after some computations we get that $c_2=0$ and $F_4=c_4{f_2}^2$ where c_4 is a constant. Hence $F_2=F_3=0$.

Now, we will prove that by mathematical inductions that for $n\geq 3$

$$F_{2n}=c_{2n}{f_2}^n,\quad F_{2n+1}=c_{2n+1}{f_2}^{n-1}f_3,\ F_j=0, \tag{11}$$

where c_{2n} and c_{2n+1} are constants and, $j=1,2,...,n-1$. We assume that equation (11) holds for $n=3,4,5,...,$*N-1* where $N\geq 4$. We will prove that it is true for $n=N$. The homogeneous polynomial F_{2n-1} of degree $2n-1$ in equation (10) satisfy partial differential equation

$$-\frac{kk_3}{k_4}Y\frac{\partial F_{2n-1}}{\partial X}+\left(-\left(Y+Z\right)k_3+kZ\right)\frac{\partial F_{2n-1}}{\partial Y}+\left(k_4X+\left(k_3-k\right)Z\right)\frac{\partial F_{2n-1}}{\partial Z}$$
$$=k_2XY\frac{\partial F_{2n-2}}{\partial Z}.$$

It follows from Lemma 2, that $F_{2n-2}=c_{2n-2}{f_2}^{n-1}$, where c_{2n-2} is constant. Then the above equation becomes

$$-\frac{k\ k_3}{k_4}Y\frac{\partial F_{2n-1}}{\partial X}+\left(-\left(Y+Z\right)k_3+k\ Z\right)\frac{\partial F_{2n-1}}{\partial Y}+\left(k_4\ X+\left(k_3-k\right)Z\right)\frac{\partial F_{2n-1}}{\partial Z}$$
$$=\left(n-1\right)c_{2n-2}k_2XY{f_2}^{n-2}\frac{\partial f_2}{\partial Z}. \tag{12}$$

We want to show that $F_{2n-1}=c_{2n-1}{f_2}^{n-2}g_3$, where g_3 is a homogeneous polynomial of degree 3. We consider two different cases:

1. If F_{2n-1} is not divisible by f_2. Then, since the equation (12) when $f_2=0$ must be zero or a polynomial first integral of linear part system (6) and we already know that this last case is not possible, we have that F_{2n-1} restricted to $f_2=0$ must be zero. Since f_2 is irreducible, we have that $F_{2n-1}=f_2f_{2n-3}$ where f_{2n-3} is a homogeneous polynomial of degree $2n-3$ in the variables X,Y,Z, that is contradiction.

2. If F_{2n-1} is divisible by f_2. Then $F_{2n-1}={f_2}^mH$, where $H(X,Y,Z)$ is a homogeneous polynomial of degree $2n-1-2m$ and is not divisible by f_2 with $1\le m\le n-3$, then the partial differentia equation (12) becomes

$$\begin{aligned}&-\frac{kk_3}{k_4}Y\left(m{f_2}^{m-1}H\frac{\partial f_2}{\partial X}+{f_2}^m\frac{\partial H}{\partial X}\right)\\&+\left(k_4\ X+\left(k_3-k\right)Z\right)\left(m\ {f_2}^{m-1}H\frac{\partial f_2}{\partial Z}+{f_2}^m\frac{\partial H}{\partial Z}\right)\\&+\left(kZ-k_3\left(Y+Z\right)\right)\left(m{f_2}^{m-1}H\frac{\partial f_2}{\partial Y}+{f_2}^m\frac{\partial H}{\partial Y}\right)\\&=\left(n-1\right)c_{2n-2}k_2XY{f_2}^{n-2}\frac{\partial f_2}{\partial Z}.\end{aligned}$$

Since f_2 is first integral of linear part of system (6) and after simplification we obtain

$$\frac{-kk_3}{k_4}Y\frac{\partial H}{\partial X}+\left(-\left(Y+Z\right)k_3+kZ\right)\frac{\partial H}{\partial Y}+\left(k_4X+\left(k_3-k\right)Z\right)\frac{\partial H}{\partial Z}$$
$$=\left(n-1\right)c_{2n-2}k_2XYf_2{}^{n-m-2}\frac{\partial f_2}{\partial Z}.$$

Since $1 \leq m \leq n-3$, and the same arguments used in Case 1 imply a contradiction. Then must be $F_{2n-1} = f_2{}^{n-2}g_3$ where g_3 is homogeneous polynomial of degree three in the variables X, Y and Z. Again since f_2 is first integral of linear part of system (6) then equation (12) as

$$\begin{aligned}&\frac{-kk_3}{k_4}Y\frac{\partial g_3}{\partial X}+\left(-\left(Y+Z\right)k_3+kZ\right)\frac{\partial g_3}{\partial Y}+\left(k_4X+\left(k_3-k\right)Z\right)\frac{\partial g_3}{\partial Z}\\&\qquad=\left(n-1\right)c_{2n-2}k_2XY\frac{\partial f_2}{\partial Z},\end{aligned}\tag{13}$$

and solving it, we obtain

$$g_3 = (n-1)\frac{c_{2n-2}}{c_2}f_3.$$

Now computing the homogeneous polynomial F_{2n} of the terms of degree $2n$ in equation (10) satisfies differential equation

$$-\frac{kk_3}{k_4}Y\frac{\partial F_{2n}}{\partial X}+\left(-\left(Y+Z\right)k_3+kZ\right)\frac{\partial F_{2n}}{\partial Y}+\left(k_4X+\left(k_3-k\right)Z\right)\frac{\partial F_{2n}}{\partial Z}$$
$$=k_2XY\frac{\partial F_{2n-1}}{\partial Z},$$

that is

$$-\frac{kk_3}{k_4}Y\frac{\partial F_{2n}}{\partial X}+\left(-\left(Y+Z\right)k_3+kZ\right)\frac{\partial F_{2n}}{\partial Y}+\left(k_4X+\left(k_3-k\right)Z\right)\frac{\partial F_{2n}}{\partial Z}$$
$$=k_2c_{2n-1}XY\left(\left(n-2\right)f_2{}^{n-3}g_3\frac{\partial f_2}{\partial Z}+f_2{}^{n-2}\frac{\partial g_3}{\partial Z}\right).$$

By the same argument in computing F_{2n-1} we get that $F_{2n} = f_2{}^{n-3}g_6$, where g_6 is homogeneous polynomial of degree six in the variables X, Y and Z. Since f_2 is first integral of linear part system (6) and after simplification we can rewrite the above equation as

$$-\frac{kk_3}{k_4}Y\frac{\partial g_6}{\partial X}+(-(Y+Z)\,k_3+kZ)\,\frac{\partial g_6}{\partial Y}+(k_4X+(k_3-k)\,Z)\,\frac{\partial g_6}{\partial Z} \\ = k_2c_{2n-1}XY\,(n-2)\,g_3\frac{\partial f_2}{\partial Z}, \tag{14}$$

calculating the homogeneous polynomials g_3 and g_6 with maple from both equation (13) and (14) we get $c_{2n-2}=0$ and $g_3=0$, this gives as $F_{2n-1}=F_{2n-2}=0$. Then via Lemma 2 we obtain $F_{2n}=c_{2n}f_{2n}^n$, where c_{2n} is constant.

Finally, computing the homogeneous polynomial F_{2n+1} of degree $2n+1$ in equation (10) which satisfies

$$-\frac{kk_3}{k_4}Y\frac{\partial F_{2n+1}}{\partial X}+(-(Y+Z)\,k_3+kZ)\,\frac{\partial F_{2n+1}}{\partial Y}+(k_4X+(k_3-k)\,Z)\,\frac{\partial F_{2n+1}}{\partial Z} \\ = nc_{2n}k_2XY{f_2}^{n-1}\frac{\partial f_2}{\partial Z},$$

similar argument implies $F_{2n+1}={f_2}^{n-1}f_3$ where $f_3(X,Y,Z)$ is homogeneous polynomial of degree three. We have equation (11) is true for $n=N$. This completes the proof.

3. Since the parameters k_5, k_3 and k in system (1) are positive numbers then the coefficients of characteristic equation (7) are positive. Applying Routh Hurwitz criteria we conclude that equilibrium point E_1 is an attractor if and only if $k_5+k_3>k$. By using Theorem 12, then system (1) has no C^1-first integrals defined in a neighborhood of E_1. ■

Proof of Theorem 7. To apply Theorem 13, first we replace $y \leftrightarrow z$. Thus, system (1) becomes

$$\begin{aligned} \dot{x} &= kx-k_2xz, \\ \dot{y} &= k_4x-k_5y, \\ \dot{z} &= -k_3z+k_5y. \end{aligned}$$

Comparing the above system with system in Theorem 13, we obtain

$$P = kx,\ Q = -k_2 x,\ R = k_4 x - k_5 y,\ S = 0,\ T = k_5 y \text{ and } G = -k_3.$$

We take $c = (x_1, y_1) = (0, 0)$ then all conditions of part 1, Theorem 13 are satisfied, we directly conclude that the system (1) is not completely integrable with two functionally independent rational first integrals.
Since at the point $c = (0, 0)$, then $G(c) = -k_3 < 0$, $\triangle = -k_2 < 0$ and $\triangle_1(c) = 0 \notin \mathbb{N}$. So all conditions of part 2, Theorem 13 are satisfied. Thus, system (1) is not B-integrable in a class of rational functions in variables x, y, z and does not possess any rational first integral for all values of the parameters. ■

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