If the distance is very small (M) s are equal $M_1 = M_2 = M$

$M\Delta\emptyset = ds$

If the distance between P_1 and P_2 very small we can compute cord not arc

$X_1 - X_2 = -dX$

$Ds = dx/\sin\emptyset$ but $-dx$

$M.d\emptyset = -dx/\sin\emptyset$ $\emptyset_1 = \emptyset_2 = \emptyset$ the changes very small

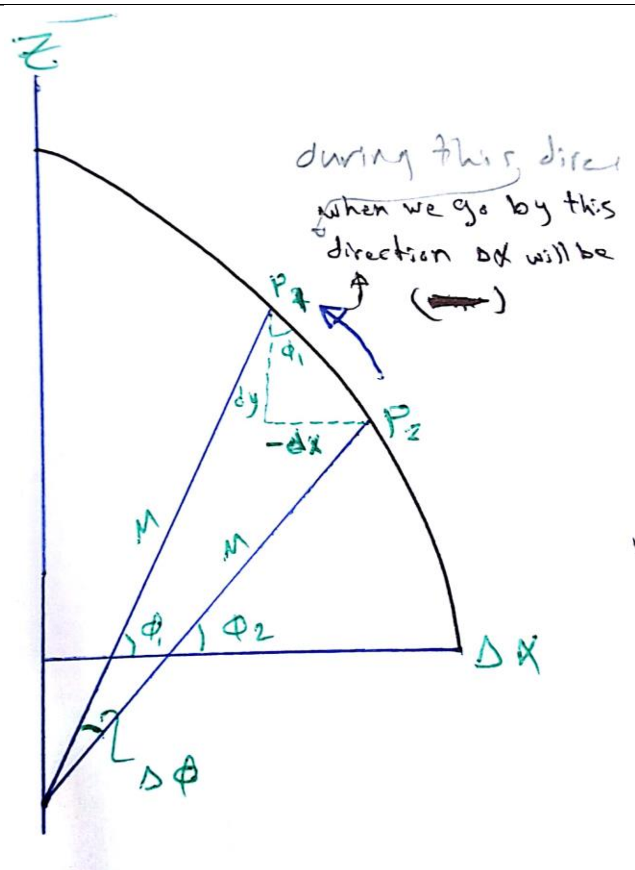

Figure 1: the radius of curvature of meridian

$$M = \frac{-1}{\sin\emptyset} * \frac{dx}{d\emptyset}$$

$$x = \frac{a * cos\emptyset}{(1 - e^2\, sin^2\, \emptyset)^{1/2}}$$

$$\frac{dx}{d\emptyset} = \frac{-a * sin\emptyset}{(1 - e^2\, sin^2\, \emptyset)^{3/2}}$$

$$M = \frac{-1}{sin\emptyset} * \frac{-a * \sin\emptyset * (1 - e^2)}{(1 - e^2\, sin^2\, \emptyset)^{3/2}}$$

$$M = \frac{\mathrm{a}(1 - e^2)}{(1 - e^2\, sin^2\, \emptyset)^{3/2}}$$

By another way to prove the radius of M

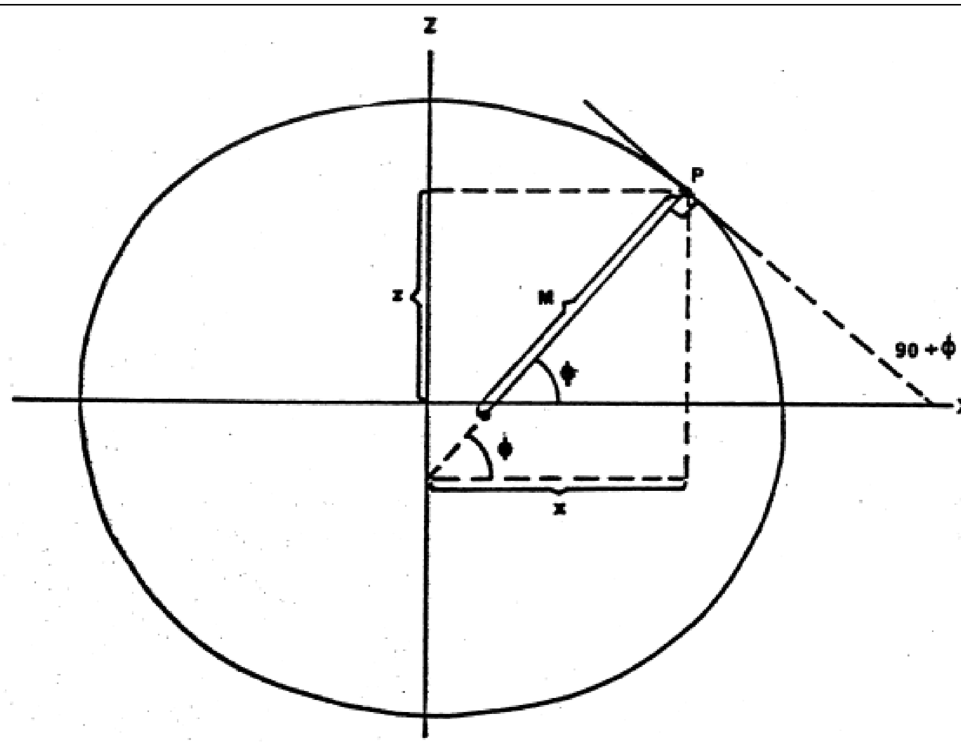

Figure 2: Meridian radius of curvature (M)

Form figure-18 - ΔOCB ≈ ΔODA →Z/b = S/a → $\frac{z^2}{b^2} = \frac{s^2}{a^2}$

But $x^2 + s^2 = a^2 \rightarrow 0 = \frac{z^2}{b^2} - \frac{a^2-x^2}{a^2}$

$$\frac{z^2}{b^2} - 1 + \frac{x^2}{a^2} = 0$$

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \ldots\ldots\ldots (1)$$

The radius of curvature of this curve, at any point P is given by

$$M = \frac{(1 + (dz/dx)^2)^{3/2}}{dz^2/dx^2} \ldots\ldots\ldots (2)$$

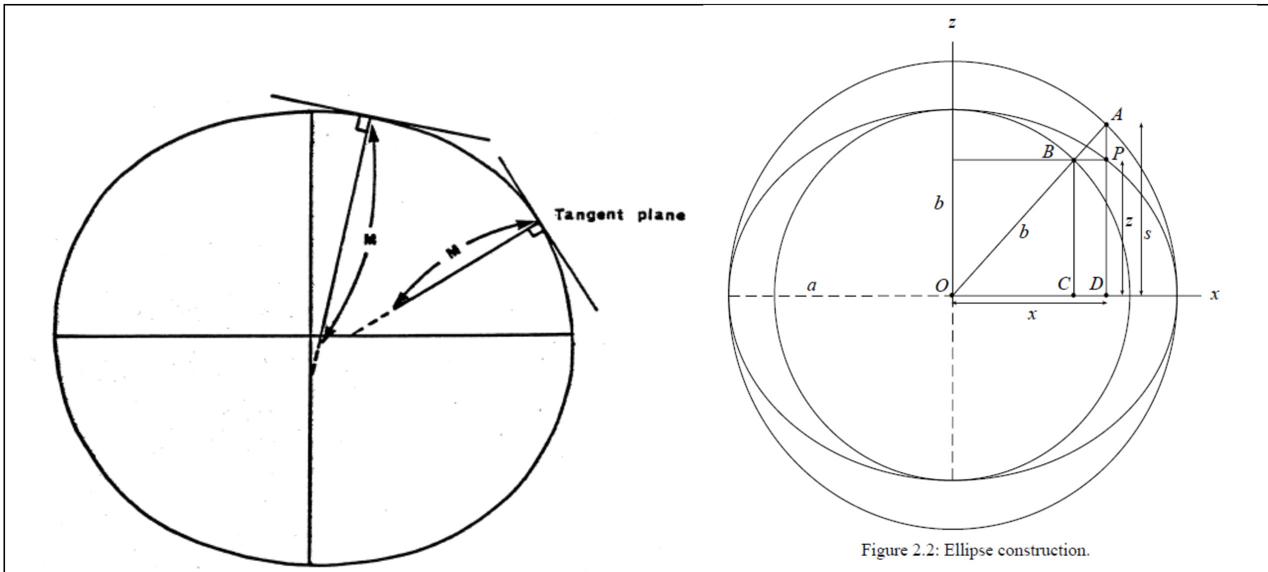

Figure 3: Meridian normal section

In the case of meridian ellipse

$$\frac{dz}{dx} = \frac{-x}{z} * \frac{b^2}{a^2} \ldots\ldots.. (3)$$

$$\frac{d^2z}{dx^2} = -\frac{b^2}{a^2} * \left(\frac{z - x\, dz/dx}{z^2}\right)$$

Put the value of dz/dx in the formula we get

$$\frac{d^2z}{dx^2} = -\frac{b^2}{a^2} * \left(z + \frac{x^2}{z} * \frac{b^2}{a^2}\right) \dots\dots\dots. (4)$$

From figure-, we can also see that the slope of the tangent to P is given by

$$\tan(90 + \emptyset) = \frac{dz}{dx} = -\cot\emptyset \dots\dots\dots (5)$$

Equating (3) and (5)

$$-\cot(\emptyset) = -\frac{x}{z} * \frac{b^2}{a^2} \quad or\ tan\emptyset = \frac{a^2}{b^2} * \frac{z}{x} \dots\dots\dots\dots (6)$$

Substituting

$$b = a(1 - e^2)^{1/2} \dots\dots\dots\dots. (7)$$

In equation (6) yields

$$z = x(1 - e^2)\tan\emptyset \dots\dots\dots\dots (8)$$

Then, after replacing b and z in equation 1 with 7 and 8 respectively, some simple manipulation results in

$$x = \frac{a * cos\emptyset}{(1 - e^2\, sin^2\,\emptyset)^{1/2}} \dots\dots\dots\dots.. (9)$$

Substituting the above expression for x in equation (8)

$$z = \frac{a(1 - e^2)\sin\emptyset}{(1 - e^2\, sin^2\,\emptyset)^{1/2}} \dots\dots\dots\dots (10)$$

Finally, replacing x and z in (3) and (4) and placing these values in (2) for (dz/dx) and ($\frac{d^2z}{dx^2}$), the expression for the meridian radius of curvatures

$$M = \frac{a(1 - e^2)}{(1 - e^2\, sin^2\,\emptyset)^{3/2}} \dots\dots\dots\dots\dots. (11)$$

In equation (11), the only variable parameters is the geodetic latitude (∅), thus

At the equator (∅ = 0°)

M = a (1-e^2)

At the pole (∅ = 90°)

$$M = a/(1-e^2)^{1/2}$$

Each latitude has the same (N) and different (M) but each longitude has the same (M) and different (N).

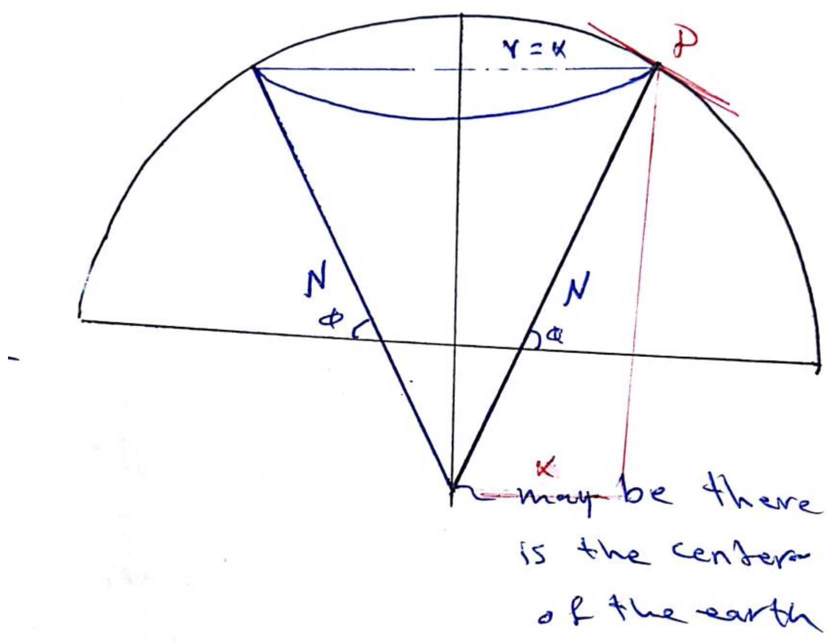

Figure 4: Radius of curvature in the prime vertical

$$N = \frac{a}{(1-e^2 \sin^2 \emptyset)^{1/2}}$$

$$r = x = N \cos \emptyset$$

$N = \frac{x}{\cos \emptyset}$ Substituting the value of (x) we get

$$N = \frac{\dfrac{a\cos\emptyset}{(1 - e^2\, sin^2\,\emptyset)^{1/2}}}{\cos\emptyset}$$

$$N = \frac{\mathrm{a}}{(1 - e^2\, sin^2\,\emptyset)^{1/2}}$$

At the equator (Ø = 0°)

N = a

At the pole (Ø = 90°)

$$N = M = a/(1 - e^2)^{1/2}$$

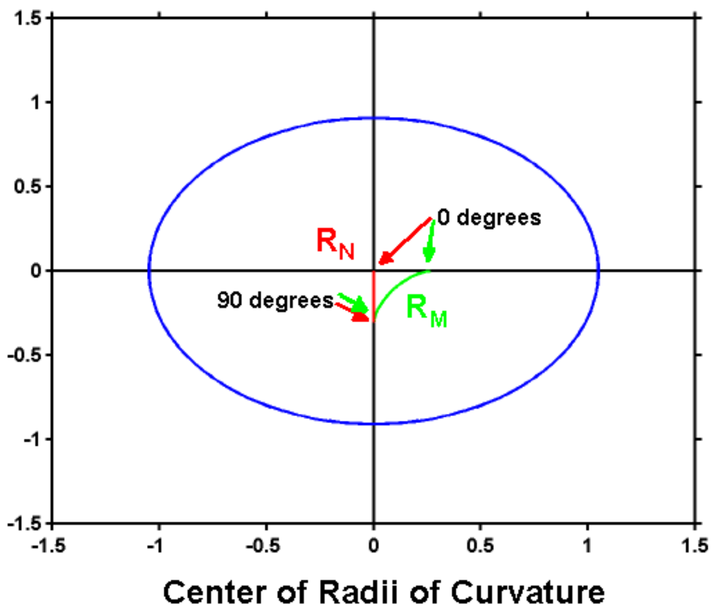

Figure 5: Center of radii of curvature at zero and ninety degree

An important quantity that is used very often in geometric geodetic computations is the (Gaussian Mean Radius of curvature), which is given by

$$R = \sqrt{MN}$$

$$R = \frac{\sqrt{(1-e^2)^2 \sin^2 \emptyset + \cos^2 \emptyset}}{\sqrt{1 - e^2 \sin^2 \emptyset}}$$

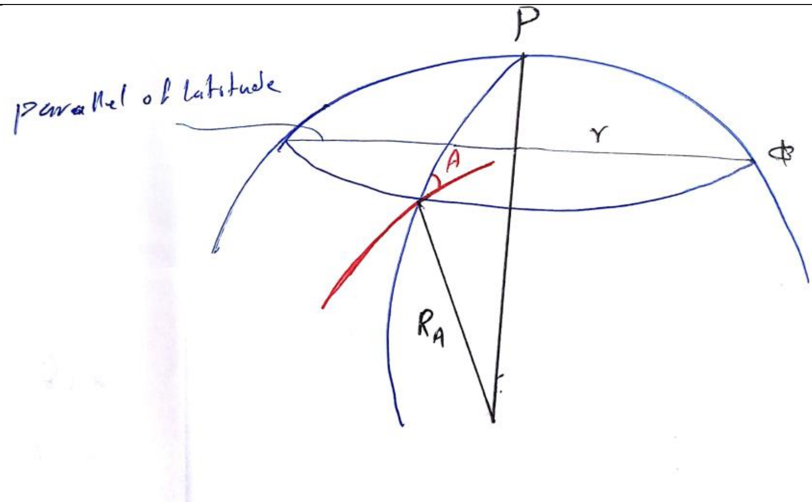

Figure 6: radius along both latitude and longitude

In many instances, the mean radius is sufficiently accurate for position computations.

Radius of curvature at azimuth [Oiler law]

$$\frac{1}{R_a} = \frac{\cos^2 a}{M} + \frac{\sin^2 a}{N} \quad Oiler\ law$$

$$R_A = \frac{N}{1 + e'^2 * \cos^2 \emptyset_m * \cos^2 A}$$

Another radius of curvature that may be needed from time to time is that of a **parallel of latitude**, any parallel of latitude viewed from the north pole of the ellipsoid (z axis) describes a circle. It is equal to x-coordinate then the radius of curvature of a parallel of latitude is given by

$R_\emptyset = N * \cos \emptyset$

It is easily seen that when $\emptyset = 0°$ (equator), $R_\emptyset = N$, thus $R_\emptyset = a$ (since N = a at $\emptyset = 0°$), and at either pole ($\emptyset = 90°$), $\cos \emptyset = 0$ and the radius disappears

And we have another radius in the field of geodesy

- **On the sphere**

$$R = \frac{2a + b}{3}$$

Why we have 2a but one b in the formula of radius on the sphere?

There are 2a and one b, because the shape of the earth has 3D, and rotate about b (z-axis) from left and right contain (a).

- **When surface area is equal**

$$A_{sphere} = 4\pi R^2$$

$$A_{ellipse} = ab^2\pi(1 + \frac{2}{3}e^2 + \frac{3}{5}e^4 + \cdots)$$

$$R = a(1 - \frac{e^2}{6} - \frac{17}{360}e^4 + \cdots)$$

- **In equal volume**

$$V_{sphere} = \frac{4}{3}\pi R^3$$

$$V_{ellipse} = \frac{4}{3}a^2 b\,\pi$$

$$R = \sqrt[3]{a^2 b}$$

Example-1

If point ‘P’ is at 40° 15’ 24” N and 76° 00’ 08” W, e^2 = 0.0066943800229 and a = 6378137 m, what is the radius of the curvature in the meridian of the point ‘P’?

$$M = \frac{a(1 - e^2)}{(1 - e^2\, sin^2\, \emptyset)^{3/2}} = \frac{6378137(1 - 0.0066943800229)}{(1 - 0.0066943800229 sin40°15'24"^2)^{3/2}}$$

M = 6362098.666 m

What is the radius of curvature in the prime vertical?

$N = \frac{\text{a}}{(1-e^2\, sin^2\, \emptyset)^{1/2}}$ = 6387070. 818 m

Example-2

What are the values M and N at the poles ($\emptyset = 90°$)?

$$M = \frac{\text{a}(1-e^2)}{(1-e^2\, sin^2\, \emptyset)^{3/2}} = \frac{a(1-e^2)}{(1-e*1)^{3/2}} = \frac{a}{(1-e)^{1/2}}$$

$$N = \frac{\text{a}}{(1-e^2\, sin^2\, \emptyset)^{1/2}} = \frac{\text{a}}{(1-e^2)^{1/2}}$$

$R_M = R_N$ = 6399593.627 m

Example-3

What are the values of M and N at the equator ($\emptyset = 0°$)?

$$M = \frac{\text{a}(1-e^2)}{(1-e^2\, sin^2\, \emptyset)^{3/2}} = \frac{a(1-e^2)}{(1-e*0)^{3/2}} = a(1-e^2) = 6335439.327\, m$$

$$N = \frac{\text{a}}{(1-e^2\, sin^2\, \emptyset)^{1/2}} = \frac{\text{a}}{(1-e^2*0)^{1/2}} = a = 6378137\, m$$

Note: the maximum difference between M and N at the equator.

Example-4

What is the length of the arc on the ellipsoid? Between (A = 40° 15’ 24” N, 76° 00’ 08”W) and (B = 40° 15’ 24” N, 77° 23’ 42” W)

N = 6387070.818 m (from the data of the first example).

$dE = N * \cos\emptyset * \Delta\lambda$

$\Delta\lambda = 77° \ 23' \ 42'' - 76° \ 00' \ 08'' = 1° \ 23' \ 34''$

$d = 6387070.818 * \cos(40° \ 15' \ 24'') * [(1° \ 23' \ 34'') * (\pi/180)]$

$d = 118488.167$ m