The Wiener and Hyper Wiener indices of Zero-divisor Graph of the Ring of Integers Modulo n

Research Project

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Certification of the Supervisor

I certify that this work was prepared under my supervision at the Department of Mathematics / College of Education / Salahaddin University- Erbil in partial fulfillment of the requirements for the degree of Bachelor of philosophy of Science in Mathematics.

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Date: / /2023

In view of available recommendations, I forward this word for debate by the examining committee.

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Acknowledgement

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Finally, I would like to state my thanks to all my teachers for helping me during difficult times.

Abstract

Let Z_n be the ring of integers modulo n. The purpose of this work is to study some zero-divisor graphs of Z_n. If n is a prime number, then Z_n has no zero-divisors; so $\Gamma(Z_n)$ is the null graph. Hence in this work, we only consider the case that n is a composite. We calculate the Wiener and Hyper Wiener indices of $\Gamma(Z_n)$. Note that all figures are drawn via website http://graphonline.ru/en/

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Introduction

Let Z_n be the ring of integers modulo n and Z(Z_n) the set of nonzero zero divisors of Z_n. The zero-divisor graph of Z_n, denoted by Γ(Z_n), is the simple graph with vertex set Z(Z_n), and for distinct a , b ∈ Z(Z_n), a and b are adjacent if and only if a b = 0. Clearly, Γ(Z_n) is the null graph if and only if Z_n is an integral domain.

The Wiener index for a graph Γ which is simple undirected graph defined by

W(Γ) =$\sum_{\{u,v\}\in(v(\Gamma)} d(u,v)$. The hyper Wiener index of Γ is defined by

WW(Γ) = $\frac{1}{2}$ W(Γ) + $\frac{1}{2}$ $\sum_{\{u,v\}\in(v(\Gamma)} (d(u,v))^2$ [10].

This thesis is divided into two chapters. The first chapter contains some basic definitions, which we need in the present work and some examples to illustrating the definitions.

The second chapter, includes the definition of Wiener index, hyper Wiener index and we find the zero divisors of the ring of integers modulo n, where n is composite, also we draw the graph of some rings and determined the distance between all vertices of this graphs. Finally, we calculate Wiener and hyper Wiener indices of each graphs of a ring of integers modulo n.

CHAPTER ONE

BACKGROUND MATERIALS

In this chapter some basic definitions that we need in our work and examples to illustrating this definition.

Definition 1.1: (Marlo Anderson and Todd Feil, 2015)

Group is a pair (G ,*) consisting of nonempty set G and binary operation * defund on G satisfying the following conditions

G_1 —closed under the operation *

G_2 —the operation * is associative on G

G_3—contains an identity element for the operation *, i.e there is an element e in G such that e*a=a=a*e for all a∈ G.

G_4 —for each a ∈ G the exist a^{-1} such that a^{-1}*a=e=a*a^{-1}

Example: $(Z, +)$, $(R, +)$, $(R - \{0\}, .\ )$ are groups.

Definition 1.2: (Marlo Anderson and Todd Feil, 2015)

A ring R is a nonempty set together with two binary operation + and .(called addition and multiplication defined on R) if satisfying the following axioms:

I. $(R, +)$ is an abelian group,

II. $(R, .\ )$ is semi-group,

III. the distributive law hold in R:

$for\ all\ a, b, c \in R, a.(b + c) = a . b + a. c$ and $(a + b). c = a. c + b$.

Example: (Z , ⊕ , ⊙) is a ring,

Definition1.3: (Marlo Anderson and Todd Feil, 2015)

A nonzero element a in a ring R is called a zero divisor if there exists $b \in R$ such that $b \neq 0$ and $ab = 0$. In particular, a is a left divisor of zero and b is a right divisor of zero .

Example: In the ring Z_6={0,1,2,3,4,5}

Since $2.3 = 6 = 0$ $3.4 = 12 = 0$, then 2,3,4 are zero divisors of Z_6

Definition1.4: (Burton, 1980) Prime numbers are numbers greater than 1, they only have two factors 1 and the numbers cannot be divide by any number other than 1.

Example: The number 2,3,5,7,11,13,17,… are prime numbers

Definition 1.5: (Behzad & CHartrand, 1979) A graphs is a finite non empty set of objects called vertices (the singular word is vertex) together with a (Possibly empty) set of un order pairs of distinct vertices of called edges.

Definition1.6: (Naduvath, 2017): The order of a graph G, denoted by V(G), is the number of its vertices and the size of G, denoted by E(G), is the number of its edges. A graph with p-vertices and q-edges is called a (p,q)-graph.

Definition1.7 : (Naduvath, 2017) An edge of a graph that joins a node to itself is called loop or a self-loop. That is, a loop is an edge uv, where u = v

Definition1.8: (Naduvath, 2017) The edges connecting the same pair of vertices are called multiple edges or parallel edges.

Definition 1.9: (Naduvath, 2017):A graph G which does not have loops or parallel edges is called a simple graph. A graph which is not simple is generally called a multigraph.

Definition 1.10: (Naduvath, 2017) A walk in a graph G is an alternating sequence of vertices and connecting edges in G. In other words, a walk is any route through a graph from vertex to vertex along edges.

Definition 1.11: (Naduvath, 2017) A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last.

Definition 1.12: (Naduvath, 2017) The distance between two vertices u and v in a graph G, denoted by d_G(u, v) or simply d(u, v), is the length (number of edges) of a shortest path connecting them.

CHAPTER TWO

This chapter includes the definition of Wiener index and hyper Wiener index of Γ we find the zero divisors of the ring of integers modulo n, where n is composite, also we draw the graph of some rings and determined the distance between each two vertices of this graphs. Finally we calculate the Wiener index and the hyper Wiener index of each graphs of the ring of integers modulo n, n from 6 to 30.

Definition 2.1: (Winer, 1947 The Wiener index for a graph Γ which is simple undirected graph defined by

$W(\Gamma) = \sum_{\{u,v\} \in (v(\Gamma)} d(u,v).$

Definition 2.2: (J. Klein, 1995The hyper Wiener index of Γ is defined by

$WW(\Gamma) = \frac{1}{2} W(\Gamma) + \frac{1}{2} \sum_{\{u,v\} \in (v(\Gamma)} (d(u,v))^2$ [10].

Use following rings

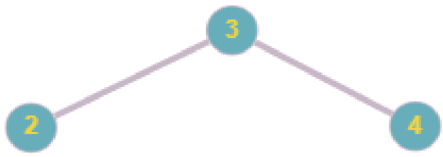
1) Z_6={0,1,2,3,4,5}

Since 2.3 = 6 = 0 3.4 = 12 = 0

The set of zero divisors of Z_6 is Z(Z_6) = { 2,3,4 } = V(Z_6), where V(Z_6) the set of vertex of the graph of Z_6.

The set of edges of Γ(Z_6) is E(Z_6)={(2,3),(3,4)}

The graph of the ring Z_6



The distance between any two vertices of $\Gamma(Z_6)$ are

d(4,3)=1 d(3,2)=1 d(4,2)=2

The Wiener index of the graph of Z_6

$W(\Gamma(Z_6)) = \sum_{\{u,v\}\in(v(\Gamma(Z_6))} d(u,v).$

$=1 + 2 + 1$

$= 4$

The hyper Wiener index of the graph of Z_6

$WW(\Gamma(Z_6)) = \frac{1}{2} W(\Gamma(Z_6)) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_6))} (d(u,v))^2$

$= \frac{1}{2}(4) + \frac{1}{2}[(1)^2 + (2)^2 + (1)^2]$

$=2+\frac{1}{2}[6]$

$= 2+3$

$=5$

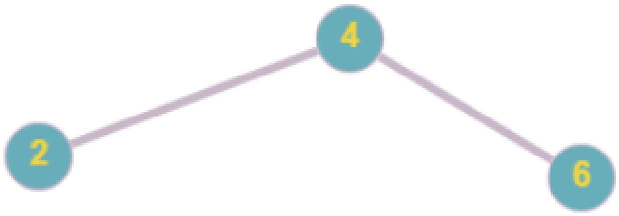
2) Z_8={0,1,2,3,4,5,6,7}

Since 2.4=8=0 and 6.4=24=0

The set of zero divisors of Z_8 is Z(Z_8) = { 2,4,6 } = V(Z_8), where V(Z_8) the set of vertex of the graph of Z_8.

The set of edges of Z_8 is E(Z_8)={(2,4),(4,6)}

The graph of the ring Z_8



The distance between any two vertices of $\Gamma(Z_8)$

$d(2,4) = 1$ $d(2,6) = 2$ $d(4,6) = 1$

The Wiener index of the graph of Z_8

$W(\Gamma(Z_8)) = \sum_{\{u,v\}\in(v(\Gamma(Z_8))} d(u,v).$

$= 1 + 2 + 1$

$= 4.$

The hyper Wiener index of the graph of Z_8

$WW(\Gamma(Z_8)) = \frac{1}{2} W(\Gamma(Z_8) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_8))} (d(u,v))^2$

$= \frac{1}{2}(4) + \frac{1}{2}[(1)^2 + (2)^2 + (1)^2]$

$=2+\frac{1}{2}[6]$

$= 2+3=5$

3) Z_9={0,1,2,3,4,5,6,7,8}

Since 3.6=18=0

The set of zero divisors of Z_9 is Z(Z_9) = { 3,6 } = V(Z_9), where V(Z_9) the set of vertex of the graph of Z_9.

The set of edges of Z_9 is E(Z_9)={(3,6)}

The graph of the ring Z_9

The distance between any two vertices of $\Gamma(Z_9)$ is d(3,6)=1

The Wiener index of the graph of Z_9

$W(\Gamma(Z_9)) = \sum_{\{u,v\}\in(v(\Gamma(Z_9))} d(u,v).$

$=1$

The hyper Wiener index of the graph of Z_9

$WW(\Gamma(Z_9)) = \frac{1}{2} W(\Gamma(Z_9) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_9))} (d(u,v))^2$

$= \frac{1}{2}(1) + \frac{1}{2}[(1)^2]$

$=1$

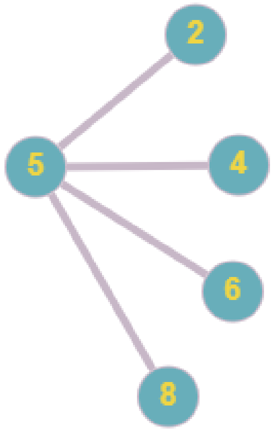
4) Z_{10}={0,1,2,3,4,5,6,7,8,9}

$$Since\ 2.5 = 10 = 0, \quad 4.5 = 20 = 0, \quad 6.5 = 30 = 0,$$
$$8.5 = 40 = 0$$

The set of zero divisors of Z_{10} is Z(Z_{10}) = { 2,4,5,6,8 } = V(Z_{10}), where V(Z_{10}) the set of vertex of the graph of Z_{10}.

The set of edges of Z_{10} is E(Z_{10})={(2,5),(4,5),(6,5),(8,5)}3,6)}

The graph of the ring Z_{10}



The distance between any two vertices of $\Gamma(Z_{10})$ are

$$d(2,4) = d(2,6) = d(2,8) = d(4,6) = d(4,8) = d(6,8) = 2$$
$$d(5,2) = d(5,4) = d(5,6) = d(5,8) = 1$$

The Wiener index of the graph of Z_{10}

W($\Gamma(Z_{10})$) =$\sum_{\{u,v\}\in(v(\Gamma(Z_{10}))} d(u,v)$.

$$= 2 + 2 + 2 + 2 + 2 + 2 + 1 + 1 + 1 + 1$$

= 6(2) + 4(1)

= 16

The hyper Wiener index of the graph of Z_{10}

$$WW(\Gamma(Z_{10})) = \frac{1}{2}W(\Gamma(Z_{10}) + \frac{1}{2}\sum_{\{u,v\}\in(v(\Gamma(Z_{10}))}(d(u,v))^2$$

$$=\frac{1}{2}(16) + \frac{1}{2}[\,6(2)^2 + 4(1)^2]$$

=8+14

=22

5) Z_{12}={0,1,2,3,4,5,6,7,8,9,10,11}

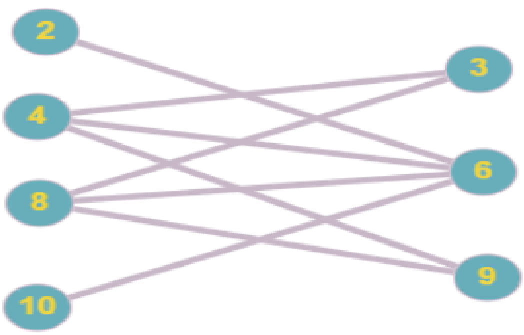
Since 2.6=12=0 3.4=12=0 8.3=24=0 4.6=24=0

9.4=36=0 6.10=60=0 6.8=48=0 8.9=72=0

The set of zero divisors of Z_{12} is Z(Z_{12}) = {2,3,4,6,8,9,10}= V(Z_{12}), where V(Z_{12}) the set of vertex of the graph of Z_{12}.

The set of edges of Z_{12} is E(Z_{12})={(2,6),(3,4),(3,8),(4,9),(6,8),(6,10),(4,6),(8,9)}

The graph of the ring Z_{12}



The distance between any two vertices of $\Gamma(Z_{12})$ are

d(2,3)= 3 d(2,4)= 2 d(2,6)= 1 d(2,8)= 2 d(2,9)= 3 d(2,10)= 2 d(3,4)=1
d(3,6)= 2 d(3,8)= 1 d(3,9)= 2 d(3,10)= 3 d(4,6)= 1 d(4,8)= 2 d(4,9)= 1
d(4,10)= 2 d(6,8)= 1 d(6,9)= 2 d(6,10)= 1 d(8,9)= 1 d(8,10)= 2 d(9,10)= 3

The Wiener index of the graph of Z_{12}

$$W(\Gamma(Z_{12})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{12}))} d(u,v).$$

$$= 4(3) + 9(2) + 8(1)$$

$$= 12 + 18 + 8$$

$$= 38$$

The hyper Wiener index of the graph of Z_{12}

$$WW(\Gamma(Z_{12})) = \frac{1}{2} W(\Gamma(Z_{12}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{12}))} (d(u,v))^2$$

$$= \frac{1}{2}(38) + \frac{1}{2}[4(3)^2 + 9(2)^2 + 8(1)^2]$$

$$=19+\frac{1}{2}[36 + 36 + 8]$$

$$=19+40$$

$$=59$$

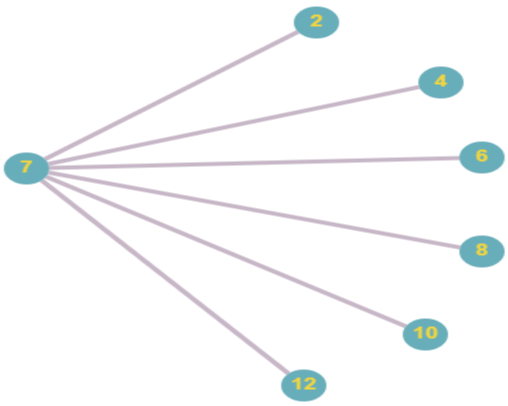
6) Z_{14}={0,1,2,3,4,5,6,7,8,9,10,11,12,13}

Since 2.7=14=0 4.7=28=0 6.7=42=0 8.7=56=0 10.7=70=0
12.7=84=0

The set of zero divisors of Z_{14} is Z(Z_{14}) = {2,4,6,7,8,10,12}= V(Z_{14}), where V(Z_{14}) the set of vertex of the graph of Z_{14}.

The set of edges of Z_{14} is E(Z_{14}) ={(2,7),(4,7),(6,7),(8,7),(10,7),(12,7)}

The graph of the ring Z_{14}



The distance between any two vertices of $\Gamma(Z_{14})$ are

d(2,7)= d(4,7)= d(6,7)= d(8,7)= d(10,7)= d(12,7)= 1

d(2,4)= d(2,6)= d(2,8)= d(2,10)= d(2,12)= d(4,6)= d(4,8)= d(4,10)

= d(4,12)= d(6,8)= d(6,10)= d(6,12)= d(8,10)= d(8,12)= d(10,12)= 2

The Wiener index of the graph of Z_{14}

$W(\Gamma(Z_{14})) = \sum_{\{u,v\} \in (v(\Gamma(Z_{14}))} d(u,v).$

= 6(1) + 15(2)

= 6+30

= 36

The hyper Wiener index of the graph of Z_{14}

$WW(\Gamma(Z_{14})) = \frac{1}{2} W(\Gamma(Z_{14}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{14}))} (d(u,v))^2$

$= \frac{1}{2}(36) + \frac{1}{2}[6(1)^2 + 15(2)^2]$

$=18+\frac{1}{2}[6 + 60]$

$=18+33$

$=51=(7)^2 + 3$

7) Z_{15}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14}

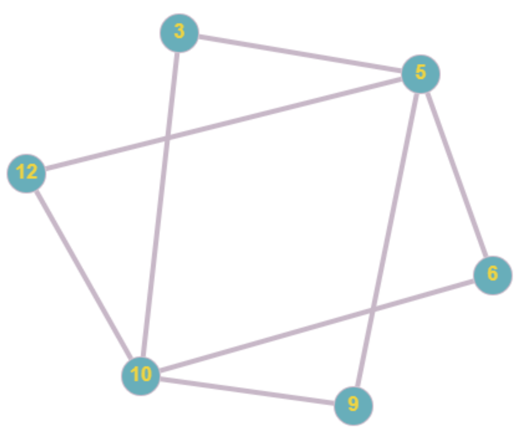
Since 3.5=15=0   6.5=30=0   10.6=60=0   12.5=60=0

9.5=45=0   10.12=120=0   9.10=90=0   3.10=30=0

The set of zero divisors of Z_{15} is Z(Z_{15}) = {3,5,6,9,10,12}= V(Z_{15}), where V(Z_{15}) the set of vertex of the graph of Z_{15}.

The set of edges of Z_{15} is E(Z_{15}) ={(3,5),(3,10),(5,12),(5,9),(5,6),(6,10),(9,10),(10,12)}

The graph of the ring Z_{15}



The distance between any two vertices of $\Gamma(Z_{15})$ are

d(3,5) = d(3,10) = d(5,6) = d(10,12) = d(9,10) = d(5,9)= d(5,12)= d(6,10)= 1

d(3,6) = d(3,9) = d(3,12) = d(5,10) = d(6,9) =(9,12) = d(6,12) =2

The Wiener index of the graph of Z_{15}

$$W(\Gamma(Z_{15})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{15}))} d(u,v).$$

$$= 8(1) + 7(2)$$

$$= 8+14$$

$$= 22$$

The hyper Wiener index of the graph of Z_{15}

$$WW(\Gamma(Z_{15})) = \frac{1}{2} W(\Gamma(Z_{15}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{15}))} (d(u,v))^2$$

$$= \frac{1}{2}(22) + \frac{1}{2}[8(1)^2 + 7(2)^2]$$

$$= 11+\frac{1}{2}[8 + 28]$$

$$= 11+18$$

$$= 29$$

8) Z_{16}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}

Since 2.8=16=0 4.8=24=0 6.8=48=0

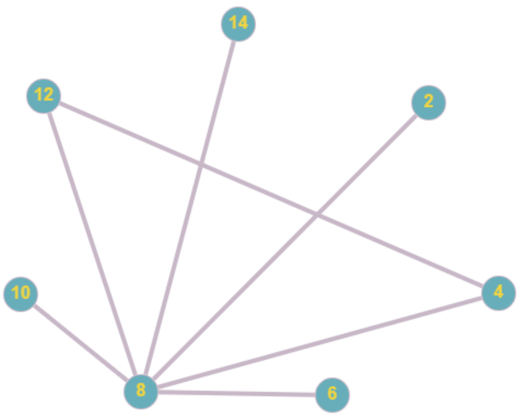
10.8=80=0 12.4=48=0 8.12=96=0 14.8=112=0

The set of zero divisors of Z_{16} is Z(Z_{16}) = {2,4,6,8,10,12,14}

= V(Z_{16}), where V(Z_{16}) the set of vertex of the graph of Z_{16}.

The set of edges of Z_{16} is $E(Z_{16})$ ={(2,8),(4,8),(8,6),(8,10),(8,14),(8,12),(12,4)}

The graph of the ring Z_{16}



The distance between any two vertices of $\Gamma(Z_{16})$ are

d(2,8)= d(8,4)=d(4,12)=d(8,6)= d(8,10)=d(8,12)=d(8,14)= 1

d(2,14)= d(2,10)= d(2,6)= d(2,12)= d(2,4)= d(14,10)= d(14,6)=d(14,12)=d(14,4)=d(10,6)=d(10,12)=d(10,4)=d(6,12)=d(6,4)= 2

The Wiener index of the graph of Z_{16}

$W(\Gamma(Z_{16})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{16}))} d(u,v).$

= 7(1) + 14(2)

= 7+28

= 35

The hyper Wiener index of the graph of Z_{16}

$WW(\Gamma(Z_{16})) = \frac{1}{2} W(\Gamma(Z_{16}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{16}))} (d(u,v))^2$

$= \frac{1}{2}(35) + \frac{1}{2}[\,7(1)^2 + 14(2)^2]$

$= \frac{1}{2}[35 + 63]$

$=\frac{1}{2}[98]$

=49

9) Z_{18}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17}

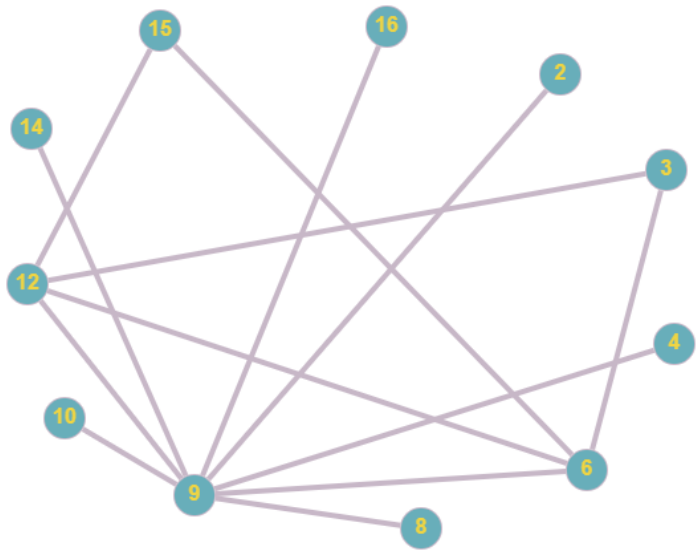
Since 2.9=18=0 3.6=18=0 12.3=36=0 4.9=36=0 6.9=54=0
6.12=72=0 6.15=90=0 9.10=90=0 8.9=72=0 9.12=108=0
9.14=126=0 9.16=144=0 12.15=180=0

The set of zero divisors of Z_{18} is Z(Z_{18}) = {2,3,4,6,8,9,10,12,14,15,16}

= V(Z_{18}), where V(Z_{18}) the set of vertex of the graph of Z_{18}.

The set of edges of Z_{18} is E(Z_{18})
={(2,9),(3,6),(4,9),(9,6),(6,12),(6,15),(9,10),(9,12),(9,14),(12,3),(8,9),(15,10),(16,9),(12,15)}

The graph of the ring Z_{18}



The distance between any two vertices of $\Gamma(Z_{18})$ are

d(2,9)=1 d(2,3)=3 d(2,4)=2 d(2,6)=2 d(2,8)=2 d(2,10)=2 d(2,12)=2

d(2,14)=2 d(2,15)=3 d(2,16)=2 d(3,9)=2 d(3,4)=3 d(3,6)=1 d(3,8)=3

d(3,10)=3 d(3,12)=1 d(3,14)=3 d(3,15)=2 d(3,16)=3 d(4,9)=1 d(4,6)=2 d(4,8)=2 d(4,10)=2 d(4,12)=2 d(4,14)=2 d(4,16)=2 d(6,9)=1 d(6,8)=2 d(6,10)=2 d(6,12)=1 d(6,14)=2 d(6,15)=1 d(6,16)=2 d(8,9)=1 d(8,10)=2 d(8,12)=2 d(8,14)=2 d(8,15)=3 d(8,16)=2 d(10,9)=1 d(10,12)=2 d(10,14)=2 d(10,15)=3 d(10,16)=2 d(12,9)=1 d(12,14)=2 d(12,15)=1 d(12,16)=2 d(14,9)=1 d(14,15)=3 d(14,16)=2 d(15,16)=3 d(15,9)=2 d(16,9)=1

The Wiener index of the graph of Z_{18}

W($\Gamma(Z_{18})$) =$\sum_{\{u,v\}\in(v(\Gamma(Z_{18}))} d(u,v)$.

= 13(1) + 30(2) + 11(3)

= 13 + 60 + 33

= 106

The hyper Wiener index of the graph of Z_{18}

$$WW(\Gamma(Z_{18})) = \frac{1}{2} W(\Gamma(Z_{18}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{18}))} (d(u,v))^2$$

$$= \frac{1}{2}(106) + \frac{1}{2}[13(1)^2 + 30(2)^2 + 11(3)^2]$$

$$=53+\frac{1}{2}[13 + 120 + 99]$$

$$=53 + \frac{1}{2}[232]$$

=53+116=169

10) Z_{20}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19}

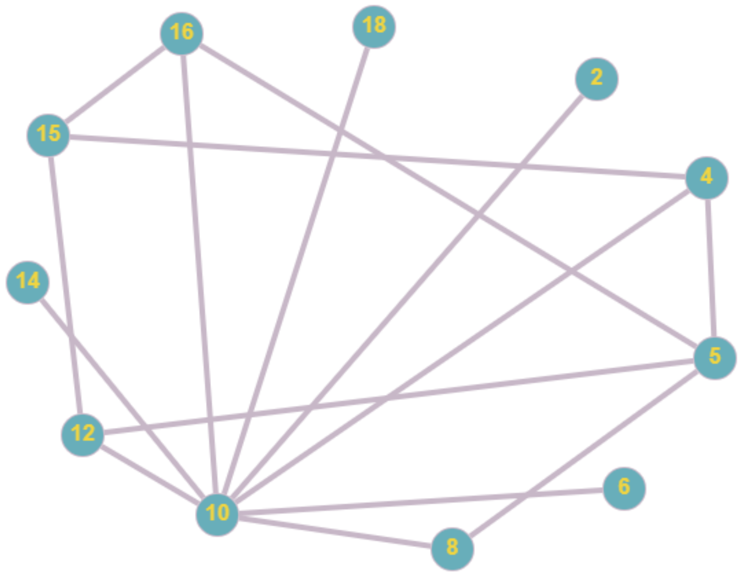
Since 2.10=20=0 4.5=20=0 4.10=40=0 4.15=60=0 10.14=140=0
10.16=160=0 10.28=180=0 5.8=40=0 5.16=80=0 6.10=160=0
12.15=180=0 16.15=240=0 8.10=80=0 8.15=120=0 10.12=120=0

The set of zero divisors of Z_{20} is Z(Z_{20}) {2,4,5,6,8,10,12,14,15,16,18}

= V(Z_{20}), where V(Z_{20}) the set of vertex of the graph of Z_{20}.

The set of edges of Z_{20} is E(Z_{20})
={(2,10),(4,5),(4,10),(4,15),(10,14),(10,16),(10,18),(5,8),(5,12),(5,16),(6,10),(12,15),(16,15),(8,10),(8,15),(10,12)}

The graph of the ring Z_{20}



The distance between any two vertices of $\Gamma(Z_{20})$ are

d(2,4)=2 d(2,5)=3 d(2,6)=2 d(2,8)=2 d(2,12)=2 d(2,14)=2 d(2,15)=3 d(2,16)=2 d(2,18)=2 d(4,5)=1 d(4,6)=2 d(4,8)=2 d(4,12)=2 d(4,14)=2 d(4,15)=1 d(4,16)=2 d(4,18)=2 d(4,10)=1 d(5,6)=3 d(5,8)=1 d(5,10)=2 d(5,12)=1 d(5,14)=3 d(5,15)=2 d(5,16)=1 d(5,18)=3 d(6,8)=2 d(6,10)=1 d(6,12)=2 d(6,14)=2 d(6,15)=2 d(6,16)=2 d(6,18)=2 d(8,10)=1 d(8,12)=2 d(8,14)=2 d(8,15)=2 d(8,16)=2 d(8,18)=2 d(10,12)=1 d(10,14)=1 d(10,15)=1 d(10,16)=1 d(10,18)=1 d(12,14)=2 d(12,15)=2 d(12,16)=2 d(12,18)=2 d(14,15)=2 d(14,16)=2 d(14,18)=2 d(15,16)=1 d(15,18)=2 d(16,18)=2

The Wiener index of the graph of Z_{20}

$W(\Gamma(Z_{20})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{20}))} d(u,v).$

= 14(1) + 35(2) + 5(3)

= 14 + 70 + 15

= 99

The hyper Wiener index of the graph of Z_{20}

$$WW(\Gamma(Z_{20})) = \frac{1}{2} W(\Gamma(Z_{20}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{20}))} (d(u,v))^2$$

$$= \frac{1}{2}(99) + \frac{1}{2}[\,14(1)^2 + 35(2)^2 + 5(3)^2]$$

$$= \frac{1}{2}[99 + 14 + 140 + 45] = \frac{1}{2}[99 + 14 + 140 + 45]$$

$$= \frac{1}{2}[298]$$

$$=149$$

11) Z_{21}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20}

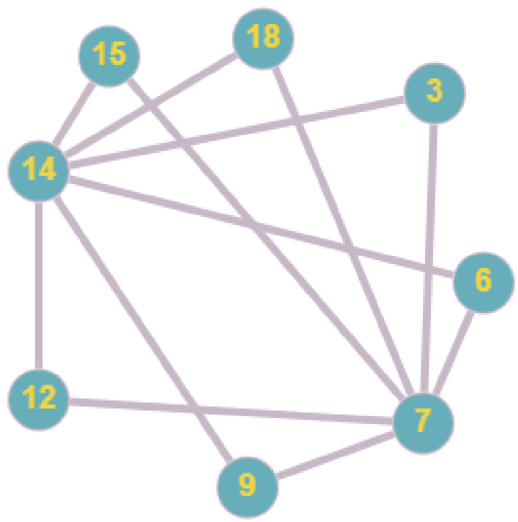
Since 3.7=21=0 3.14=42=0 6.7=42=0 6.14=84=07.9=63=0 7.12=84=0 7.15=105=0 14.18=252=0 7.18=126=0 9.14=126=0 12.14=168=0 14.15=210=0

The set of zero divisors of Z_{21} is Z(Z_{21})= {3,6,7,9,12,14,15,18}

= V(Z_{21}), where V(Z_{21}) the set of vertex of the graph of Z_{21}.

The set of edges of Z_{21} is E(Z_{21})={(3,7),(3,14),(6,7),(6,14),(7,9),(7,12),(7,15),(14,18),(7,18),(9,14),(12,14),(14,15)}

The graph of the ring Z_{21}



The distance between any two vertices of $\Gamma(Z_{21})$ are

d(3,6)=2 d(3,7)=1 d(3,9)=2 d(3,12)=2 d(3,14)=1 d(3,15)=2
d(3,18)=2 d(6,7)=1 d(6,9)=2 d(6,12)=2 d(6,14)=1 d(6,15)=2
d(6,18)=2 d(7,9)=1 d(7,12)=1 d(7,14)=2 d(7,15)=1 d(7,18)=1
d(9,12)=2 d(9,14)=1 d(9,15)=2 d(9,18)=2 d(12,14)=1 d(12,15)=2
d(12,18)=2 d(14,15)=1 d(14,18)=1 d(15,18)=2

The Wiener index of the graph of Z_{21}

$$W(\Gamma(Z_{21})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{21}))} d(u,v).$$

$$= 12(1) + 16(2)$$

$$= 12 + 32$$

$$= 44$$

The hyper Wiener index of the graph of Z_{21}

$$WW(\Gamma(Z_{21})) = \frac{1}{2} W(\Gamma(Z_{21}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{21}))} (d(u,v))^2$$

$$= \frac{1}{2}(44) + \frac{1}{2}[\,12(1)^2 + 16(2)^2]$$

$= \frac{1}{2}[44 + 12 + 64]$

$= \frac{1}{2}[120]$

=60

12) Z_{22}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21}

Since 2.11=22=0 4.11=44=0 6.11=66=0 8.11=88=0 1O.11=110=0

12.11=132=0 14.11=154=0 16.11=176=0 18.11=198=0 20.11=220=0

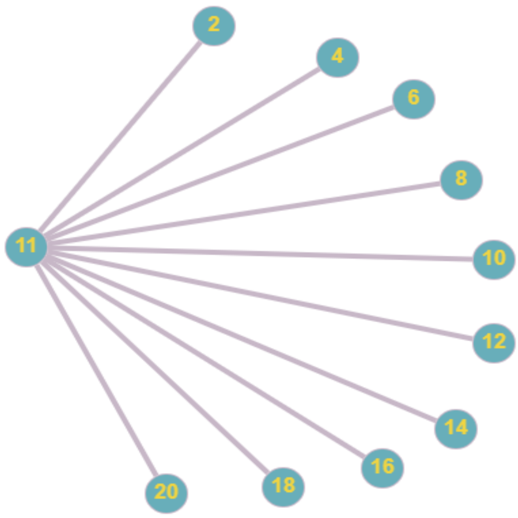
The set of zero divisors of Z_{22} is Z(Z_{22})= ={2,4,6,8,10,11,12,14,16,18,20}

= V(Z_{22}), where V(Z_{22}) the set of vertex of the graph of Z_{22}.

The set of edges of Z_{22} is

E(Z_{22})={(2,11),(4,11),(6,11),(8,11),(10,11),(12,11),(14,11),(16,11),(18,11),(20,11)
}

The graph of the ring Z_{22}



The distance between any two vertices of $\Gamma(Z_{22})$ are

d(2,4)=2 d(2,6)=2 d(2,8)=2 d(2,10)=2 d(2,11)=1 d(2,12)=2 d(2,14)=2 d(2,16)=2 d(2,18)=2 d(2,20)=2 d(4,6)=2 d(4,8)=2 d(4,10)=2 d(4,11)=1 d(4,12)=2 d(4,14)=2 d(4,16)=2 d(4,18)=2 d(4,20)=2 d(6,8)=2 d(6,10)=2 d(6,11)=1 d(6,12)=2 d(6,14)=2 d(6,16)=2 d(6,18)=2 d(6,20)=2 d(8,10)=2 d(8,11)=1 d(8,12)=2 d(8,14)=2 d(8,16)=2 d(8,18)=2 d(8,20)=2 d(10,11)=1 d(10,12)=2 d(10,14)=2 d(10,16)=2 d(10,18)=2 d(10,20)=2 d(12,11)=1 d(12,14)=2 d(12,16)=2 d(12,18)=2 d(12,20)=2 d(14,11)=1 d(14,16)=2 d(14,18)=2 d(14,20)=2 d(16,11)=1 d(16,18)=2 d(16,20)=2 d(18,11)=1 d(18,20)=2 d(20,11)=1

The Wiener index of the graph of Z_{22}

W($\Gamma(Z_{22})$) =$\sum_{\{u,v\}\in(v(\Gamma(Z_{22}))} d(u,v)$.

= 10(1) + 45(2)

= 10 + 90

= 100

The hyper Wiener index of the graph of Z_{22}

$$WW(\Gamma(Z_{22})) = \frac{1}{2} W(\Gamma(Z_{22}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{22}))} (d(u,v))^2$$

$$= \frac{1}{2}(100) + \frac{1}{2}[\,10(1)^2 + 45(2)^2]$$

$$= \frac{1}{2}[100 + 10 + 180]$$

$$= \frac{1}{2}[290]$$

=95

13) Z_{24}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23}

Since 2.12=24=0 3.8=24=0 4.6=24=0 4.12=48=0 4.18=72=0 6.8=48=0

6.12=72=0 6.16=0 6.20=120=0 6.24=144=0 12.20=240=0

12.22=264=0 6.12=72=0 6.20=120=0 6.24=144=0 8.9=72=0

8.12=96=0 8.15=120=0 15.16=240=0 16.18=288=0 16.21=336=0

8.21=168=0 9.16=144=0 10.12=120=0 12.14=168=0 12.16=192=0

12.18=216=0 18.20=360=0

The set of zero divisors of Z_{24} is

Z(Z_{24})={2,3,4,6,8,9,10,12,14,15,16,18,20,21,22}

= V(Z_{24}), where V(Z_{24}) the set of vertex of the graph of Z_{24}.

The set of edges of Z_{24} is

E(Z_{24})={(2,12),(3,8),(4,6),(4,12),(4,18),(8,6),(12,20),(12,22),(6,12),(6,20),(8,9),(8,12),(8.15),(16,15),(16,18),(16,21),(8,21),(9,16),(12,10),(12,14),(12,16),(18,12),(18,20)}

The graph of the ring Z_{24}

The distance between any two vertices of $\Gamma(Z_{24})$ are

d(2,3)=3	*d(3,4)=3*	*d(4,8)=2*	*d(6,12)=1*
d2,4)=2	*d(3,6)=2*	*d(4,9)=2*	*d(6,14)=2*
d(2,6)=2	*d(3,8)=1*	*d(4,10)=2*	*d(6,15)=2*
d(2,8)=2	*d(3,9)=2*	*d(4,12)=1*	*d(6,16)=2*
d(2,9)=3	*d(3,10)=3*	*d(4,14)=2*	*d(6,18)=2*
d(2,10)=2	*d)3,12)=2*	*d(4,15)=3*	*d(6,20)=1*
d(2,12)=1	*d(3,14)=3*	*d(4,16)=2*	*d(6,21)=2*
d(2,14)=2	*d(3,15)=2*	*d(4,18)=1*	*d(6,22)=2*
d(2,15)=3	*d(2,16)=3*	*d(4,20)=2*	*d(8,9)=1*
d(2,16)=2	*d(3,18)=3*	*d(4,21)=2*	*d(8,10)=2*
d(2,18)=2	*d(3,20)=3*	*d(4,22)=2*	*d(8,12)=1*
d(2,20)=2.	*d(3,21)=3*	*d(6,8)=1*	*d(8,14)=2*
d(2,21)=2	*d(3,22)=3*	*d(6,9)=2*	*d(8,15)=1*
d(2,22)=2	*d(4,6)=1*	*d(8,16)=2*	*d(8,20)=2*
d(8,21)=1			

d(4,6)=1	d(4,8)=2	d(4,9)=1	d(4,10)=2	d(4,12)=1	d(2,14)=2
d(4,15)=2	d(4,16)=2	d(4,18)=1	d(4,20)=2	d(4,21)=2	d(4,22)=2
d(6,8)=1	d(6,9)=2	d(6,10)=2	d(6,12)=1	d(6,14)=2	d(6,15)=2
d(6,18)=2	d(6,20)=1	d(6,21)=2	d(6,22)=2	d(4,16)=1	d(8,9)=1
d(8,10)=2	d(8,12)=1	d(8,14)=2	d(8,15)=1	d(8,16)=2	d(8,18)=1
d(8,20)=2	d(8,21)=1	d(8,22)=2	d(9,10)=3	d(9,12)=2	d(9,14)=3
d(9,15)=2	d(9,16)=1	d(9,18)=2	d(9,20)=3	d(9,21)=2	d(9,22)=3
d(10,12)=1	d(10,14)=2	d(10,15)=3	d(10,16)=2	d(10,18)=2	d(10,20)=2
d(10,21)=3	d(10,22)=2	d(12,14)=1	d(12,15)=2	d(12,16)=1	d(12,18)=1
d(12,20)=1	d(12,21)=2	d(12,22)=1	d(14,15)=3	d(14,16)=2	d(14,18)=2
d(14,20)=2	d(14,21)=3	d(14,22)=2	d(15,16)=1	d(15,18)=2	d(15,20)=3
d(15,21)=2	d(15,22)=3	d(16,18)=1	d(16,20)=2	d(16,21)=1	d(16,22)=2
d(18,20)=1	d(18,21)=2	d(18,22)=2	d(20,21)=3	d(20,22)=3	d(20,21)=3

The Wiener index of the graph of Z_{24}

$$W(\Gamma(Z_{24})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{24}))} d(u,v).$$

$$= 29(1) + 62(2) + 25(3)$$

$$= 29 + 124 + 75$$

$$= 228$$

The hyper Wiener index of the graph of Z_{24}

$$WW(\Gamma(Z_{24})) = \frac{1}{2} W(\Gamma(Z_{24}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{24}))} (d(u,v))^2$$

$= \frac{1}{2}(228) + \frac{1}{2}[\ 29(1)^2 + 62((2)^2) + 25(3)^2]$

$= \frac{1}{2}[228 + 29 + 228 + 225]$

$= \frac{1}{2}[710]$

=355

14) Z_{25}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24}

Since 10.5=50=0 5.15=75=0 20.5=100=0

10.15=150=0 10.20=200=0 15.20=300=0

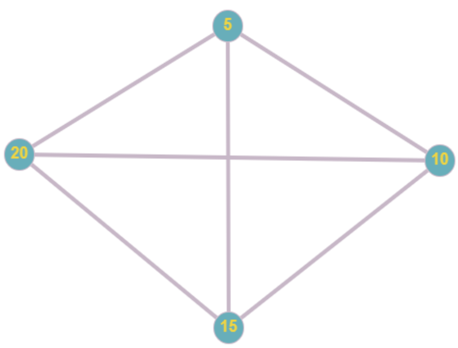
The set of zero divisors of Z_{25} is Z(Z_{25})= {5,10,15,20}

= V(Z_{25}), where V(Z_{25}) the set of vertex of the graph of Z_{25}.

The set of edges of Z_{25} is

E(Z_{25})={(5,10)(5,15),(5,20),(10,15),(10,20),(15,20)}

The graph of the ring Z_{25}



The distance between any two vertices of $\Gamma(Z_{25})$ are

d(5,10)=1 d(5,15)=1 d(5,20)=1

d(15,10)=1 d(20,10)=1 d(15,20)=1

The Wiener index of the graph of Z_{25}

$$W(\Gamma(Z_{25})) = \sum_{\{u,v\} \in (v(\Gamma(Z_{25}))} d(u,v).$$

$$= 6(1)$$

$$= 6$$

The hyper Wiener index of the graph of Z_{25}

$$WW(\Gamma(Z_{25})) = \frac{1}{2} W(\Gamma(Z_{25}) + \frac{1}{2} \sum_{\{u,v\} \in (v(\Gamma(Z_{25}))} (d(u,v))^2$$

$$= \frac{1}{2}(6) + \frac{1}{2}[6(1)^2]$$

$$= \frac{1}{2}[6 + 6]$$

$$= \frac{1}{2}[12]$$

$$=6$$

15) Z_{26}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25}

Since 2.13=26=0 4.13=52=0 6.13=78=0 8.13=104=0 10.13=130=0 14.13=182=0 16.13=208=0 18.13=234=0 20.13=260=0 22.13=286=0 24.13=312=0

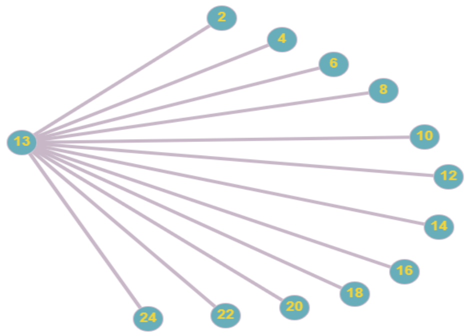
The set of zero divisors of Z_{26} is Z(Z_{26})= {2,4,6,8,10,12,13,14,16,18,20,22,24}

= V(Z_{26}), where V(Z_{26}) the set of vertex of the graph of Z_{26}.

The set of edges of Z_{26} is

E(Z_{26})={(13,2)(13,4),(13,6),(13,8),(13,10),(13,12),(13,14),(13,16),(13,18),(13,20),(13,22),(13,24)}

The graph of the ring Z_{26}



The distance between any two vertices of $\Gamma(Z_{26})$ are

d(13,x)=1 $\forall\, x \in V(Z_{26})$

d(x , y)=2 $\forall x, y \in V(Z_{26}) - 13$

The Wiener index of the graph of Z_{26}

$$W(\Gamma(Z_{26})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{26}))} d(u,v).$$

$$= 12(1) + 11(2) + 10(2) + 9(2) + 8(2) + 7(2) + 6(2) + 5(2) + 4(2) + 3(2) + 2(2) + 2$$

$$= 12 + 22 + 20 + 18 + 16 + 14 + 12 + 10 + 8 + 6 + 4 + 2$$

$$= 144$$

The hyper Wiener index of the graph of Z_{26}

$$WW(\Gamma(Z_{26})) = \frac{1}{2} W(\Gamma(Z_{26}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{26}))} (d(u,v))^2$$

$= \frac{1}{2}(144) + \frac{1}{2}[\,12(1)^2 + 66(2)^2]$

$= \frac{1}{2}[144 + 12 + 224]$

$= \frac{1}{2}[390]$

=195

16)

Z_{27}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26}

Since 3.9=0 3.18=0 6.9=0 6.18=0 9.12=0 9.15=0 9.18=0

9.21=0 9.24=0 15.18=0 18.21=0 18.24=0

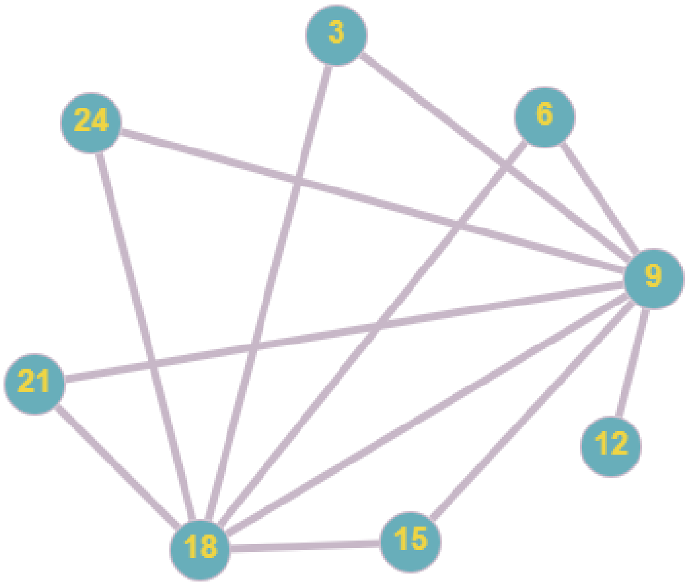
The set of zero divisors of Z_{27} is Z(Z_{27})={3,6,9,12,15,18,21,24}

= V(Z_{27}), where V(Z_{27}) the set of vertex of the graph of Z_{27}.

The set of edges of Z_{27} is

E(Z_{27})={(3,9),(3,18),(6,9),(6,18),(9,12),(9,15),(9,18),(9,21),(9,24),(15,18),(18,21),(18,24)}

The graph of the ring Z_{27}



The distance between any two vertices of $\Gamma(Z_{27})$ are

d(3,6)=2 d(3,9)=1 d(3,12)=2 d(3,15)=2 d(3,18)=1 d(3,21)=2 d(3,24)=2
d(6,9)=1 d(6,12)=2 d(6,15)=2 d(6,18)=1 d(6,21)=2 d(6,24)=2 d(9,12)=1
d(9,15)=1 d(9,18)=1 d(9,21)=1 d(9,24)=1 d(12,15)=2 d(12,18)=2 d(12,21)=2
d(12,24)=2 d(15,18)=1 d(15,21)=2 d(15,24)=2 d(18,21)=1 d(18,24)=1 d(21,24)=2

The Wiener index of the graph of Z_{27}

$$W(\Gamma(Z_{27})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{27}))} d(u,v).$$

$= 12(1) + 16(2)$

$= 12 + 32$

$= 44$

The hyper Wiener index of the graph of Z_{27}

$$WW(\Gamma(Z_{27})) = \frac{1}{2} W(\Gamma(Z_{27}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{27}))} (d(u,v))^2$$

$= \frac{1}{2}(44) + \frac{1}{2}[12(1)^2 + 16(2)^2]$

$= \frac{1}{2}[44 + 12 + 64]$

$= \frac{1}{2}[120]$

=195

17)

Z_{28}=
{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27}

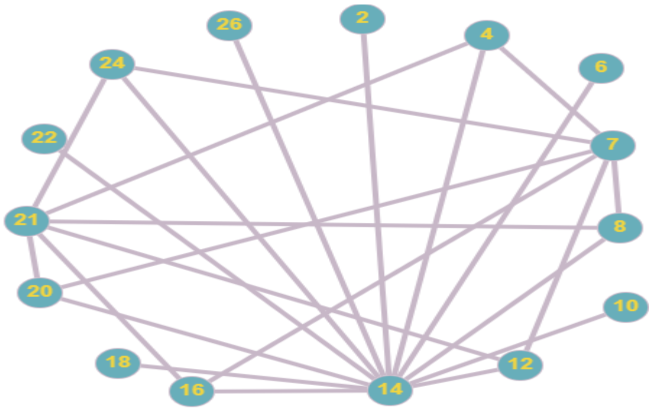
Since 4.14=28=0 4.7=28=0 4.14=56=0 4.21=84=0 6.14=84=0
7.8=56=0 7.12=84=0 7.16=112=0 7.20=140=0 7.24=168=0
8.14=112=0 8.21=168=0 10.14=140=0 12.14=168=0 12.21=252=0
14.16=224=0 14.22=308=0 16.21=336=0 14.24=336=0 14.20=280=0
14.26=364=0 21.24=506=0

The set of zero divisors of Z_{28} is Z(Z_{28})=

{ 2,4,6,7,8,10,12,14,16,18,20,21,22,24,26}= V(Z_{28}), where V(Z_{28}) the set of vertex of the graph of Z_{28}.

The set of edges of Z_{28} is
E(Z_{28})={(3,9),(3,18),(6,9),(6,18),(9,12),(9,15),(9,18),(9,21),(9,24),(15,18),(18,21),(18,24)}

The graph of the ring Z_{28}



The Wiener index of the graph of Z_{28}

$$W(\Gamma(Z_{28})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{28}))} d(u,v).$$

$$= 24(1) + 69(2) + 12(3)$$

$$= 24 + 138 + 36$$

$$= 198$$

The hyper Wiener index of the graph of Z_{28}

$$WW(\Gamma(Z_{28})) = \frac{1}{2} W(\Gamma(Z_{28}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{28}))} (d(u,v))^2$$

$$= \frac{1}{2}(198) + \frac{1}{2}[\, 24(1)^2 + 69(2)^2 + 12(3)^2]$$

$$= \frac{1}{2}[198 + 24 + 276 + 108]$$

$$= \frac{1}{2}[606]$$

$$=303$$

18)

Z_{30}=

{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29}

2.15=30=0 3.10=30=0 3.20=60=0 4.14=60=0

6.5=30=0 5.12=60=0 5.18=90=0 5.24=120=0

6.10=60=0 6.15=90=0 6.20=120=0 6.25=150=0

8.15=120=0 9.10=90=0 10.12=120=0 10.15=150=0

10.18=180=0

10.21=110=0 10.24=240=0 10.27=270=0 12.15=180=0

12.20=240=0 12.25=300=0 14.15=210=0 15.16=240=0

15.18=270=0 15.20=300=0 15.22=330=0 15.24=360=0

15.26=390=0 15.28=420=0 18.20=360=0 18.25=450=0

20.21=420=0 20.24=480=0 20.27=540=0 24.25=600=0

The set of zero divisors of Z_{30} is Z(Z_{30})=

{2,3,4,5,6,8,9,10,12,14,15,16,18,20,21,22,25,26,27,28}= V(Z_{30}), where V(Z_{30}) the set of vertex of the graph of Z_{30}.

The set of edges of Z_{30} is

E(Z_{30})={(2,15),(3,10),(3,20),(4,15),(6,5),(5,12),(5,18),(5,24),(6,10),(18,20),(18,25),(6,15),(6,20),(6,25),(8,15),(9,10),(10,12),(10,15),(10,18),(10,21),(10,24),(20,21),(20,24),(10,27),(12,15),(12,20),(12,25),(14,15),(15,16),(15,18),(15,20),(15,22),(15,24),(15,26),(15,28),(20,27),(25,24)}

The graph of the ring Z_{30}

The Wiener index of the graph of Z_{30}

$$W(\Gamma(Z_{30})) = \sum_{\{u,v\}\in(v(\Gamma(Z_{30}))} d(u,v).$$

$= 34(1) + 119(2) + 57(3)$

$= 34 + 138 + 171$

$= 243$

The hyper Wiener index of the graph of Z_{30}

$$WW(\Gamma(Z_{30})) = \frac{1}{2} W(\Gamma(Z_{30}) + \frac{1}{2} \sum_{\{u,v\}\in(v(\Gamma(Z_{30}))} (d(u,v))^2$$

$= \frac{1}{2}(243) + \frac{1}{2}[34(1)^2 + 119(2)^2 + 57(3)^2]$

$= \frac{1}{2}[243 + 34 + 476 + 513]$

$= \frac{1}{2}[1266]$

$=633$

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پوخته

با بازنەی ژمارەکان n بێت ، مەبەستمان ئەو ئیشە بۆ ئەوەیە تا گرافی بەسەر سفر دابەشدا بخوێنین ، ئەگەر n ژمارەی سەرەیە کەوایە هیچ بەسەر دابەشکراو سفر نیە ، کەوایە () گرافی سفرە .

لەو ئیشەوە پێمان وایە کە تەنیا لەکاتێکدا کە n ژمارەی ئاوێتە بێت .

تێبینی ئەمە بکە هەموو گرافەکان هەیە لەم وێبسایتە http://graphonline.ru/en .

الملخص

تجريدى فلتكن حلقة الاعداد الصحيحة n modulo الغرچ من هذا العمل هو دراسة بعض الرسوم البيانية ذات القاسم الصفرى ل إذا كان n عددا أوليا ، فلا يوجد به مقسومات صفرية ، لذا فإن FO هو الرسم البيانى الفارغ . و من ثم فى هذا العمل ، فإننا نعتبر فقط الحالة التى يكون فيها n مركبا . نحسب مؤشرات wiener و Hyper wiener ل FO لاحظ أن جميع الارقام يتم رسمها عبر الموقع الالكترونى http://graphonline.ru/en