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Finite Mathematics

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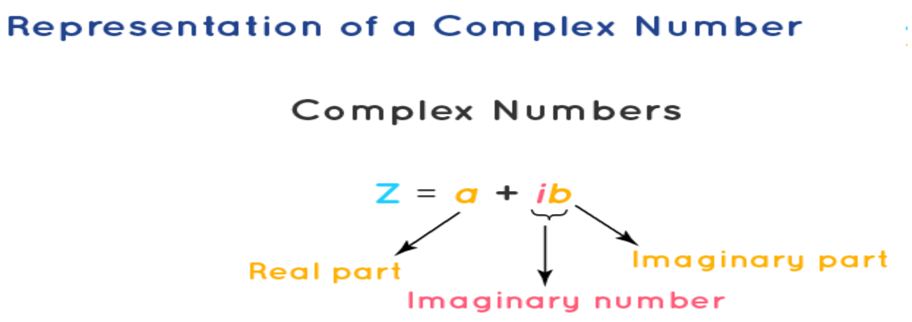
Chapter one

The Complex numbers

A complex number is the sum of a real number and an imaginary number. A complex number is of the form $a + ib$ or (a, b) and is usually represented by z. Here both a and b are real numbers. The value $'a'$ is called the ***real part*** which is denoted by **Re(z)**, and $'b'$ is called the ***imaginary part*** **Im(z).** Also, ib is called an ***imaginary number***.

$$\mathbb{C} = \{a + bi : a, b \text{ are real numbers}\} = \{(a, b) : a, b \text{ are real numbers}\}$$



Some of the examples of complex numbers are

$2 + 3i, \quad -6 - 5i, \quad 12 + i32 \quad \sqrt{7} + 3i, \quad \frac{15}{2} - 5i$ etc.

Power of i

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = i \cdot i^2 = i(-1) = -i$
- $i^4 = (i^2)(i^2) = (-1)(-1) = 1$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

Complex Number Formulas

Addition

Two Complex numbers are **added** by adding the real and imaginary parts of the summands. That is to say:

$$(a + ib) + (c + id) = (a + c) + (b + d)i$$

$$(2 + 3i) + (5 - 4i) = (2 + 5) + (3 - 4)i = 7 + (-1)i = 7 - i$$

Subtraction

Two Complex numbers are **subtraction** by subtracting the real and imaginary parts. This is defined by

$$(a + ib) - (c + id) = (a - c) + (b - d)i$$

$$\left(\frac{2}{3} + i\right) - \left(\frac{5}{3} + 4i\right) = \left(\frac{2}{3} - \frac{5}{3}\right) + (1 - 4)i = \frac{-3}{3} - (3)i = -1 - 3i$$

Multiplication

The multiplication of two complex numbers is defined by the following formula:

$$(a + ib)(c + id) = (ac - bd) + (bc + ad)i$$

$$(2 + 4i)(1 + 3i) = (2 * 1 + 2 \times 3i + 4i * 1 + 4i \times 3i$$

$$= 2 + 6i + 4i + 12i^2$$

$$= 2 + 10i - 12 \qquad \text{because } i^2 = -1$$

$$= -10 + 10i$$

Division

The division of two complex numbers is defined in terms of complex multiplication, which is described above, and real division. Where at least one of ***c*** and ***d*** is **non-zero**:

$$\frac{a+bi}{c+di} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{cb-ad}{c^2+d^2}\right)i$$

$$\frac{3-2i}{1+3i} = \frac{3-2i}{1+3i} \times \frac{1-3i}{1-3i} = \frac{3-9i-2i+6i^2}{1+3i-3i-9i^2} = \frac{-3-11i}{10}$$

Example: (H.W) **Simplify** the following

1- $4(3+2i) - 5(2-6i) + (5+8i)$.

2- $3(1+2i) + 5(3+2i) - (6+2i)$.

3- $(5-2i)(3+5i)$.

4- $(2+3i)(2-5i)(3+2i)$.

5- $3(3,-1) - (-6,3) + 2(-2,-3)$.

Example: (H.W) **Find** the value of $\boldsymbol{x}$ and $\boldsymbol{y}$ of the equation

1- $(3,4)2 - 2(x,-y) = (x,y)$.

2- $\left(\frac{2+i}{2-i}\right)^2 + \frac{1}{2x+iy} = 2+i$ **.**

3- $(1+4i)^2 - 2(x-yi) = x+yi$.

4- **If** $\frac{x-iy}{x+iy} = a+ib$, **then prove** that $a^2+b^2 = 1$.

5- ***Prove that*** $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = 1$.

Conjugate and absolute value of a Complex Numbers

Let $z = a + ib$ be a complex number.

The ***absolute value*** of Z is represented by $|Z|$.

Mathematically, $|Z| = \sqrt{a^2 + b^2} \geq 0$.

The ***conjugate*** of "z" is denoted by $\overline{Z}$

Mathematically, $\overline{Z} = a - ib$

Example: **Find** the ***absolute value*** and the ***conjugate*** of $z = 3 - 4i$.

Solution:

$$|z| = \sqrt{a^2 + b^2} = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

And $\overline{Z} = a - ib = 3 + 4i$

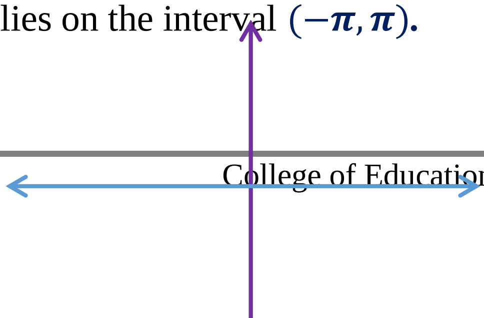
Graphing of Complex Numbers

The complex number consists of a ***real part*** and an ***imaginary part***, which can be considered as an ordered pair $(Re(z), Im(z))$ and can be represented as coordinates points in the complex plane. The complex number $z = a + ib$ is represented with the ***real part-a***, with reference to the ***x-axis***, and the ***imaginary part-b***, with reference to the ***y-axis***.

Polar Coordinates of a Complex Numbers

Let $z = x + y\,i = (x, y)$ be a complex number. Then $|z| = \sqrt{x^2 + y^2}$ and the argument of z, θ denoted by **arg**(z) is the angle between the ***positive direction for the real axis*** with ***the vector*** $\overrightarrow{Oz}$, where O is the ***origin*** and $z = (x, y)$, and the direction of θ with anti-clockwise and it's positive. Thus a complex numbers has an infinite number of arguments, any two of which differ by an integral multiple of 2π. The ***principal argument*** of z is the unique argument that lies on the interval $(-\pi, \pi)$.

If $z = (x, y)$, then

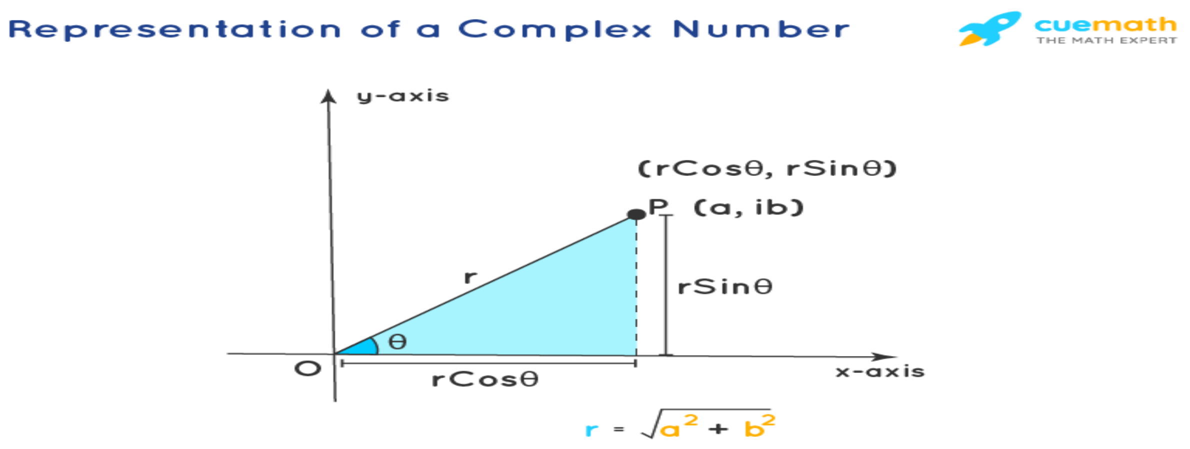


1- $x = r\cos\theta,\ y = r\sin\theta,$ $\theta = \pi - \alpha$ $\alpha = \theta$

2- $r = \sqrt{x^2 + y^2} = |z|,$

3- $\alpha = tan^{-1}\left|\frac{y}{x}\right|.$ $\theta = \alpha - \pi$ $\theta = -\alpha$

Hence $z = (x, y) = (r\cos\theta, r\sin\theta) = r(\cos\theta, \sin\theta) = r(\cos\theta + i\sin\theta).$



Example: ***Write*** the number **1− *i*** as in ***polar forms***.

Solution:

$$z = 1 - i,\ r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

And $\alpha = tan^{-1}\left(\frac{-1}{1}\right) = tan^{-1}(-1)$,

$$z = 1 - i = \sqrt{2}\,(\cos\frac{-\pi}{4} + \text{i}\sin\frac{-\pi}{4}).$$

Examples. ***Find*** the ***polar forms*** of these numbers:

1) $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ 2) $(\sqrt{3} - i)$ 3) $4 + 3i$ 4) $-6i$

Theorem: If z_1 and z_2 be two complex numbers, where

$z_1 = r_1(cos\theta_1, sin\theta_1)$ and $z_2 = r_2(cos\theta_2, sin\theta_2)$, then

1- $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$

2- $arg(z_1 z_2) = arg z_1 + arg z_2$

3- $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$

4- $\boldsymbol{arg}\left(\frac{z_1}{z_2}\right) = \boldsymbol{arg} z_1 - \boldsymbol{arg} z_2$

.

Example:

If we have $z_1 = 12(cos85, sin85)$ and $z_2 = 3(cos50, sin50)$, then find

1- $|z_1 z_2|$ 2- $\boldsymbol{arg}(z_1 z_2)$ 3 $-$ $\left|\frac{z_1}{z_2}\right|$ 4- $\boldsymbol{arg}\left(\frac{z_1}{z_2}\right)$.

Solution:

$z_1 z_2 = 12 \times 3(cos135, sin135)$,

$|z_1 z_2|$= 36 and $arg(z_1 z_2) = 135$

$\frac{z_1}{z_2}$= $4(cos35, sin35)$,

$\left|\frac{z_1}{z_2}\right|$ =4

And $arg\left(\frac{z_1}{z_2}\right) = 35$.

Theorem: (*Demoiver's theorem*) If $\boldsymbol{n}$ is *a positive integer*, then

$z^{\boldsymbol{n}} = (r(cos\theta + \boldsymbol{i}sin\theta))^{\boldsymbol{n}} = (r(cos\theta,\ sin\theta))^{\boldsymbol{n}} = r^{\boldsymbol{n}}\ (cos\,\boldsymbol{n}\theta, sin\,\boldsymbol{n}\theta)$.

Example: By using ***Demoiver's theorem*** **Find** $z = (-\sqrt{3} + i)^5$.

Solution:

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2,$$

$$\theta = tan^{-1}\frac{1}{\sqrt{3}} = 150^{\circ}$$

$$z = 2(\cos 150^{\circ} + \ isin\ 150^{\circ})$$

$$(-\sqrt{3} + i)^5 = \ 2^5(\cos 150^{\circ} + \ isin\ 150^{\circ})^5$$

$$= 32(\cos 5(150^{\circ}) + \ isin\ 5(150^{\circ}))$$

$$= 32(\cos\ 750 + \ isin\ 750)$$

$$= 32(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = 16\sqrt{3} + \ i16.$$

H.W. ***Find*** the value of the following

1) $((14(cos\ 50, isin\ 50))^{\mathbf{3}}$

2) $z = (3-\sqrt{3}i)^4$

3) $(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i)^{40}$

4) Let n be a ***positive integer*** then *prove that*

$$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$$