Chapter Two

Matrices

Definition: A matrix A=$[a_{ij}]$is a rectangle array of numbers or variables denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

OR

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad i = 1,2,\cdots,m \text{ and } j = 1,2,\cdots,n$$

In which m is the number of ***rows*** and n is the number of ***columns***. A matrix A with m ***rows*** and n ***columns*** is denoted by $A_{m\times n}$ (we say that A is m by n matrix) and the ***elements*** of the matrix A is denoted by a_{ij}.

That is i^{th} ***row*** of A is $[a_{i1} \quad a_{i2} \quad \dots \quad a_{in}]$, $1 \le i \le m$ and

j^{th} ***Column*** of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, $1 \le j \le n$.

Example 2.1:

$$A = \begin{pmatrix} 1 & 0 \\ -3 & 4 \\ -10 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 12 & 3 & 7 \\ 6 & -2 & -1 \\ -11 & 3 & -2 \end{pmatrix} \quad C = \begin{pmatrix} \frac{3}{4} & \sqrt{4} & 6 \end{pmatrix} \quad D = \begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Matrix A with 3 ***rows*** and 2 ***columns***, matrix B with 3 ***rows*** and 3 ***columns***, matrix C with 1 ***rows*** and 3 ***columns*** and matrix D ***with*** 4 ***rows*** and 1 ***column***. $A_{3\times 2}$, $B_{3\times 3}$, $C_{3\times 1}$, $D_{4\times 1}$. The ***elements*** of matrix A are

$a_{11} = 1,\ a_{12} = 0,\ a_{21} = -3,\ a_{22} = 4,\ a_{31} = -10 \text{ and } a_{32} = 3.$

Example 2.2:

Construct a matrix of degree 2×3 such that its ***elements*** are of the form $a_{ij} = (i + 2j) \quad \forall\, i, j$.

Example 2.3: H.W,

Find the matrix $A = (a_{ij})_{3\times 2}$ such that $a_{ij} = 2j + i^2$.

Operations on Matrices

-Matrix Addition

Let $A = (a_{ij})$ and $B = (b_{ij})$ be any two $m \times n$ matrices, then the ***sum*** of two matrix A and B defined by $A + B = (a_{ij}) + (b_{ij})$ is also $m \times n$ matrix.

$$A + B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{21} & a_{m2} + b_{21} & \cdots & a_{mn} + b_{21} \end{bmatrix}$$

Example 2.4:

1) Let $A = \begin{pmatrix} 3 & -1 & 0 \\ 5 & 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \end{pmatrix}$

Then $A + B = \begin{pmatrix} 3 & -1 & 0 \\ 5 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \end{pmatrix} =$

$$\begin{pmatrix} 3+2 & -1+3 & 0+(-1) \\ 5+1 & 3+1 & 2+0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 & -1 \\ 6 & 4 & 2 \end{pmatrix}.$$

2) H.W.

Find $A + B$, $A + C$,$B + C$ and $A + B + C$ where

$$A=\begin{pmatrix}1&2&3&4\\4&3&2&1\\-1&2&9&0\end{pmatrix}\quad B=\begin{pmatrix}-1&6&3&7\\2&1&4&-2\\-1&3&8&1\end{pmatrix}\text{ and } C=\begin{pmatrix}1&0&9&4\\4&2&2&11\\5&2&9&0\end{pmatrix}$$

-Matrix Subtraction

Let $\boldsymbol{A}=(\boldsymbol{a}_{ij})$ and $\boldsymbol{B}=(\boldsymbol{a}_{ij})$ be any two $\boldsymbol{m}\times\boldsymbol{n}$ matrices, then the ***subtraction*** of two matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ defined by $\boldsymbol{A}-\boldsymbol{B}=(\boldsymbol{a}_{ij})-(\boldsymbol{b}_{ij})$ is also $\boldsymbol{m}\times\boldsymbol{n}$ matrix.

$$\boldsymbol{A}-\boldsymbol{B}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}-\begin{bmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&&\vdots\\b_{m1}&b_{m2}&\cdots&b_{mn}\end{bmatrix}$$

$$=\begin{bmatrix}a_{11}-b_{11}&a_{12}-b_{12}&\cdots&a_{1n}-b_{1n}\\a_{21}-b_{21}&a_{22}-b_{22}&\cdots&a_{2n}-b_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}-b_{21}&a_{m2}-b_{21}&\cdots&a_{mn}-b_{21}\end{bmatrix},\ \boldsymbol{i}=1,2,\cdots,\boldsymbol{m}\text{ and }\boldsymbol{j}=1,2,\cdots,\boldsymbol{n}$$

Example 2.5:

Let $\boldsymbol{A}=\begin{pmatrix}2&-1\\0&5\end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix}2&0\\3&-2\end{pmatrix}$ then find $\boldsymbol{A}-\boldsymbol{B}$

$$\boldsymbol{A}-\boldsymbol{B}=\begin{pmatrix}2&-1\\0&5\end{pmatrix}-\begin{pmatrix}2&0\\3&-2\end{pmatrix}=\begin{pmatrix}2-2&-1-0\\0-3&5-(-2)\end{pmatrix}=\begin{pmatrix}0&-1\\-3&7\end{pmatrix}$$

-Scalar multiplication:

Let $\boldsymbol{A}=(\boldsymbol{a}_{ij})$ be any $\boldsymbol{m}\times\boldsymbol{n}$ matrix and $\boldsymbol{c}$ be any real number, then the ***scalar multiplication*** c by A is anm $\times n$ matrix D=c A= c (a_{ij})

$$\boldsymbol{D}=\boldsymbol{c}\cdot\boldsymbol{A}=\boldsymbol{c}\cdot\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}=\begin{bmatrix}ca_{11}&ca_{12}&\cdots&ca_{1n}\\ca_{21}&ca_{22}&\cdots&ca_{2n}\\\vdots&\vdots&&\vdots\\ca_{m1}&ca_{m2}&\cdots&ca_{mn}\end{bmatrix}$$

Example 2.6:

$$\boldsymbol{2}\cdot\begin{bmatrix}1&8&-3\\4&-2&5\end{bmatrix}=\begin{bmatrix}2.1&2.8&2(-3)\\2.4&2.(-2)&2.5\end{bmatrix}=\begin{bmatrix}2&16&-6\\8&-4&10\end{bmatrix}$$

Example 2.7:

$$\textbf{\textit{Find}}\ \sqrt{3}\begin{pmatrix} 6 & 12 \\ 3 & 1 \\ -15 & 0 \end{pmatrix}.$$

-Matrix Equality

Let $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ and $\boldsymbol{B} = (\boldsymbol{b}_{ij})$ be any two $\boldsymbol{m} \times \boldsymbol{n}$ matrices, then the ***equal*** of two matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ defined by

$$\boldsymbol{A} = \boldsymbol{B} \Longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

$\boldsymbol{a}_{ij} = \boldsymbol{b}_{ij}$ for all $\boldsymbol{i} = 1,2,\cdots,\boldsymbol{m}$ and $\boldsymbol{j} = 1,2,\cdots,\boldsymbol{n}$

Example 2.8:

Given that the following matrices are ***equal***, ***find*** the values of $\boldsymbol{x}$ and $\boldsymbol{y}$.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\ and\ \boldsymbol{B} = \begin{pmatrix} \boldsymbol{x} & 2 \\ 3 & \boldsymbol{y} \end{pmatrix}$$

***Solution*:** Since $\boldsymbol{A} = \boldsymbol{B}$

Then $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} x & 2 \\ 3 & y \end{pmatrix}$, $\quad \boldsymbol{x} = \mathbf{1}\ and\ \boldsymbol{y} = \mathbf{4}.$

Here are two matrices which ***are not equal*** even though they have the same elements.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3\times 2} \neq \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$$

Example 2.9:

Given that the following matrices are ***equal***, ***find*** the values of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

$$\boldsymbol{A} = \begin{pmatrix} 4 & 0 \\ 6 & -2 \\ 3 & 1 \end{pmatrix}\ and\ \boldsymbol{B} = \begin{pmatrix} \boldsymbol{x} & 0 \\ 6 & \boldsymbol{y}+4 \\ \frac{\boldsymbol{z}}{3} & 1 \end{pmatrix}$$

-Multiplication of a Matrix by a Scalar

Let $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix and $\boldsymbol{B} = (\boldsymbol{b}_{ij})$be an $\boldsymbol{n} \times \boldsymbol{p}$ matrix, then the ***product*** of $\boldsymbol{A}$ and $\boldsymbol{B}$ defined by $\boldsymbol{A} \times \boldsymbol{B}\ (\boldsymbol{A} \cdot \boldsymbol{B})$ is an $\boldsymbol{m} \times \boldsymbol{p}$ matrix $\boldsymbol{C}$.

$$C = (a_{ij})_{m\times n} \cdot (b_{ij})_{n\times p} = \left(\sum_{t=1}^{n} a_{it} \cdot b_{tj}\right)_{m\times p},$$

for all $i = 1,2,\cdots,m$ *and* $j = 1,2,\cdots,p.$

Example 2.10:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2} \text{ and } \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{23} & b_{23} \end{pmatrix}_{2\times 3}$$

$$A \cdot B = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{23} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{11} + a_{22}b_{23} & a_{21}b_{13} + a_{22}b_{23} \end{pmatrix}_{2\times 3}$$

Example 2.11:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Then $A \cdot B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 4 & 2 \\ 1 & 3 \end{pmatrix} =$

$$\begin{pmatrix} 1 * -1 + 2 * 4 + 3 * 1 & 1 * 0 + 2 * 2 + 3 * 3 \\ -1 * -1 + 0 * 4 + 4 * 1 & -1 * 0 + 0 * 2 + 4 * 3 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 5 & 12 \end{pmatrix}$$

Remark:

In the matrix ***multiplication*** the ***number of columns of the first matrix*** is ***equal*** to the ***number of rows in the second matrix***.

Example 2.12:

1) ***Find*** $A \cdot B$ where $A = \begin{pmatrix} 1 & 2 \\ 6 & -3 \\ 0 & 1 \end{pmatrix}_{3\times 2}$ and $B = \begin{pmatrix} -1 \\ -1 \end{pmatrix}_{2\times 1}$ and $B \cdot A$ if it is possible

2) ***Determine*** the matrices A and B

Where $A + 2B = \begin{pmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{pmatrix}$ and $2A - B = \begin{pmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{pmatrix}$.

-Transpose

If $A = (a_{ij})$ be any $m \times n$ matrix, then the $n \times m$ matrix $B = (b_{ij})$, where $b_{ij} = a_{ij}$ is called the ***transpose of*** A and is denoted by A^T.

For example $\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 5 \\ 6 & 4 & 7 \end{pmatrix}$, then the ***transpose*** of $\boldsymbol{A}$ is $\begin{pmatrix} 2 & 6 \\ 3 & 4 \\ 5 & 7 \end{pmatrix} = \boldsymbol{A}^{\boldsymbol{T}}$.

Properties of transpose

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $(\alpha \cdot A)^T = \alpha \cdot A^T$
4) $(A \cdot B)^T = B^T \cdot A^T$

Types of Matrices

1) Row and Column Matrix

Matrices with ***only one row*** and ***any number of columns*** are known *as* ***row matrices*** and matrices with ***one column*** and ***any number of rows*** are called ***column matrices***.

Let's look at two examples below:

Row Matrix	Column Matrix
$\boldsymbol{A} = [\mathbf{1} \quad \mathbf{2} \quad \mathbf{3}]$	$\boldsymbol{B} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$
There is *only one row*, **so** $\boldsymbol{A}$ **is a** *row matrix*.	There is ***only one column***, so B is a ***column matrix***.

2) Rectangular and Square Matrix

Any matrix that ***does not have an equal number of rows and columns*** is called a ***rectangular matrix*** and a rectangular matrix can be denoted by $[\boldsymbol{B}]_{m\times n}$. Any matrix that ***has an equal number of rows and columns*** is called a ***square matrix*** and a square matrix can be denoted by $[\boldsymbol{B}]_{n\times n}$.

Let's look at the examples below:

Rectangular Matrix	Square Matrix
$A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 2 & 1 \end{bmatrix}$ $B = \begin{pmatrix} 12 & 5 \\ 2 & -3 \\ 0 & 11 \end{pmatrix}_{3\times 2}$	$C = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 6 & 7 \\ 3 & 2 & 1 \end{bmatrix}$ $D = \begin{pmatrix} 3 & -1 & 0 \\ 7 & -4 & 6 \\ 0 & 1 & 2 \end{pmatrix}_{3\times 3}$
The matrix $\boldsymbol{A}$ have ***two rows*** and **three** ***columns*** in this matrix, and $\boldsymbol{B}$ have ***three rows*** and ***two columns*, so** $\boldsymbol{A}$ and $\boldsymbol{B}$ are ***rectangular matrices*.**	The matrix C have ***three rows*** and ***three columns*** and so $D.C$ and are ***square matrices***.

3) Identity matrix:

Let $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ be any $\boldsymbol{m} \times \boldsymbol{m}$ matrix, then $\boldsymbol{A}$ is said to be ***identity matrix if the diagonal elements are equal to*** **1** and ***other all elements are zero***. Thus, a square matrix

A$= [\boldsymbol{a}_{ij}]$ is an $\boldsymbol{identity\ matrix}$ if $\boldsymbol{a}_{ij} = \begin{cases} 0 & if\ \boldsymbol{i} \neq \boldsymbol{j} \\ 1 & if\ \boldsymbol{i} = \boldsymbol{j} \end{cases}$ It is denoted by $\boldsymbol{I}_{\boldsymbol{m}\times\boldsymbol{m}}$.

$$\boldsymbol{I}_{\mathbf{1}\times\mathbf{1}} = (\mathbf{1}),\ I_{2\times 2} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix},\ I_{3\times 3} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}.$$

5) Zero matrix:

Let $A = (\boldsymbol{a}_{ij})$ be any $\boldsymbol{m} \times \boldsymbol{n}$ matrix, then A is said to be ***zero matrix if all elements of A are zero*** (***if*** $\boldsymbol{a}_{ij} = \mathbf{0}$ for all $\boldsymbol{i} = 1,2,\cdots,\boldsymbol{m}$ and $\boldsymbol{j} = 1,2,\cdots,\boldsymbol{n}$).

For example $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times 3}$ is a ***zero matrix***.

6) Diagonal matrix:

A square matrix $\boldsymbol{A}$ is said to be ***diagonal matrix*** if ***all elements expect the diagonal elements are zero*** (***if*** $\boldsymbol{a}_{ij} = \mathbf{0}$ for $\boldsymbol{i} \neq \boldsymbol{j}$).

For example

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{3\times 3}$$ is a *diagonal matrix*.

7) Scalar matrix:

A diagonal matrix A is said to be *scalar matrix* ***if the diagonal elements are all equal***. Thus, a square matrix $A = [a_{ij}]$ a *scalar matrix* ***if*** $a_{ij} = \begin{Bmatrix} 0 & if\ i \neq j \\ k & if\ i = j \end{Bmatrix}$

where k is a constant. For example

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}_{3\times 3}$$ is *a scalar matrix*.

Conclusions:

- All ***identity*** matrices are *scalar* matrices
- All ***scalar*** matrices are ***diagonal*** matrices
- All ***diagonal*** matrices are ***square*** matrices

8) Triangular Matrix:

A ***square matrix*** is said to be ***a triangular matrix if*** the ***elements above or below the principal diagonal are zero***. There are two types:

- ***Upper Triangular Matrix***

A ***square matrix*** A is called an ***upper triangular matrix***, ***if*** $a_{ij} = 0$ ***when*** $i > j$

For example $A = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 2 & 7 \\ 0 & 0 & -9 \end{pmatrix}_{3\times 3}$ is an ***upper triangular matrix***

- ***Lower Triangular Matrix***

A ***square matrix*** A is called a ***lower triangular matrix, If*** $a_{ij} = 0$ when $i < j$.

. For example $A = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 5 & 8 & -9 \end{pmatrix}_{3\times 3}$ is a ***lower triangular matrix***.

9) Symmetric matrix:

An $m \times m$ matrix $A = (a_{ij})$ is said to be *symmetric matrix* if $A = A^T$ $(i.e.\ \ a_{ij} = a_{ji}$ for all $i, j)$.

For example $A = \begin{pmatrix} 4 & -3 & 1 \\ -3 & 2 & 8 \\ 1 & 8 & -9 \end{pmatrix}_{3\times 3}$ is *a symmetric matrix.*

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 7 \\ 3 & 7 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 0 \\ 4 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 4 \\ -1 & 3 & 5 & -2 \\ 0 & 4 & -2 & 9 \end{pmatrix}$$

10) Skew symmetric matrix:

An $m \times m$ matrix $A = (a_{ij})$ is said to be ***skew-symmetric*** matrix

if $= -A^T$ $\left(i.e.\ \ a_{ij} = \begin{Bmatrix} -a_{ij} & if\ i \neq j \\ 0 & if\ i = j \end{Bmatrix} \right).$

For example $A = \begin{pmatrix} 0 & 5 & -1 \\ -5 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}_{3\times 3}$ is ***a skew-symmetric*** matrix.

11) Idempotent matrix:

If $A^2 = A$, then A is called ***idempotent*** matrix.

Example:

Show that $A = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}_{3\times 3}$ is ***idempotent*** matrix.

12) Nilpotent matrix:

If $A^k = 0$, where k is a ***positive integer***, then A is called a ***nilpotent*** matrix. The least value of k is the index of it.

For example $A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$ is a ***nilpotent*** matrix.

Fundamental properties of matrix multiplication:

Let A, B and C are $m \times n$ matrices and r, k are real numbers, then

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A + 0 = 0 + A = A, (\,where\ 0\ is\ \textbf{zero matrix})$
4. $A + (-A) = (-A) + A = 0$
5. $A(BC) = (AB)C$
6. $r(kA) = (rk)A = k(rA)$
7. $A(B + C) = AB + AC$.
8. $(r + k)A = rA + kA$
9. $r(A + B) = rA + rB$
10. $A(rB) = r(AB)$.

Definition: Two square matrices A and B of the some degree are said to be ***commutative for multiplication if*** $A \cdot B = B \cdot A$, and is said to be ***skew commutative if*** $A \cdot B = -B \cdot A$

Example:

1) $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ are commutative.

2) $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$ are not commutative.

3) $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$ are skew commutative.