Chapter Four

System of Linear Equations

Definition -1-: The equation of the straight line in the xy-plane can be represented algebraically by an equation of the form $a_1x + a_2y = b$

Where a_1, a_2 *and* b are real constants and a_1 *and* a_2 are not both zero. An equation of this form is called a linear equation in the variables x and y.

Definition -2- **:** A linear equation in the n variables $x_1, x_2, \dots, x_n$ to be one that can be expressed in the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

Where $a_1, a_2, \dots, a_n$ *and* b are real constants. The variables in a linear equation are sometimes called unknown.

Example: Which of the following equations are linear?

- $4x_1 - 5x_2 + 2 = x_1$ linear: $3x_1 - 5x_2 = -2$
- $x_2 = 2(\sqrt{6} - x_1) + x_3$ linear: $2x_1 + x_2 - x_3 = 2\sqrt{6}$
- $4x_1 - 6x_2 = x_1x_2$ not linear: x_1x_2
- $x_2 = 2\sqrt{x_1} - 7$ not linear: $\sqrt{x_1}$

Definition -3- : A finite set of linear equations in the variables $x_1, x_2, \dots, x_n$ is called a system of linear equations or linear system. An arbitrary system of m linear equations in n unknowns can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \qquad \cdots\cdots\cdots \quad (1) \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Where $x_1, x_2, \dots, x_n$ a are unknowns and a_{ij}'s and b_j's denote constants , for $i = 1,2, \dots, n$ and $j = 1,2, \dots, m.$

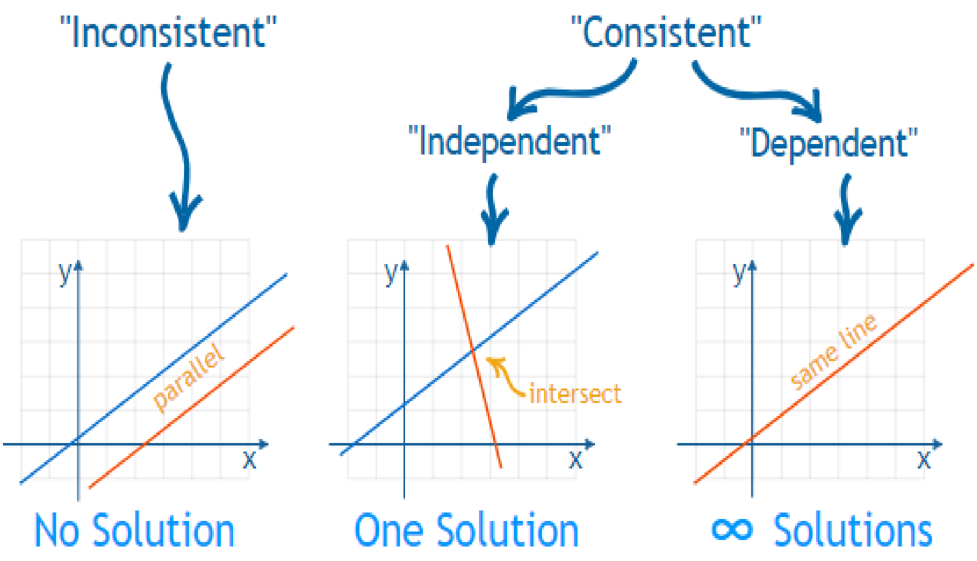
Definition -4- : A solution of a linear system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \dots, s_n$ are substituted for $x_1, x_2, \dots, x_n$, respectively.

Remark: Every system of linear equations has either no solutions, exactly one solution or infinitely many solutions.

When there is **no solution** the equations are called **"inconsistent"**.

One or **infinitely many solutions** are called **"consistent"**

Here is a diagram for **2 equations in 2 variables**:



"Independent" means that each equation gives new information. Otherwise they are **"Dependent"**.

Ex 1: $6x + 14y = 6 \longrightarrow 6x + 14y = 6 \longrightarrow 6x + 14y = 6$

$-4x - 7y = -11 \longrightarrow 2[-4x - 7y = -11] \longrightarrow -8x - 14y = -22$

$-2x = -16$

$\frac{-2x}{-2} = \frac{-16}{-2}$

$x = 8$

I choose the first equation to substitute x = 8 into.

$6(8) + 14y = 6$

$48 + 14y = 6$

$-48 \quad -48$

$14y = -42$

$\frac{14y}{14} = \frac{-42}{14}$

$y = -3$

The solution is (8, −3). It is consistent and independent.

Ex 2: $-8x + 2y = -10 \longrightarrow -8x + 2y = -10 \longrightarrow -8x + 2y = -10$

$-4x + y = -2 \longrightarrow -2[-4x + y = -2] \longrightarrow 8x - 2y = 4$

$0 = -6$

This is a false statement and has no solution. The lines are parallel. It is inconsistent.

Ex 3: $6x + 8y = -28 \longrightarrow 6x + 8y = -28 \longrightarrow 6x + 8y = -28$

$-3x - 4y = 14 \longrightarrow 2[-3x - 4y = 14] \longrightarrow -6x - 8y = 28$

$0 = 0$

This is a **true** statement and has infinitely many solutions. The equations are the exact same line. It is consistent and dependent.

Example:

- x + y = 3
- 2x + 2y = 6

Those equations are **"Dependent"**, because they are really the **same equation**, just multiplied by 2.

Example:

- 3x + 2y = 19
- x + y = 8

We can start with **any equation** and **any variable**.

** The solution of linear system.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \cdots\cdots\cdots \quad (1)$$

We define the matrix A, X and C as follows

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \text{ X=} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} \text{ and B=} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$$

So the system (1) becomes AX=B. The matrix A is called the coefficient matrix of the system (1) and the matrix of the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

is called augmented of system (1)

Methods for solving system of linear equation

1- Gaussian –Jordan method

Operations for used to solve systems of linear equations.

These operations correspond to the following operations on the rows of the augmented matrix.

a- Multiply a row through by a nonzero constant.

b- Interchange two rows

c- Add a multiple of one row to another row.

Example: Solve the system of linear equation by using Gaussian –Jordan method

$$x-2y+2z=6$$
$$2x+y+3z=2$$
$$3x-2y+z=5$$

Solution: First we find the augmented matrix of this system

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -2 & 2 & \vdots & 6 \\ 2 & 1 & 3 & \vdots & 2 \\ 3 & -2 & 1 & \vdots & 5 \end{pmatrix} \quad -2R_1+R_2=R_2 \Longrightarrow$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -2 & 2 & \vdots & 6 \\ 0 & 5 & -1 & \vdots & -10 \\ 3 & -2 & 1 & \vdots & 5 \end{pmatrix} \quad -3R_1+R_3=R_3 \Longrightarrow$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -2 & 2 & \vdots & 6 \\ 0 & 5 & -1 & \vdots & -10 \\ 0 & 4 & -5 & \vdots & -13 \end{pmatrix} \quad \frac{-4}{5}R*_2+R_3=R_3 \Longrightarrow$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -2 & 2 & \vdots & 6 \\ 0 & 5 & -1 & \vdots & -10 \\ 0 & 0 & \frac{-21}{5} & \vdots & -5 \end{pmatrix} \Longrightarrow$$

$$x-2y+2z=6$$
$$5y-z=-10$$
$$\frac{-21}{5}z=-5 \Longrightarrow z=-5\ \frac{-5}{21}=\boxed{\frac{25}{21}}$$

$y=\boxed{\frac{185}{21}}$ and $x=\boxed{-\frac{294}{21}}$.

Example: Solve the system of linear equation by using Gaussian –Jordan method

$$x-5y+2z=13$$
$$3x-14y+3z=29$$
$$4x-18y+3z=35$$

Solution: First we find the augmented matrix of this system

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -5 & 2 & \vdots & 13 \\ 3 & -14 & 3 & \vdots & 29 \\ 4 & -18 & 3 & \vdots & 35 \end{pmatrix} \quad -3R_1+R_2=R_2 \Longrightarrow\text{-}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{pmatrix} 1 & -5 & 2 & \vdots & 13 \\ 0 & 1 & -3 & \vdots & -10 \\ 4 & -18 & 3 & \vdots & 35 \end{pmatrix} \; -4R_1 + R_3 = R_3 \Longrightarrow$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{pmatrix} 1 & -5 & 2 & \vdots & 13 \\ 0 & 1 & -3 & \vdots & -10 \\ 0 & 2 & -5 & \vdots & -17 \end{pmatrix} \; -2R_2 + R_3 = R_3 \Longrightarrow$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{pmatrix} 1 & -5 & 2 & \vdots & 13 \\ 0 & 1 & -3 & \vdots & -10 \\ 0 & 0 & 1 & \vdots & 3 \end{pmatrix} \Longrightarrow$$

$x - 5y + 2z = 13$

$y - 3z = -10$

$z = 3,$

Z=3 Y=-1 X=2

Solve the following equations using matrix methods

1-

$x + y + z = 9$

$2x + 5y + 7z = 52$

$2x + y - z = 0$

2-

$2x + 4y - z = 9,$

$3x - y + 5z = 5,$

$8x + 2y + 9z = 19.$

2- Grammars' rule

If $AX = B$ is a system of n linear equations in n unknowns such that det(A) $\neq$0, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, x_n = \frac{\det(A_n)}{\det(A)}$$

Where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix B, where

$$\boldsymbol{A_1} = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_m & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \boldsymbol{A_2} = \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & b_m & \cdots & a_{mn} \end{pmatrix}, \dots,$$

$$\boldsymbol{A_n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & b_m \end{pmatrix}.$$

Example: Using Crammer rule to solve the system of linear equations

$$x + 2z = 6$$

$$-3x + 4y + 6z = 30$$

$$-x - 2y + 3z = 8$$

Solution.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix}, A_1 = \begin{pmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} \begin{matrix} 1 & 0 \\ -3 & 4 \\ -1 & -2 \end{matrix}$$

$$= 12 + 0 + 12 + 8 + 12 + 0$$

$$= 44$$

$$\det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} \begin{matrix} 6 & 0 \\ 30 & 4 \\ 8 & -2 \end{matrix}$$

$$= 72 + 0 - 120 - 64 + 72 + 0$$

$$= 40$$

$$\det(A_2) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} \begin{matrix} 1 & 6 \\ -3 & 30 \\ -1 & 8 \end{matrix}$$

$$= 90 - 36 - 48 + 60 - 48 + 54$$

$$= 72$$

$$\det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} \begin{matrix} 1 & 0 \\ -3 & 4 \\ -1 & -2 \end{matrix}$$

$= 32 + 0 + 36 + 24 + 60 + 0$

$= 152$

Therefore $x = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}$,

$y = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$,

$z = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$

Example: Using Cramer rule to solve the system of linear equations

$x + 2y + 3z = 5$

$2x + 5y + 3z = 3$

$x + 8z = 17$

Solution: $\boldsymbol{x = 1,\ y = -1,\ z = 2.}$

Example / use the Cramer's rule to solve the following equations

$2x_1 - 3x_2 = 5$

$x_1 + x_2 = 5$

Solution:

$A = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 5 & -3 \\ 5 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}, and\ B = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$|A| = [(2 * 1) - (1 * -3)] \rightarrow |A| = 5$

$|A_1| = [(5 * 1) - (5 * -3)] \rightarrow |A_1| = 20$

$|A_2| = [(2 * 5) - (1 * 5)] \rightarrow |A_2| = 5$

$\rightarrow$

$x_1 = \frac{|A_1|}{|A|} \rightarrow x_1 = \frac{20}{5}$

$= 4$

$x_1 = \frac{|A_1|}{|A|} \rightarrow x_1 = \frac{20}{5}$

$= 1$

Eigen values and Eigen vectors.

Definition -5-: Characteristic polynomial**:** Let A be an $n \times n$ matrix,

$P(\lambda) = \det(A - \lambda I) = |A - \lambda I|$, when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

Definition-6-: Eigenvalues: Let A be an $n \times n$ matrix.

The characteristic equation of A is $|A - \lambda I| = 0$. The roots of the characteristic equation are called Eigenvalues of A.

Definition -7-: Eigenvectors: Let A be an $n \times n$ matrix. If there exist a non zero vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $AX = \lambda X$, then the vector X is called an Eigenvector of A corresponding to the Eigenvalue λ .

Method of finding characteristic equation of a 3x3 matrix and 2x2 matrix

The characteristic equation of a 3x3 matrix is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1= sum of main diagonal elements.

S_2 = sum of minor of main diagonal elements.

S_3 = Det (A) =|A|

The characteristic equation of a 2x2 matrix is $\lambda^2 - S_1\lambda + S_2 = 0$

Where, S_1 = sum of main diagonal elements.

S_2 = Det (A) =|A|

1. Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

S_1 = sum of main diagonal elements

$= 1+2=3$

S_2 = Det (A) =|A|

$= \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$

$S_2 = 2-0 = 2$

The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$.

2. Find the characteristic equation of $\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = sum of the main diagonal elements

$= 2+1-4 = -1$

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$

$= (-4-6)+(-8+5)+(2+9) = -10+(-3)+11 = -2$

$S_3 = \text{Det}(A) = |A|$

$= \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix}$

$= 2(-4-6)-(-3)(-12+15)+1(6+5)$

$= 2(-10)+3(3)+1(11) = -20+9+11 = 0$

The characteristic equation is $\lambda^3 + \lambda^2 - 2\lambda = 0$

Example: Find the characteristic polynomial, characteristic equation, eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

Solution: The characteristic polynomial of A is

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2 = 4 - 5\lambda + \lambda^2.$$

The eigenvalues of A satisfy the equation $4 - 5\lambda + \lambda^2 = 0$. To solve the equation we obtain $\lambda_1 = 1$ and $\lambda_2 = 4$, that is $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

If $\lambda=1$, then $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then $x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

If $x_2 = -1 \Rightarrow x_1 = 1 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 1$

If $\lambda=4$, then $\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then $-x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_2$

If $x_2 = 2 \Rightarrow x_1 = 4 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 4$.

Example: Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$.

Solution: The characteristic polynomial of A is

$$P(\lambda) = \det(A - \lambda I) = \left| \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} \right|$$

$$= (2-\lambda)(3-\lambda)(4-\lambda)$$

The eigenvalues of A satisfy the equation $(2-\lambda)(3-\lambda)(4-\lambda) = 0$. To solve the equation we obtain $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 4$ are the eigenvalue of a matrix A

Example: Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$

Solution: The characteristic polynomial of A is

$$P(\lambda) = \det(A - \lambda I) = \left| \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} -\lambda & 0 & 3 \\ 0 & 3-\lambda & 0 \\ 3 & 0 & -\lambda \end{pmatrix} \right|$$

$$= (\lambda-3)(\lambda+3)(\lambda-3)$$

The eigenvalues of A satisfy the equation $(\lambda-3)(\lambda+3)(\lambda-3) = 0$. To solve the equation we obtain $\lambda_1 = 3$, $\lambda_2 = 3$ and $\lambda_3 = -3$ are the eigenvalue of a matrix A

If $\lambda_1 = 3$, then $\begin{pmatrix} -3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ then $-3x_1 + 3x_3 = 0 \Longrightarrow x_1 = x_3$ and $x_2 \in R$, if $x_3 = 1$ *and* $x_2 = 2$ *then* $x_1 = 1 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 3$.

And if $x_3 = 2$ *and* $x_2 = 4$ *then* $x_1 = 2 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 3$.

If $\lambda_3 = -3$, then $\begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ then $3x_1 + 3x_3 = 0 \Longrightarrow x_1 = -x_3$ and

$x_2 \in R$, if $x_3 = -1$ *and* $x_2 = 2$ *then* $x_1 = 1 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_3 = -3$.

Theorem1: If n is a positive integer, λ is an eigenvalue of a matrix A and x is a corresponding eigenvector, then λ^n is an eigenvalue of A^n and x is a acorresponding eigenvector.

Theorem 2: If λ is an eigenvalue of an invertible matrix A and x is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and x is a corresponding eigenvector

Theorem 3: If λ is an eigenvalue of matrix A then k λ is an eigenvalue of matrix kA where k is a constant.

Exercise: 1) Find the characteristic equations of the following matrices:

(a) $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$.

2) Let $A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$. Then find the eigenvalues of

(a) A^{-1} using Theorem 2 (b) A^4 using Theorem 1 (c) $A + 2I$

(d) 3A using Theorem 3