Chapter Five

Mathematical induction

Mathematical induction is a proof technique that allows us to prove some mathematical statement, let's call it P(n), that depends on n. Sometimes it is an equation, or an inequality, and sometimes it is a sentence with mathematical meaning. Notice that in both cases n should be a variable in the statement. So that we can plug n into P(n). We get a statement that we can often check by hand.

Step of mathematical induction:

Step (1): Show that the result is true for n=1, P(1) is true.

Step (2): Assume the result is true for n=k, P(k) is true,

Step (3): Show that the result is true for n=k+1, P(k + 1) is true.

Example 1: Use mathematical induction to prove that

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + 5 + 6 + \cdots + n = \frac{n(n+1)}{2}$$

Solution: Note: $S_1=1$

$S_2=1+2=3=S_1+2$

$S_3=1+2+3=6=S_2+3$

$S_4=1+2+3+4=10=S_3+4$

$S_5=1+2+3+4+5=15=S_4+5$

$S_6=1+2+3+4+5+6=21=S_5+6$

$\vdots$

$S_{k+1}=S_k+t_{k+1}$

Step (1): Verify the result is true for n=1, then the left-hand side of (1) is equal to 1. The right hand side of (1) is equal to $\frac{1(2)}{2} = 1$. So the result is true for n=1.

Step(2) Assume that the result is true for n=k, $S_k = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$,

Step (3): Show that the result is true for n=k+1,

$$S_{k+1} = 1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Now, $S_{k+1}=S_k+t_{k+1} = \underbrace{1 + 2 + 3 + \cdots + k}_{S_k} + \underbrace{(k+1)}_{t_{k+1}}$

$= \frac{k(k+1)}{2} + (k+1)$. Since P(k) is true

$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$

Thus we have proven that P(k + 1) is true.

Therefore, the result is true for all n.

Example 2: Prove

$$6 + 12 + 18 + \cdots + 6n = 3n(n+1) \text{ for all } n \in \mathbb{N}$$

Solution:

Step (1): Verify the result is true for n=1, then the left-hand side of (1) is equal to 6. The right hand side of (1) is equal to $3(1)(1+1) = (3)(2) = 6$. So the result is true for n=1.

Step(2) Assume that the result is true for n=k,

$S_k = 6 + 12 + 18 + \cdots + 6k = 3k(k+1)$,

Step (3): Show that the result is true for n=k+1,

$$S_{k+1} = 6 + 12 + \cdots + 6k + 6(k+1) = 3(k+1)((k+1)+1)$$

$$= 3(k+1)(k+2)$$

Now, $S_{k+1}=S_k+t_{k+1}= \underbrace{6 + 12 + 18 + \cdots + 6k}_{S_k} + 6\underbrace{(k+1)}_{t_{k+1}} = 3k(k+1) + 6(k+1)$

$$= 3(k+1)(k+2).$$

Thus we have proven that P(k + 1) is true. Therefore the result is true for all n.

Example 3:

$1 + 3 + 5 + \cdots + (2n-1) = n^2$, for all $n \in \mathbb{Z}^+$

Solution:

Step (1): Verify the result is true for n=1, then the left-hand side of (1) is equal to 1. The right hand side of (1) is equal to 1^2

Step(2): Assume that the result is true for n=k, $S_k = 1 + 3 + 5 + \cdots + (2k-1) = k^2$,

Step (3): Show that the result is true for n=k+1,

$S_{k+1} = 1 + 3 + 5 + \cdots + (2k-1) + (2(k+1)-1) = (k+1)^2$

Now, S_{k+1}=S_k+t_{k+1}= $\underbrace{1 + 3 + 5 + \cdots + (2k-1)}_{S_k} + \underbrace{(2(k+1)-1)}_{t_{k+1}}$

$$=k^2 + (2k + 2 - 1)$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

Thus we have proven that P(k + 1) is true.

Therefore, the result is true for all n.

Example 4: Prove

$2 + 8 + 14 + \cdots + (6n-4) = n(3n-1)$ for all positive integer n.

Example 5.

For all $n \in \mathbb{N}$, $2^n > n$.

Solution:

Step (1): Verify the result is true for n=1, $then\ 2^1 > 1$.

Step(2): Assume that the result is true for n=k, then $2^k > k$.

Step (3): To show that the result is true for n=k+1, $2^{k+1} > k+1$

Now, since $2^k > k$ then $2^k > 1 \Longrightarrow 2^k + 2^k > k + 1$

$$\Longrightarrow 2(2^k) > k + 1$$

$$\Longrightarrow 2^{k+1} > k + 1.$$

Thus we have proven that P(k + 1) is true.

Therefore, the result is true for all n.

Example 6: Prove $n^3 - n$ is divisible by 3 for all positive integers

Solution:

Step (1): Verify the result is true for n=1. $Then\ 1^3 - 1 = 0$ is divisible by 3.

Step(2): Assume that the result is true for n=k, then P(k) = $k^3 - k$ is divisible by 3.

Step(3): To show that the result is true for n=k+1, P(k+1) = $(k+1)^3 - (k+1)$ is divisible by 3

Now, $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3\ k + 1 - k - 1$

$$= (k^3 - k) + 3(k^2 + \ k)$$

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is divisible by 3 is divisible by 3

Thus $P(k+1) == (k+1)^3 - (k+1)$ is divisible by 3

Therefore, the result is true for all positive integer n.

Example: Prove that

$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$, for all positive integer n

Exercises.

Prove the following by mathematical induction:

(a) $\sum_{i=1}^{n} i\,(i+1)\,(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

(b) $\sum_{i=1}^{n} (i+2)\,(3i+1) = n(n+2)(n+3)$

(c) $\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n-1)}{r-1}$