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Chapter One

1- Matrices

Definition: A matrix A=$[\boldsymbol{a}_{ij}]$is a rectangle array of numbers or variables denoted by

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

OR

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad i=1,2,\cdots,m \text{ and } j=1,2,\cdots,n$$

In which $\boldsymbol{m}$ is the number of ***rows*** and $\boldsymbol{n}$ is the number of ***columns***. A matrix $\boldsymbol{A}$ with $\boldsymbol{m}$ ***rows*** and $\boldsymbol{n}$ ***columns*** is denoted by $A_{m\times n}$ (we say that $\boldsymbol{A}$ is $\boldsymbol{m}$ by $\boldsymbol{n}$ matrix) and the ***elements*** of the matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{a}_{ij}$.

That is i^{th} *row* of $\boldsymbol{A}$ is $[\boldsymbol{a}_{i1} \quad \boldsymbol{a}_{i2} \quad \ldots \quad \boldsymbol{a}_{in}]$, $\quad 1\le i \le m$ and

j^{th} ***Column*** of $\boldsymbol{A}$ is $\begin{bmatrix} \boldsymbol{a}_{1j} \\ \boldsymbol{a}_{2j} \\ \vdots \\ \boldsymbol{a}_{mj} \end{bmatrix}$, $\quad 1\le j \le n$.

Example:

$$A=\begin{pmatrix} 1 & 0 \\ -3 & 4 \\ -10 & 3 \end{pmatrix} \quad B=\begin{pmatrix} 12 & 3 & 7 \\ 6 & -2 & -1 \\ -11 & 3 & -2 \end{pmatrix} \quad C=\begin{pmatrix} \frac{3}{4} & \sqrt{4} & 6 \end{pmatrix} \quad D=\begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Matrix $\boldsymbol{A}$ with 3 ***rows*** and 2 ***columns***, matrix $\boldsymbol{B}$ with 3 ***rows*** and 3 ***columns***, matrix $\boldsymbol{C}$ with 1 ***rows*** and 3 ***columns*** and matrix $\boldsymbol{D}$ ***with*** 4 ***rows*** and 1 ***column***. $A_{3\times 2}$, $B_{3\times 3}$, $C_{3\times 1}$, $D_{4\times 1}$. The ***elements*** of matrix $\boldsymbol{A}$ are

$a_{11}=1,\ a_{12}=0,\ a_{21}=-3,\ a_{22}=4,\ a_{31}=-10 \text{ and } a_{32}=3.$

Operations on Matrices

-Matrix Addition

Let $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ and $\boldsymbol{B} = (\boldsymbol{b_{ij}})$ be any two $\boldsymbol{m} \times \boldsymbol{n}$ matrices, then the ***sum*** of two matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ defined by $\boldsymbol{A} + \boldsymbol{B} = (\boldsymbol{a_{ij}}) + (\boldsymbol{b_{ij}})$ is also $\boldsymbol{m} \times \boldsymbol{n}$ matrix.

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{21} & a_{m2} + b_{21} & \cdots & a_{mn} + b_{21} \end{bmatrix}$$

Example:

1) Let $\boldsymbol{A} = \begin{pmatrix} 3 & -1 & 0 \\ 5 & 3 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \end{pmatrix}$

Then $\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 3 & -1 & 0 \\ 5 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3+2 & -1+3 & 0+(-1) \\ 5+1 & 3+1 & 2+0 \end{pmatrix}$

$$= \begin{pmatrix} 5 & 2 & -1 \\ 6 & 4 & 2 \end{pmatrix}.$$

2) H.W.

Find $\boldsymbol{A} + \boldsymbol{B}$, $\boldsymbol{A} + \boldsymbol{C}$,$\boldsymbol{B} + \boldsymbol{C}$ and $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ -1 & 2 & 9 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 6 & 3 & 7 \\ 2 & 1 & 4 & -2 \\ -1 & 3 & 8 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 & 9 & 4 \\ 4 & 2 & 2 & 11 \\ 5 & 2 & 9 & 0 \end{pmatrix}$$

-Matrix Subtraction

Let $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ and $\boldsymbol{B} = (\boldsymbol{a_{ij}})$ be any two $\boldsymbol{m} \times \boldsymbol{n}$ matrices, then the ***subtraction*** of two matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ defined by $\boldsymbol{A} - \boldsymbol{B} = (\boldsymbol{a_{ij}}) - (\boldsymbol{b_{ij}})$ is also $\boldsymbol{m} \times \boldsymbol{n}$ matrix.

$$\boldsymbol{A}-\boldsymbol{B}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}-\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}-b_{21} & a_{m2}-b_{21} & \cdots & a_{mn}-b_{21} \end{bmatrix}, \boldsymbol{i}=1,2,\cdots,\boldsymbol{m} \text{ and } \boldsymbol{j}=1,2,\cdots,\boldsymbol{n}$$

Example:

Let $\boldsymbol{A}=\begin{pmatrix} 2 & -1 \\ 0 & 5 \end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix} 2 & 0 \\ 3 & -2 \end{pmatrix}$ then find $\boldsymbol{A}-\boldsymbol{B}$

$$\boldsymbol{A}-\boldsymbol{B}=\begin{pmatrix} 2 & -1 \\ 0 & 5 \end{pmatrix}-\begin{pmatrix} 2 & 0 \\ 3 & -2 \end{pmatrix}=\begin{pmatrix} 2-2 & -1-0 \\ 0-3 & 5-(-2) \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ -3 & 7 \end{pmatrix}$$

-Scalar multiplication:

Let $\boldsymbol{A}=(\boldsymbol{a}_{ij})$ be any $\boldsymbol{m}\times\boldsymbol{n}$ matrix and $\boldsymbol{c}$ be any real number, then the ***scalar multiplication*** c by A is an$m\times n$ matrix D=c A= c (a_{ij})

$$\boldsymbol{D}=\boldsymbol{c}\cdot\boldsymbol{A}=\boldsymbol{c}\cdot\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

Example:

$$\boldsymbol{2}\cdot\begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix}=\begin{bmatrix} \boldsymbol{2}.1 & \boldsymbol{2}.8 & \boldsymbol{2}(-3) \\ \boldsymbol{2}.4 & \boldsymbol{2}.(-2) & \boldsymbol{2}.5 \end{bmatrix}=\begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

Example:

$$\boldsymbol{Find}\ \sqrt{3}\begin{pmatrix} 6 & 12 \\ 3 & 1 \\ -15 & 0 \end{pmatrix}.$$

-Matrix Equality

Let $\boldsymbol{A}=(\boldsymbol{a}_{ij})$ and $\boldsymbol{B}=(\boldsymbol{b}_{ij})$ be any two $\boldsymbol{m}\times\boldsymbol{n}$ matrices, then the ***equal*** of two matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ defined by

$$A = B \Longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

$a_{ij} = b_{ij}$ for all $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$

Example:

Given that the following matrices are ***equal***, ***find*** the values of x and y.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} x & 2 \\ 3 & y \end{pmatrix}$$

Solution: Since $A = B$

Then $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} x & 2 \\ 3 & y \end{pmatrix}$, $\quad x = 1$ and $y = 4$.

Here are two matrices which ***are not equal*** even though they have the same elements.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3\times 2} \neq \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$$

Example:

Given that the following matrices are ***equal***, ***find*** the values of x, y, and z. $A =$

$$\begin{pmatrix} 4 & 0 \\ 6 & -2 \\ 3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} x & 0 \\ 6 & y+4 \\ \frac{z}{3} & 1 \end{pmatrix}$$

- *Matrix Multiplication*

Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix, then the ***product*** of A and B defined by $A \times B$ $(A \cdot B)$ is an $m \times p$ matrix C.

$$C = (a_{ij})_{m\times n} \cdot (b_{ij})_{n\times p} = \left(\sum_{t=1}^{n} a_{it} \cdot b_{tj}\right)_{m\times p},$$

for all $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,p$.

Suppose $A = [a_{ik}]$ and $B = [b_{kj}]$ are matrices such that the number of columns of A is equal to the number of rows of B; say, A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product AB is the $m \times n$ matrix whose ij-entry is obtained by multiplying the ith row of A by the jth column of B. That is,

$$\begin{bmatrix} a_{11} & \dots & a_{1p} \\ . & \dots & . \\ a_{i1} & \dots & a_{ip} \\ . & \dots & . \\ a_{m1} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ . & \dots & . & \dots & . \\ . & \dots & . & \dots & . \\ . & \dots & . & \dots & . \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ . & \dots & . \\ . & c_{ij} & . \\ . & \dots & . \\ c_{m1} & \dots & c_{mn} \end{bmatrix}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$

Example:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2} \text{ and } \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{23} & b_{23} \end{pmatrix}_{2\times 3}$$

$$\boldsymbol{A \cdot B} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{23} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{11} + a_{22}b_{23} & a_{21}b_{13} + a_{22}b_{23} \end{pmatrix}_{2\times 3}$$

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Then $\boldsymbol{A \cdot B} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 4 & 2 \\ 1 & 3 \end{pmatrix} =$

$$\begin{pmatrix} 1 * -1 + 2 * 4 + 3 * 1 & 1 * 0 + 2 * 2 + 3 * 3 \\ -1 * -1 + 0 * 4 + 4 * 1 & -1 * 0 + 0 * 2 + 4 * 3 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 5 & 12 \end{pmatrix}$$

Remark:

In the matrix ***multiplication*** the ***number of columns of the first matrix*** is ***equal*** to the ***number of rows in the second matrix***.

Example:

1) ***Find*** $\boldsymbol{A\cdot B}$ where $\boldsymbol{A}=\begin{pmatrix}1&2\\6&-3\\0&1\end{pmatrix}_{3\times2}$ and $\boldsymbol{B}=\begin{pmatrix}-1\\-1\end{pmatrix}_{2\times1}$ and $\boldsymbol{B\cdot A}$ if it is possible

2) ***Determine*** the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$

Where $\boldsymbol{A+2B}=\begin{pmatrix}1&2&0\\6&-3&3\\-5&3&1\end{pmatrix}$ and $2A-B=\begin{pmatrix}2&-1&5\\2&-1&6\\0&1&2\end{pmatrix}$.

-Transpose

If $\boldsymbol{A}=(\boldsymbol{a}_{ij})$ be any $\boldsymbol{m}\times\boldsymbol{n}$ matrix, then the $\boldsymbol{n}\times\boldsymbol{m}$ matrix $\boldsymbol{B}=(\boldsymbol{b}_{ij})$,

where $\boldsymbol{b}_{ij}=\boldsymbol{a}_{ij}$ is called the ***transpose of*** $\boldsymbol{A}$ and is denoted by $\boldsymbol{A^T}$.

For example $\boldsymbol{A}=\begin{pmatrix}2&3&5\\6&4&7\end{pmatrix}$, then the ***transpose*** of $\boldsymbol{A}$ is $\begin{pmatrix}2&6\\3&4\\5&7\end{pmatrix}=\boldsymbol{A^T}$.

Properties of transpose

1) $(A^T)^T = A$

2) $(A+B)^T = A^T + B^T$

3) $(\alpha\cdot A)^T=\alpha\cdot A^T$

4) $(A\cdot B)^T = B^T\cdot A^T$

Some Types of Matrices

1) Row and Column Matrix

Matrices with ***only one row*** and ***any number of columns*** are known *as* ***row matrices*** and matrices with ***one column*** and ***any number of rows*** are called ***column matrices*.**

Let's look at two examples below:

Row Matrix	Column Matrix

$A = [1 \quad 2 \quad 3]$	$B = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$
There is *only one row*, so *A* is a *row matrix*.	There is ***only one column***, so B is a ***column matrix***.

2) Rectangular and Square Matrix

Any matrix that ***does not have an equal number of rows and columns*** is called a ***rectangular matrix*** and a rectangular matrix can be denoted by $[B]_{m\times n}$. Any matrix that ***has an equal number of rows and columns*** is called a ***square matrix*** and a square matrix can be denoted by $[B]_{n\times n}$.

Let's look at the examples below:

Rectangular Matrix	Square Matrix
$A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 2 & 1 \end{bmatrix}$ $B = \begin{pmatrix} 12 & 5 \\ 2 & -3 \\ 0 & 11 \end{pmatrix}_{3\times 2}$	$C = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 6 & 7 \\ 3 & 2 & 1 \end{bmatrix}$ $D = \begin{pmatrix} 3 & -1 & 0 \\ 7 & -4 & 6 \\ 0 & 1 & 2 \end{pmatrix}_{3\times 3}$
The matrix ***A*** have ***two rows*** and **three *columns*** in this matrix, and ***B*** have ***three rows*** and ***two columns*****, so *A*** and ***B*** are ***rectangular matrices*****.**	The matrix C have ***three rows*** and ***three columns*** and so $D.C$ and are ***square matrices***.

3) Identity matrix:

Let $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ be any $\boldsymbol{m} \times \boldsymbol{m}$ matrix, then ***A*** is said to be ***identity matrix if the diagonal elements are equal to* 1** and ***other all elements are zero***. Thus, a square matrix

A$= [\boldsymbol{a_{ij}}]$ is an ***identity matrix*** if $\boldsymbol{a_{ij}} = \begin{Bmatrix} 0 & if\ \boldsymbol{i} \neq \boldsymbol{j} \\ 1 & if\ \boldsymbol{i} = \boldsymbol{j} \end{Bmatrix}$ It is denoted by $\boldsymbol{I_{m\times m}}$.

$\boldsymbol{I_{1\times 1}} = (\mathbf{1}),\ I_{2\times 2} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix},\ I_{3\times 3} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}.$

5) Zero matrix:

Let $A = (\boldsymbol{a_{ij}})$ be any $\boldsymbol{m} \times \boldsymbol{n}$ matrix, then A is said to be ***zero matrix if all elements of A are zero*** (***if*** $\boldsymbol{a_{ij}} = \mathbf{0}$ for all $\boldsymbol{i} = 1,2,\cdots,\boldsymbol{m}$ and $\boldsymbol{j} = 1,2,\cdots,\boldsymbol{n}$).

For example $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times 3}$ is a ***zero matrix***.

6) Diagonal matrix:

A square matrix $\boldsymbol{A}$ is said to be ***diagonal matrix*** if ***all elements expect the diagonal elements are zero*** (***if*** $\boldsymbol{a_{ij}} = \mathbf{0}$ for $\boldsymbol{i} \neq \boldsymbol{j}$).

For example

$$\boldsymbol{A} = \begin{pmatrix} 3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2 \end{pmatrix}_{3\times 3} \text{ is a } \textit{diagonal matrix}.$$

7) Scalar matrix:

A diagonal matrix $\boldsymbol{A}$ is said to be ***scalar matrix if the diagonal elements are all equal***. Thus, a square matrix $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ a ***scalar matrix if*** $\boldsymbol{a_{ij}} = \begin{Bmatrix} 0 & if\ \boldsymbol{i} \neq \boldsymbol{j} \\ \boldsymbol{k} & if\ \boldsymbol{i} = \boldsymbol{j} \end{Bmatrix}$

where $\boldsymbol{k}$ is a constant. For example

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{5} & 0 & 0 \\ 0 & \mathbf{5} & 0 \\ 0 & 0 & \mathbf{5} \end{pmatrix}_{3\times 3} \text{ is } \textit{a scalar matrix}.$$

8) Triangular Matrix:

A ***square matrix*** is said to be ***a triangular matrix if*** the ***elements above or below the principal diagonal are zero***. There are two types:

- ***Upper Triangular Matrix***

A ***square matrix A*** is called an ***upper triangular matrix***, ***if*** $\boldsymbol{a_{ij}} = \mathbf{0}$ ***when*** $\boldsymbol{i} > \boldsymbol{j}$

For example $A = \begin{pmatrix} 3 & -1 & 4 \\ \mathbf{0} & 2 & 7 \\ \mathbf{0} & \mathbf{0} & -9 \end{pmatrix}_{3\times 3}$ is an ***upper triangular matrix***

- ***Lower Triangular Matrix***

A ***square matrix A*** is called a ***lower triangular matrix***, ***If*** $\boldsymbol{a_{ij}} = \mathbf{0}$ when $\boldsymbol{i} < \boldsymbol{j}$.

. For example $\boldsymbol{A} = \begin{pmatrix} 3 & \mathbf{0} & \mathbf{0} \\ -1 & 2 & \mathbf{0} \\ 5 & 8 & -9 \end{pmatrix}_{3\times 3}$ is a ***lower triangular matrix***.

9) Symmetric matrix:

An $\boldsymbol{m} \times \boldsymbol{m}$ matrix $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ is said to be *symmetric matrix* if $\boldsymbol{A} = \boldsymbol{A}^T$ $(i.e.\ \ \boldsymbol{a_{ij}} = a_{ji}$ for all $\boldsymbol{i}, \boldsymbol{j})$.

For example $\boldsymbol{A} = \begin{pmatrix} 4 & -3 & 1 \\ -3 & 2 & 8 \\ 1 & 8 & -9 \end{pmatrix}_{3\times 3}$ is ***a symmetric matrix.***

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 7 \\ 3 & 7 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 0 \\ 4 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 4 \\ -1 & 3 & 5 & -2 \\ 0 & 4 & -2 & 9 \end{pmatrix}$$

10) Skew symmetric matrix:

An $\boldsymbol{m} \times \boldsymbol{m}$ matrix $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ is said to be ***skew-symmetric*** matrix

if $= -\boldsymbol{A}^T$ $\left(i.e.\ \ \boldsymbol{a_{ij}} = \begin{Bmatrix} -\boldsymbol{a_{ij}} & if\ \boldsymbol{i} \neq \boldsymbol{j} \\ 0 & if\ \boldsymbol{i} = \boldsymbol{j} \end{Bmatrix}\right)$.

For example $\boldsymbol{A} = \begin{pmatrix} 0 & 5 & -1 \\ -5 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}_{3\times 3}$ is ***a skew-symmetric*** matrix.

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Fundamental properties of matrix multiplication:

Let $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are $\boldsymbol{m} \times \boldsymbol{n}$ matrices and $\boldsymbol{r}, \boldsymbol{k}$ are real numbers, then

1. $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
2. $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$
3. $\boldsymbol{A} + 0 = 0 + \boldsymbol{A} = \boldsymbol{A}, (where\ 0\ is\ \boldsymbol{zero\ matrix})$
4. $\boldsymbol{A} + (-\boldsymbol{A}) = (-\boldsymbol{A}) + \boldsymbol{A} = 0$
5. $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$
6. $\boldsymbol{r}(\boldsymbol{kA}) = (\boldsymbol{rk})\boldsymbol{A} = \boldsymbol{k}(\boldsymbol{rA})$
7. $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$.
8. $(\boldsymbol{r} + \boldsymbol{k})\boldsymbol{A} = \boldsymbol{rA} + \boldsymbol{kA}$
9. $\boldsymbol{r}(\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{rA} + \boldsymbol{rB}$
10. $\boldsymbol{A}(r\boldsymbol{B}) = r(\boldsymbol{AB})$.

2- Determinants and Inverses

Definition: Let A be a square matrix. The determinant of A is a function which assign for A to the number of the field F. And denoted by det(A), f(A)=|A|.

Determinants of 1×1 matrices

$f([a]_{1\times1}) = |a| = a$

For example A=$[-5]$ then $|-5| = -5$

Determinants of 2×2 matrices

The value of the determinant of a 2 × 2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{pmatrix}$, can be given as

det(A)=$|A| = a_{11}a_{21} - a_{12}a_{21}$

Let us take an example to understand this very clearly,

Example 1: The matrix is given by, $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

Find the value of |A|.

det(A)=$|A| = 3 \cdot 4 - 2 \cdot 1 = 10.$

Example 2: The matrix is given by, $A = \begin{bmatrix} 3 & -1 \\ 4 & 3 \end{bmatrix}$

Find the value of |A|.

det(A)=$|A| = 3 \cdot 3 - (-1) \cdot 4 = 13.$

Determinants of 3×3 matrices

The first Way

The value of the determinant of a 3 × 3 matrix A=$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}_{3\times3}$, can be given as

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \bullet \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \bullet \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \bullet \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Example 1: The matrix is given by, $A = \begin{pmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{pmatrix}$

Find the value of |A|.

det(A)=$|A| = 2 \cdot \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} - (-3) \cdot \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$

$= 2 \,.(4) + 3 \,.\, 11 + 1 \,.\, 8$

$= 8 + 33 + 8$

$= 49$

Example 2: The matrix is given by, $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{pmatrix}$

Find the value of |A|.

$$\begin{bmatrix}1&-2&3\\2&0&3\\1&5&4\end{bmatrix} = \begin{bmatrix}\boxed{1}&-2&3\\2&0&3\\1&5&4\end{bmatrix} - \begin{bmatrix}1&\boxed{-2}&3\\2&0&3\\1&5&4\end{bmatrix} + \begin{bmatrix}1&-2&\boxed{3}\\2&0&3\\1&5&4\end{bmatrix}$$

$$= \boxed{1} \times \begin{vmatrix}0&3\\5&4\end{vmatrix} - \boxed{-2} \times \begin{vmatrix}2&3\\1&4\end{vmatrix} + \boxed{3} \times \begin{vmatrix}2&0\\1&5\end{vmatrix}$$

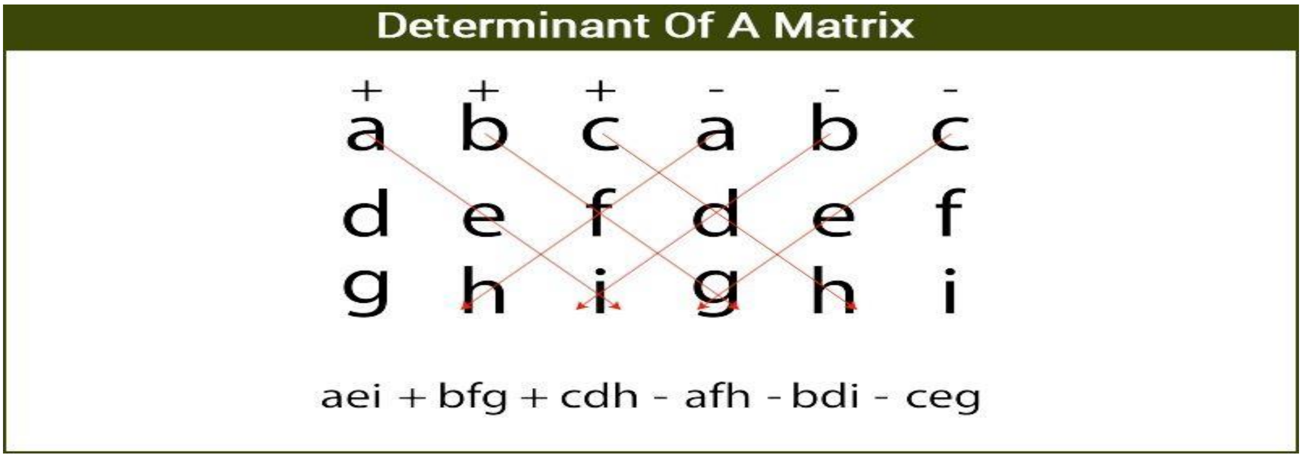
$$= 1\times(0-15)+2\times(8-3)+3\times(10-0)$$

$$= 1(-15)+2(5)+3(10)$$

$$= -15+10+30$$

$$= 25$$

The second Way



For example $A=\begin{pmatrix}1&2&-1\\3&-2&0\\5&0&-5\end{pmatrix}_{3\times3}$ then find det(A)

$$\det(A)=|A| = \begin{vmatrix}1&2&-1\\3&-2&0\\5&0&-5\end{vmatrix}\begin{matrix}1&2&-1\\3&-2&0\\5&0&-5\end{matrix}$$

$= (1)(-2)(-5) + (2)(0)(5) + (-1)(3)(0) - (1)(0)(0) - (2)(3)(-5) -$
$(-1)(-2)(5) = 10 + 0 + 0 - 0 + 30 - 10 = 30$

1- Find the det(A), where A=$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{pmatrix}$

2- Evaluate $|B|$, where $B = \begin{pmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{pmatrix}$

Cofactor Expansion

Let A=(a_{ij}) be an n *by* n matrix and C=(c_{ij}) *be an* $(n-1) \times (n-1)$ sub matrix of A obtained by deleting the i th row and j th column of A.

And the cofactor A_{ij} of a_{ij} is defined by $A_{ij} = (-1)^{i+j} \; |C_{ij}|$. Then

(Expansion of $|A|$ about the ith row) $\quad |A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$

(Expansion of $|A|$ about the jth column) $\quad |A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$

Example 1: Let A = $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$. Evaluate det(A) by cofactor expansion along the first row of A.

Solution: $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$a_{11} = 3, a_{12} = 1, a_{13} = 0$

$A_{ij} = (-1)^{i+j} \; |C_{ij}|,$

$C_{11} = \begin{pmatrix} -4 & 3 \\ 4 & -2 \end{pmatrix}, \qquad C_{12} = \begin{pmatrix} -2 & 3 \\ 5 & -2 \end{pmatrix}, \qquad C_{13} = \begin{pmatrix} -2 & -4 \\ 5 & 4 \end{pmatrix}$

$A_{11} = (-1)^{1+1} \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} = (-1)^2\big((-4)(-2) - (3)(4)\big) = -4$

$A_{12} = (-1)^{1+2} \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} = (-1)^3\big((-2)(-2) - (3)(5)\big) = (-1)(4 - 15) = 11$

$A_{13} = (-1)^{1+3} \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} = (-1)^4\big((-2)(4) - 4\big)(5) = 12$

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}, \; |A| = (3)(-4) + (1)(11) + (0)(12) = -1.$

Example 2 : Let $= \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 2 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix}$. Evaluate det(A) by cofactor expansion

Solution: $|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24}$

$a_{21} = 2,\ a_{22} = 0,\ a_{23} = 0,\ a_{23} = 1$

, $A_{ij} = (-1)^{i+j}\ |C_{ij}|,\ \ C_{21} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}, C_{22} = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix},$

$$C_{23} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix}, \qquad C_{24} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix} \begin{matrix} 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{matrix} = 0 + 9 + 1 - 6 - 2 - 0 = 2$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 3 & 1 \\ 0 & 3 \end{matrix} = 1 + 0 + 27 - 0 - 6 - 6 = 16$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix} = (-1)^3(2) = -2, \quad A_{23} = ?, \quad A_{24} = ?.$$

$$A_{24} = (-1)^{2+4} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = (-1)^6(16) = 16$$

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24}$$
$$= (2)(-2) + (0)A_{22} + (0)A_{23} + (1)(16) = 12$$

Properties of Determinants

1-If a square a matrix A has a row of zero or a column of zeros, then $|A|$= 0

For example $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ then$|A|$= 0.

2- Let A be a square matrix. Then $|A| = |A^t|$

For example A= $\begin{pmatrix} -4 & 3 \\ 2 & -2 \end{pmatrix}$ then $|A| = (-4)(-2) - (3)(2) = 2 = |A^t|$

3- If A is an $n \times n$ traianglar matrix (upper triangular, lower triangular, or diagonal), then $|A|$ is the product of the entries on the main diagonal of the matrix; that is

$|A|=a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}$.

For example i) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 1 & 3 \end{pmatrix}$, $then\ |A| = (1)(2)(2)(3) = 12$

ii) A=$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ $then\ |A| = 6.$

iii) A=$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $then\ |A| = -24.$

4- Let A be an $n \times n$ matrix. If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then det(B) = k det(A).

For example $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 3 & 3 \end{pmatrix}$ $and\ B = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 4 & 6 & 6 \end{pmatrix}$ $then\ |B| = 2|A|$

$$|B| = 2(-8) = -16$$

5- Let A be an $n \times n$ matrix. If B is the matrix that results when two rows or two columns of A is interchanging , then det(B) = − det(A).

For example A=$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -2 & 0 \\ 5 & 0 & -5 \end{pmatrix}_{3\times3}$ then det(A)=30

and B=$\begin{pmatrix} 1 & 2 & -1 \\ 5 & 0 & -5 \\ 3 & -2 & 0 \end{pmatrix}_{3\times3}$ then det(B)=−30

6- If two row(columns) of a matrix A are equal, then det(A)=0.

For example A=$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 2 \\ 5 & 0 & 0 \end{pmatrix}_{3\times3}$ then det(A)=0

7- Let A be any $n \times n$ matrix, then det (kA) = k^n det(A).

For example A= $\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ then det(A)=−4

and det(3A)=$\begin{vmatrix} 3 & 3 \\ 6 & -6 \end{vmatrix} = -36 = (3^2)(-4)$

8- If A and B are square matrices of the same size, then det(A.B) = det(A).det(B).

For example A= $\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ then det(A)=-4

and B= $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ then det(B)=2, $A \cdot B = \begin{pmatrix} 4 & 7 \\ 0 & -2 \end{pmatrix}$ and det($A\cdot B$)=-8

det(A) det(B)=$(2)(-4) = -8 = $ det($A\cdot B$).

Matrix Invers

- ## Singular matrix

A *singular matrix* is a square matrix with a determinant value equal to zero. We cannot find the inverse of a singular matrix. For a singular matrix, |A| = 0.

- ## Non-singular Matrix

A non-singular matrix is a square matrix with a non-zero determinant. To find the inverse of a matrix, the non-singular matrix property must be satisfied. For a non-singular matrix, |A| ≠ 0.

- ## Adjoint of a Matrix

Let A=$[a_{ij}]$ be a square matrix of order n. The adjoint of a matrix A is the transpose of the cofactor matrix of A. It is denoted by adj (A)

Example:

Find the adjoint of the matrix.

$$A=\begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

To find the adjoint of a matrix A, first find the cofactor matrix of the given matrix. Then find the transpose of the cofactor matrix.

Cofactor of $3 = A_{11} = \begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix} = 2$

Cofactor of $1 = A_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 2$

Cofactor of $-1 = A_{13} = \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 6$

Cofactor of $2 = A_{21} = -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1$

Cofactor of $-2 = A_{22} = \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} = -2$

Cofactor of $0 = A_{23} = -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -5$

Cofactor of $1 = A_{31} = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -2$

Cofactor of $2 = A_{32} = -\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -2$

Cofactor of $-1 = A_{33} = \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} = -8$

The cofactor matrix of A is $[A_{ij}] = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}$

Now find the transpose of A_{ij} .

$$adj\,A = (A_{ij})^T$$
$$= \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$$

Definition: A square matrix A is invertible (or nonsingular) if there exists a square matrix B such that $A_{n\times n}B_{n\times n} = I_{n\times n} = B_{n\times n}A_{n\times n}$, such that I is identity matrix and B is inverse of matrix A and denoted by $B = A^{-1}$, that is

$A \;\; A^{-1} = I = A^{-1}A$.

Inverse Matrix Method

The inverse of a matrix can be found using the three different methods. However, any of these three methods will produce the same result.

Method 1:

Let A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The inverse of a matrix A is found using the following formula

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For example A= $\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$, Find A^{-1}

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{(2)(3)-(-2)(-1)} \begin{bmatrix} 3 & -(-2) \\ -(-1) & 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

Method 2:

Let A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The inverse of a matrix A is found using the following formula

$$\mathbf{A}\ A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

Example: Find the inverse of the matrix A= $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Solution: Since $A\,A^{-1}=I$ then suppose that $A^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$,A\,A^{-1}=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a+c & b+d \\ -a+c & -b+d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $a+c=1$ $\qquad b+d=0$

$.\quad \underline{-a+c=0}$ $\qquad \underline{-b+d=1}$

$2c=1 \Rightarrow c=\frac{1}{2}$ $\qquad 2d=1 \Rightarrow d=\frac{1}{2}$

$a=\frac{1}{2}$ $\qquad b=-\frac{1}{2} \Rightarrow A^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$

Properties

A few important properties of the inverse matrix are listed below.

- If A is nonsingular, then $(A^{-1})^{-1}=A$
- If A and B are nonsingular matrices, then AB is nonsingular. Thus,
- $(AB)^{-1}=B^{-1}A^{-1}$
- If A is nonsingular then $(A^{-1})^T=(A^T)^{-1}$.
- If the inverse of the matrix A exists then it is unique.

Method 3:

One of the most important methods of finding the matrix inverse involves finding the determinants and cofactors of elements of the given matrix. Observe the below steps to understand this method clearly.

- The inverse matrix is also found using the following equation:

$$A^{-1}=\frac{1}{det(A)}\,adj(A).$$

where *adj(A)* refers to the adjoint of a matrix A, *det(A)* refers to the determinant of a matrix A.

1- Find determinant of A.
2- Find the cofactor of each element of A and arrange them in matrix C(A)
3- Find the transpose of C(A) (this new matrix is called *adjoint* of A and denoted by adj(A)), where adj(A)=$(C(A))^t$
4- Using this formula $A^{-1} = \frac{1}{det(A)}\, adj(A)$.

Example: Find the inverse of the matrix A= $\begin{pmatrix} 4 & 3 \\ -3 & -1 \end{pmatrix}$ by using *adjoint*.

Solution: We can check that $det(A) = 5$

Thus the cofactors of A are

$A_{11} = -1$ $A_{12} = 3$

$A_{21} = -3$ $A_{22} = 4$,

, $C(A) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -3 & 4 \end{pmatrix}$

$adj(A) = (C(A))^t = \begin{pmatrix} -1 & -3 \\ 3 & 4 \end{pmatrix}$.

$A^{-1} = \frac{1}{det(A)}\, adj(A)$

$= \frac{1}{5}\begin{pmatrix} -1 & -3 \\ 3 & 4 \end{pmatrix}$.

Example. Find the inverse of the matrix $\begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$ by using adjoint

Solution: We can check that det(A) = 64. Thus The cofactors of A are

$A_{11} = 12$ $A_{12} = 6$ $A_{13} = -16$

$A_{21} = 4$ $A_{22} = 2$ $A_{23} = 16$

$A_{31} = 12$ $A_{32} = -10$ $A_{33} = 16$

So the matrix C(A)=$\begin{pmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}$

and the adjoint of A is $adj(A) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$

$$A^{-1} = \frac{1}{64}\begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix} = \begin{pmatrix} 12/64 & 4/64 & 12/64 \\ 6/64 & 2/64 & -10/64 \\ -16/64 & 16/64 & 16/64 \end{pmatrix}$$

Example: Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

Exercise: 1- Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{pmatrix}$ Find the inverse of A.

2- Let $A = \begin{pmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{pmatrix}$ Find all the cofactors.