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Linear Algebra

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Chapter Two

System of Linear Equations

Definition 2.1. The equation of the straight line in the xy-plane can be represented algebraically by an equation of the form

$$a_1x + a_2y = b$$

Where a_1, a_2 *and* b are real constants and $a_1,$ *and* a_2 are not both zero. An equation of this form is called a linerar equation in the variables x and y.

Example. the equations

$2x + 5y = 7$

Definition 2.2. A linear equation in the n variables $x_1, x_2, \dots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

Where $a_1, a_2, \dots, a_n$ *and* b are real constants. The variables in a linear equation are sometimes called unknown.

Example. the equations

$y = \frac{1}{2}x + 3z + 1$ and $x_1 - 2x_2 - 3x_3 + x_4 = 7$ are linear.

But the equations

$x + 3\sqrt{y} = 5,$ $3x + 2y - z - xz = 4$ and $y = \sin x$ are not linear.

Definition 2.3. A finite set of linear equations in the variables $x_1, x_2, \dots, x_n$ is called a system of linear equations or linear system. An arbitrary system of m linear equations in n unknowns can be written as

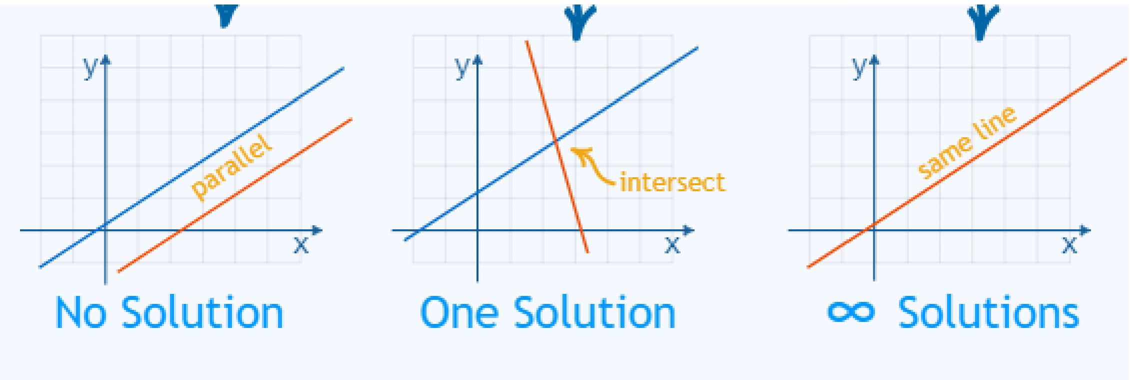
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

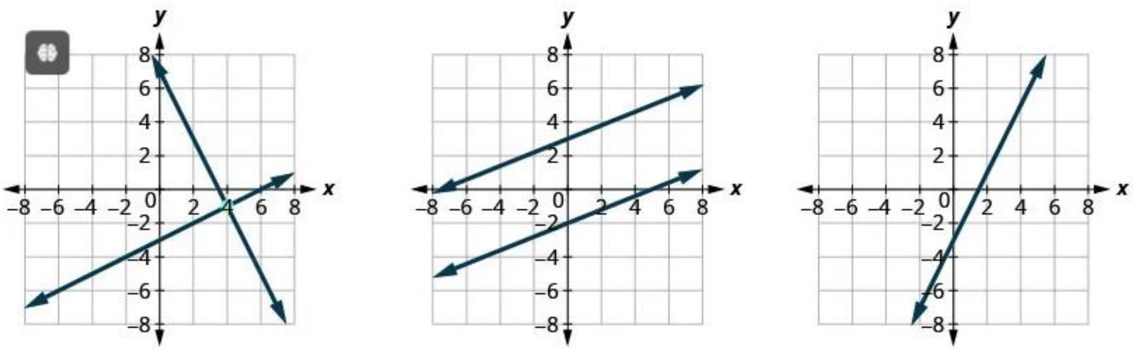
where $x_1, x_{2,\dots,} x_n$ are the unknowns, $a_1, a_{12,\dots,} a_{mn}$ are the coefficients of the system, and $b_1, b_{2,\dots,} b_m$ are the constant terms.

Solution of linear system: A solution of a linear system is an assignment of values to the variables $x_1, x_{2,\dots,} x_n$ such that each of the equations is satisfied. The set of all possible solutions is called the *solution set*

Remark: A linear system may behave in any one of three possible ways:

1. The system has *infinitely many solutions*.
2. The system has a *unique solution*.
3. The system has *no solution*.





The lines intersect. Intersecting lines have one point in common. There is one solution to this system.

The lines are parallel. Parallel lines have no points in common. There is no solution to this system.

Both equations give the same line. Because we have just one line, there are infinitely many solutions.

Matrix Form for a system of linear equations

The system

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{array}\right\} \quad \cdots\cdots\cdots (1)$$

can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad (2)$$

If we set A= $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$ is a coefficient matrix of system (1) , X= $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and

B= $\begin{bmatrix} b \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, then system(1) becomes: AX=B,

The matrix $\begin{bmatrix} a_{11} & \cdots & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & \ddots & & & \vdots & & \\ a_{m1} & \cdots & a_{m2} & \cdots & a_{mn} & \vdots & b_n \end{bmatrix}$ is called augmented matrix of system (1).

Methods for solving $n \times n$ system of linear equation

1- Gaussian –Jordan elimination **(row reduction)**

Operations for used to solve systems of linear equations.

These operations correspond to the following operations on the rows of the augmented matrix.

a- Interchange two rows

b- Multiply a row through by a nonzero constant.

c- Add a multiple of one row to another row.

Example. Using **Gaussian –Jordan elimination** to solve the system of linear equations

$$\boldsymbol{x + y + 2z = 9}$$

$$\boldsymbol{2x + 4y - 3z = 1}$$

$$\boldsymbol{3x + 6y - 5z = 0}$$

Solution: First we find the augmented matrix of this system

$$\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 2 & 4 & -3 & \vdots & 1 \\ 3 & 6 & -5 & \vdots & 0 \end{pmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 2 & 4 & -3 & \vdots & 1 \\ 3 & 6 & -5 & \vdots & 0 \end{pmatrix}$$

Add -2 times the first row to the second to obtain

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 2 & 4 & -3 & \vdots & 1 \\ 3 & 6 & -5 & \vdots & 0 \end{pmatrix} \Rightarrow -2R_1 + R_2 \Rightarrow R_2$$

Add -3 the first row to the third to obtain

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 2 & -7 & \vdots & -17 \\ 3 & 6 & -5 & \vdots & 0 \end{pmatrix} \Rightarrow -3R_1 + R_3 \Rightarrow R_3,$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 2 & -7 & \vdots & -17 \\ 0 & 3 & -11 & \vdots & -27 \end{pmatrix} \Rightarrow$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 2 & -7 & \vdots & -17 \\ 0 & 3 & -11 & \vdots & -27 \end{pmatrix} \Rightarrow \frac{1}{2}R_2$$

Add -3 times the second row to the third to obtain

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 1 & \frac{-7}{2} & \vdots & \frac{-17}{2} \\ 0 & 3 & -11 & \vdots & -27 \end{pmatrix} \Rightarrow -3R_2 + R_3 \Rightarrow R_3$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 1 & \frac{-7}{2} & \vdots & \frac{-17}{2} \\ 0 & 0 & \frac{-1}{2} & \vdots & \frac{-3}{2} \end{pmatrix} \Rightarrow$$

$$\boldsymbol{x + y + 2z = 9}$$

$$\boldsymbol{y - \frac{7}{2}z = \frac{-17}{2}}$$

$$\boldsymbol{-\frac{1}{2}z = -\frac{3}{2}}$$

Thus the solution of the above system is $z = 3$, $y = 2$ *and* $x = 1$.

2- Gramer's rule

If $AX = B$ is a system of n linear equations in n unknowns such that det(A) ≠0, then the system has a unique solution. This solution is

$$\mathrm{x}_1 = \frac{\det(A_1)}{\det(A)}, \quad \mathrm{x}_2 = \frac{\det(A_2)}{\det(A)}, \ \dots, \quad \mathrm{x}_n = \frac{\det(A_n)}{\det(A)},$$

Where A_j is the matrix obtained by replacing the entries in the jth column of A by entries in the matrix B.

Example. Using Cramer rule to solve the system of linear equations

$$\boldsymbol{x - y + 2z = 2}$$

$$\boldsymbol{-2x + y + 3z = 3}$$

$$\boldsymbol{3x - 2y + 4z = 1}$$

Solution.

$$\boldsymbol{A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & -2 & 4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 2 \\ -2 & 3 & 3 \\ 3 & 1 & 4 \end{pmatrix},}$$

$$A_3 = \begin{pmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-2} & \mathbf{1} & \mathbf{3} \\ \mathbf{3} & \mathbf{-2} & \mathbf{1} \end{pmatrix}$$

Therefore, the determinant of each of A, A_1, A_2 and A_3 are:

$\det(A)= \qquad = -5$

, $\det(A_1) = \qquad = 15$

, $\det(A_2) = \qquad = 21$

and $\det(A_3)= \qquad = 2$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{15}{-5} = -3$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{21}{-5},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{2}{-5}$$

3- Matrix inversion method:

In this way, we multiply both sides of the system AX=B by A^{-1} we get the solution $X=A^{-1}B$ of the system.

Example: Using matrix invertible to solve the system of linear equations

$$x + 2y + 3z = 5$$

$$2x + 5y + 3z = 3$$

$$x + 8z = 17$$

Solution. In matrix form this system can be written as $AX = B$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Now $X = A^{-1}B = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

The solution of the above system is $x_1 = 1,\ x_2 = -1,\ x_3 = 2$