Roots of NON- linear Equations

What is meant by the nonlinear equation: It is that equation which contains different powers for x or triangular functions or exponential functions or logarithmic.

Root of an equation

For an equation f(x) =0 to find the solution we find such value which satisfy the equation f(x)=0,these values are known as the roots of the equation .
A value a is known as the root of an equation f(x) =0 if and only if f (a) =0.

The equation : $x^4 - 3x^3 - 7x^2 + 15x = -18$ has four roots : $-2,\ 3,\ 3,$ *and* -1.

i.e., $x^4 - 3x^3 - 7x^2 + 15x + 18 = (x+2)(x-3)^2(x+1)$

Solution Methods

Several ways to solve nonlinear equations are possible:

1. Analytical Solutions Possible for special equations only.
2. Graphical Solutions Useful for providing initial guesses for other methods.
3. Numerical Solutions need initial point to start.

1-Analytical Methods

Analytical Solutions are available for special equations only.

Analytical solution of : $a x^2 + b x + c = 0$

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No analytical solution is available for: $x - e^{-x} = 0$

2. Graphical Methods

This is the simplest method to determine the root of an equation $f(x) = 0$. The procedure is quite straightforward:
- Plot the function $f(x)$
- Observe when it crosses the x-axis, this point represents the value for which $f(x) = 0$.

Note 1: This will provide only a rough approximation of the root.
Note 2: you can remark that the function has changed sign after the root.

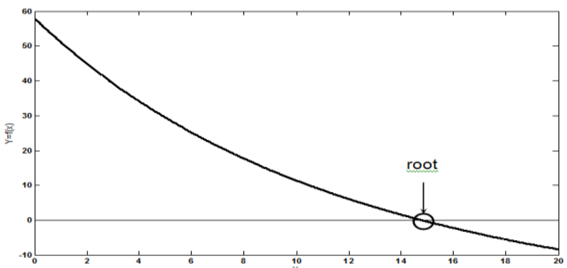
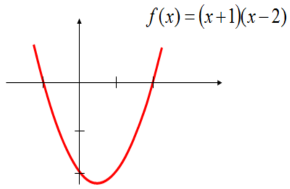


Figure. graphical method.

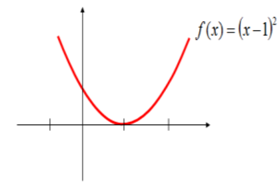
Simple Zeros



$f(x)=(x+1)(x-2)=x^2-x-2$

has two simple zeros (one at x = 2 and one at x = −1)

Multiple Zeros



$f(x)=(x-1)^2=x^2-2x+1$

has double zeros (zero with muliplicity = 2) at x = 1

Locating the position of roots (programming method):

To locate the position of roots of the function (equation) f(x)=0 by using ***programming method,*** let f(x) be continuous function on the interval [a,b]. We divide the interval [a,b] into n subintervals $a=x_0<x_1<\ldots<x_{n-1}<x_n=b$ where $x_i=a+ih$, i=0, 1, … ,n; $h=\frac{b-a}{n}$. If $f(x_i)\times f(x_{i+1})<0$ for any $0\le i\le n$, then there exits c, a<c<b for which f(c)=0.

Example 1:

Find the approximate location of the function

1. $f(x)=x^4-7x^3+3x^2+26x-10=0$ on the interval [-8,8] with n=4 and n=8..
2. $f(x)=x^3+4x^2-10=0$ on the interval [1,2] with n=5.
3. $f(x)=x^3-4x+1=0$ on the interval [-1,4] with n=5.

Solution: (1): Let n=4, $h=\frac{b-a}{n}=\frac{8-(-8)}{4}=4$

x	-8	-4	0	4	8
f(x)	+	+	-	-	+

There is a root between (-4,0) and (4,8).

If n=8, h=2:

x	-8	-6	-4	-2	0	2	4	6	8
f(x)	+	+	+	+	-	+	-	+	+

There is a root between (-2,0), (0,2), (2,4) and (4,6).

Solution: (2): Let n=5, h=0.2

x	1	1.2	1.4	1.6	1.8	2
f(x)	-	-	+	+	+	+

There is a root between (1.2,1.4).

Solution: (3): Let n=5, h=1

x	-1	0	1	2	3	4
f(x)	+	+	-	+	+	+

There is a root between (0,1) and (1,2).

3. Numerical Solutions

A. Bisection method (bracketing methods)

One of the first numerical methods developed to find the root of a nonlinear equation $f(x)=0$ was the bisection method (also called Binary-Search method). The method is based on the following theorem:

Theorem: An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_1 and x_2 if $f(x_1)f(x_2)<0$.
Note that if $f(x_1)f(x_2)>0$, there may or may not be any root between x_1 and x_2 If $f(x_1)f(x_2)<0$, then there may be more than one root between x_1 and x_2 .

Since the method is based on finding the root between two points, the method falls under the category of bracketing methods.

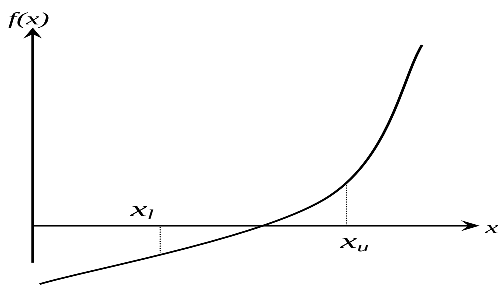


Figure. At least one root exists between two points if the function is real, continuous, and changes sign.

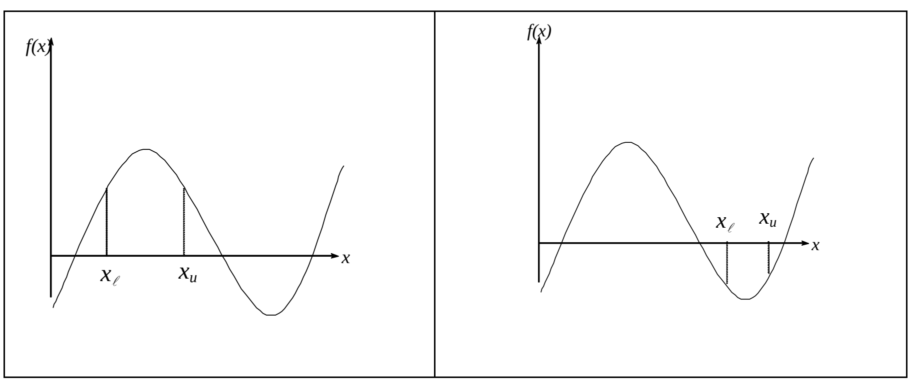


Figure. If the function $f(x)$ does not change sign between two points, there may not be any roots $f(x)=0$ between the two points.

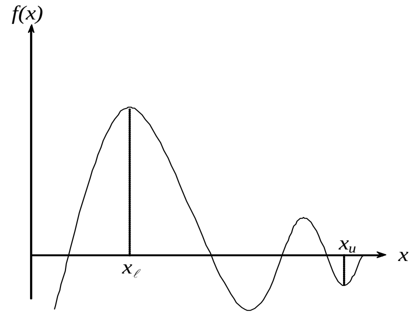


Figure. If the function $f(x)$ changes sign between two points, more than one root for $f(x)=0$ may exist between the two points.

- Multiple roots

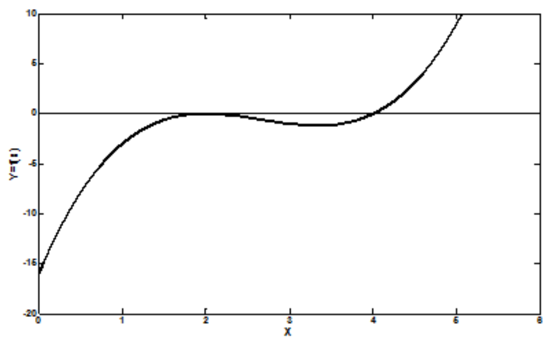


Figure. Multiple roots.

- Discontinuous functions

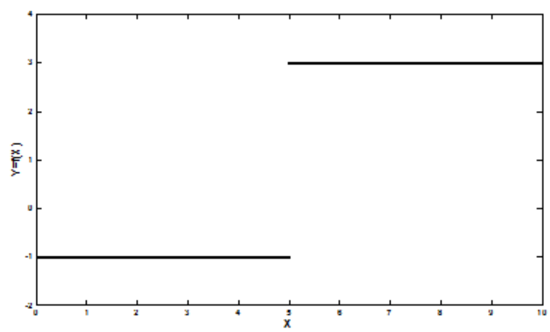
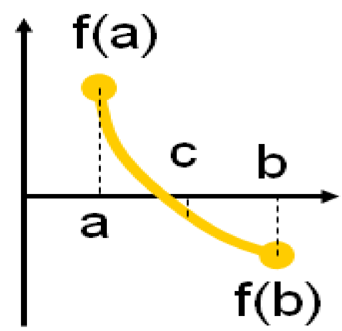


Figure. Discontinuous function.

Assumptions:
- f(x) is continuous on [a,b]
- f(a) f(b) < 0

Algorithm:



Loop

1. Compute the mid point c=(a+b)/2
2. Evaluate f(c),if f(c)=0, c is the root.
3. If f(a) f(c) < 0 then new interval [a, c]
   If f(a) f(c) > 0 then new interval [c, b]

End loop

Example

Can you use Bisection method to find a zero of :
$f(x)=x^3-3x+1$ in the interval [0,2]?

Answer:

$f(x)$ is continuous on [0,2]
and $f(0)*f(2)=(1)(3)=3>0$
$\Rightarrow$ Assumptions are not satisfied
$\Rightarrow$ Bisection method can not be used

Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval $[0,1]$?

Answer:

$f(x)$ is continuous on $[0,1]$

and $f(0) * f(1) = (1)(-1) = -1 < 0$

$\Rightarrow$ Assumptions are satisfied

$\Rightarrow$ Bisection method can be used

stopping criteria Two common stopping criteria

1. Stop after a fixed number of iterations
2. Stop when the absolute error is less than a specified value

Convergence Analysis

Given $f(x)$, a, b, *and* ε

How many iterations are needed such that: $|a - b| \le \varepsilon$

where a is the zero of *f(x)* and b is the bisection estimate (i.e., $b = c_k$)?

$$n \ge \frac{\log(b-a) - \log(\varepsilon)}{\log(2)}$$

Example

$a = 6,\ b = 7,\ \varepsilon = 0.0005$

How many iterations are needed such that: $|a - b| \le \varepsilon$?

$$n \ge \frac{\log(b-a) - \log(\varepsilon)}{\log(2)} = \frac{\log(1) - \log(0.0005)}{\log(2)} = 10.9658$$

$$\Rightarrow n \ge 11$$

Example:

- Find the root of $f(x) = x^2 - 7$ in the interval $[1, 4]$

n Iteration	a_n	b_n	c_n	$f(a_n)$	$f(b_n)$	$f(c_n)$	E_i
1	1	4	2.5	-6	9	-0.75	1.5
2	2.5	4	3.25	-0.75	9	3.56	0.75
3	2.5	3.25	2.875	-0.75	3.56	1.266	0.375
4	2.5	2.875	2.6875	-0.75	1.266	0.2226	0.1875
5	2.5	2.6875	2.59375	-0.75	0.226	-0.27246	0.0937

Example : Find the root of:

$f(x) = x^3 - 3x + 1$ in the interval: $[0,1]$

* $f(x)$ is continuous

* $f(0) = 1, \; f(1) = -1 \Rightarrow f(a)\, f(b) < 0$

$\Rightarrow$ Bisection method can be used to find the root

Iteration	a	b	c= $\frac{(a+b)}{2}$	f(c)
1	0	1	0.5	-0.375
2	0	0.5	0.25	0.266
3	0.25	0.5	.375	-0.0723

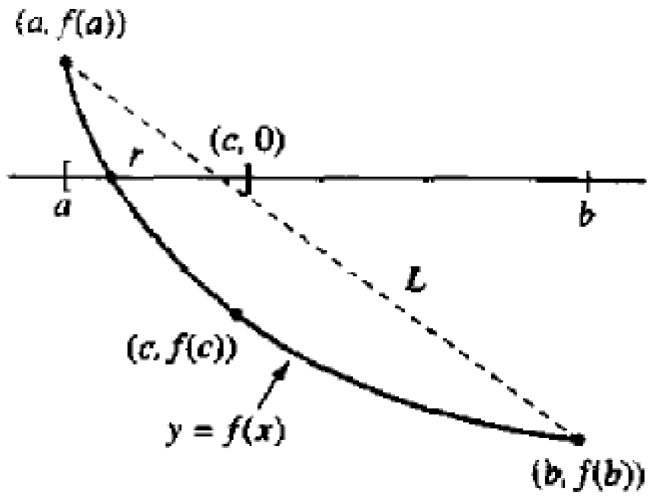
Example for $y = x - \cos(x)$ on [0.0,4.0] for eps = 1.0e-6

i	a	f(a)	b	f(b)	c	f(c)
1	0.00000	-1.00000	4.00000	4.65364	2.00000	2.41615
2	0.00000	-1.00000	2.00000	2.41615	1.00000	0.45970
3	0.00000	-1.00000	1.00000	0.45970	0.50000	-0.37758
4	0.50000	-0.37758	1.00000	0.45970	0.75000	0.01831
5	0.50000	-0.37758	0.75000	0.01831	0.62500	-0.18596
6	0.62500	-0.18596	0.75000	0.01831	0.68750	-0.08533
7	0.68750	-0.08533	0.75000	0.01831	0.71875	-0.03388
8	0.71875	-0.03388	0.75000	0.01831	0.73438	-0.00787
9	0.73438	-0.00787	0.75000	0.01831	0.74219	0.00520
10	0.73438	-0.00787	0.74219	0.00520	0.73828	-0.00135
11	0.73828	-0.00135	0.74219	0.00520	0.74023	0.00192
12	0.73828	-0.00135	0.74023	0.00192	0.73926	0.00029
13	0.73828	-0.00135	0.73926	0.00029	0.73877	-0.00053
14	0.73877	-0.00053	0.73926	0.00029	0.73901	-0.00012
15	0.73901	-0.00012	0.73926	0.00029	0.73914	0.00008
16	0.73901	-0.00012	0.73914	0.00008	0.73907	-0.00002
17	0.73907	-0.00002	0.73914	0.00008	0.73911	0.00003
18	0.73907	-0.00002	0.73911	0.00003	0.73909	0.00001
19	0.73907	-0.00002	0.73909	0.00001	0.73908	-0.00000
20	0.73908	-0.00000	0.73909	0.00001	0.73909	0.00000
21	0.73908	-0.00000	0.73909	0.00000	0.73908	-0.00000
22	0.73908	-0.00000	0.73909	0.00000	0.73909	0.00000

iterations	root
22	0.73909

B- False Position Method

Another popular algorithm is the method of false position or the regula false method. It was developed because the bisection method converges at a fairly slow speed. As before, we assume that $f(a)$ and $f(b)$ have opposite signs. The bisection method used the midpoint of the point [a,b] as the next iterate. A better approximation is obtained if we find the point $(c, 0)$ where the line L joining the point $(a, f(a))$ and $(b, f(b))$ crosses the x-axis.



$$c = b - \frac{f(b)(b-a)}{f(b) - f(a)}.$$

The three possibilities are the same as before:

(a) If $f(a)$ and $f(c)$ have opposite signs, a zero lies in $[a, c]$.

(b) If $f(c)$ and $f(b)$ have opposite signs, a zero lies in $[c, b]$.

(c) If $f(c)$ =0, then zero is c.

Example for y = x - cos(x) on [0.0,4.0] for eps = 1.0e-6

i	a	f(a)	b	f(b)	c	f(c)
1	0.00000	-1.00000	4.00000	4.65364	0.70751	-0.05248
2	0.70751	-0.05248	4.00000	4.65364	0.74422	0.00861
3	0.70751	-0.05248	0.74422	0.00861	0.73905	-0.00006
4	0.73905	-0.00006	0.74422	0.00861	0.73909	-0.00000
5	0.73909	-0.00000	0.74422	0.00861	0.73909	-0.00000
6	0.73909	-0.00000	0.74422	0.00861	0.73909	-0.00000
7	0.73909	-0.00000	0.74422	0.00861	0.73909	-0.00000
8	0.73909	-0.00000	0.74422	0.00861	0.73909	0.00000

iterations	root
8	0.73909

Note: radians and degrees are both units of measurement of the unit circle.

radians = (degrees x pi) / 180

degrees = (radians x 180) / pi

Example : Use the false position method to find the root of $x\sin(x) - 1 = 0$ that is located in the interval [0, 2]. (Use radians unit)

Starting with $a_0 = 0$ and $b_0 = 2$, we have $f(0) = -1.00000000$ and $f(2) = 0.81859485$, so a root lies in the interval [0, 2].

$c_0 = 2 - \frac{0.81859485(2-0)}{0.81859485-(-1)} = 1.09975017$ and $f(c_0) = -0.02001921.$

The function changes sign on the interval $[c_0, b_0] = [1.09975017, 2]$, so we squeeze from the left and set $a_1 = c_0$ and $b_1 = b_0$. the next approximation:

$c_1 = 2 - \frac{0.81859485(2-1.09975017)}{0.81859485-(-0.02001921)} = 1.12124074$

and $f(c_1) = 0.00983461.$

Next $f(x)$ changes sign on $[a_1, c_1] = [1.09975017, 1.12124074]$, and the next decision is to squeeze from the right and set $a_2 = a_1$ and $b_2 = c_1$. A summary of the calculations is gives in Table below

Table: False Position Method Solution of $x\sin(x) - 1 = 0$

K	a	b	c	Function value, $f(c)$
1	0.00000000	2.00000000	1.09975017	– 0.02001921
2	1.09975017	2.00000000	1.12124074	0.00983461
3	1.09975017	1.12124074	1.11416120	0.00000563
4	1.09975017	1.11416120	1.11415714	0.00000000