Numerical Analysis

Lagrange Method of Interpolation

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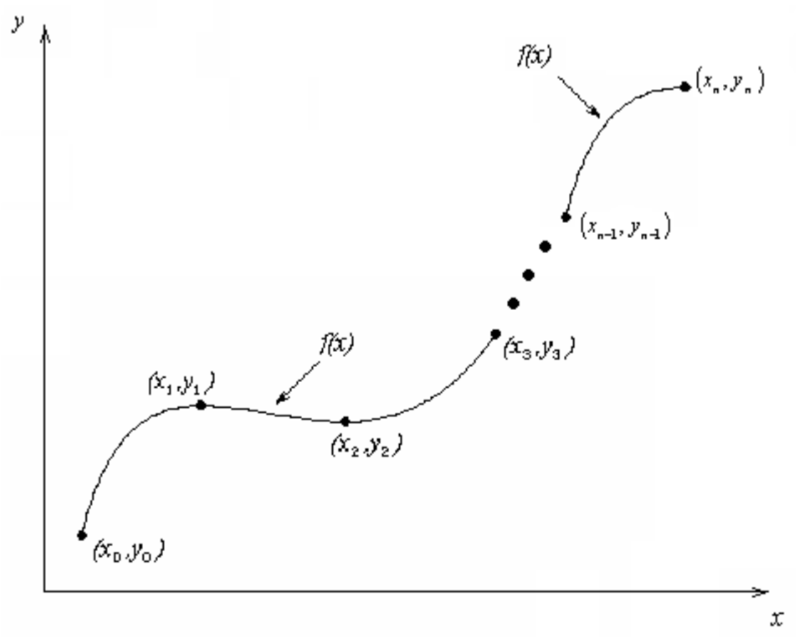
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where ‘ n ’ in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$

given at $(n+1)$ data points as $x_0, y_0, x_1, y_1, \ldots\ldots, x_{n-1}, y_{n-1}, x_n, y_n$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Linear interpolation

For $n = 1$, we have the data

x	x_0	x_1
$f(x)$	$f(x_0)$	$f(x_1)$

The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}, \; l_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}.$$

The Lagrange linear interpolation polynomial is given by

$$P_1(x) = l_0(x) \, f(x_0) + l_1(x) \, f(x_1).$$

Example *Using the data* $\sin(0.1) = 0.09983$ *and* $\sin(0.2) = 0.19867$, *find an approximate value of* $\sin(0.15)$ *by Lagrange interpolation. Obtain a bound on the error at* $x = 0.15$.

Solution We have two data values. The Lagrange linear polynomial is given by

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

$$= \frac{(x - 0.2)}{(0.1 - 0.2)} (0.09983) + \frac{(x - 0.1)}{(0.2 - 0.1)} (0.19867).$$

$$f(0.15) = P_1(0.15) = \frac{(0.15 - 0.2)}{(0.1 - 0.2)} (0.09983) + \frac{(0.15 - 0.1)}{(0.2 - 0.1)} (0.19867)$$

$$= (0.5)\,(0.09983) + (0.5)\,(0.19867) = 0.14925.$$

Quadratic interpolation

For $n = 2$, we have the data

x	x_0	x_1	x_2
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$

The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)},\ l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)},\ l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

The Lagrange quadratic interpolation polynomial is given by

$$P_1(x) = l_0(x)\, f(x_0) + l_1(x)\, f(x_1) + l_2(x)\, f(x_2).$$

Example *Use Lagrange's formula, to find the quadratic polynomial that takes the values*

x	0	1	3
y	0	1	0

Solution Since $f_0 = 0$ and $f_2 = 0$, we need to compute $l_1(x)$ only. We have

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x-3)}{(1)(-2)} = \frac{1}{2}\,(3x - x^2).$$

The Lagrange quadratic polynomial is given by

$$f(x) = l_1(x)\, f(x_1) = \frac{1}{2}\,(3x - x^2)\,(1) = \frac{1}{2}\,(3x - x^2).$$

Example *Construct the Lagrange interpolation polynomial for the data*

x	-1	1	4	7
$f(x)$	-2	0	63	342

Hence, interpolate at $x = 5$.

Solution The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{(x - 1)(x - 4)(x - 7)}{(-1 - 1)(-1 - 4)(-1 - 7)}$$

$$= -\frac{1}{80}(x^3 - 12x^2 + 39x - 28).$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{(x + 1)(x - 4)(x - 7)}{(1 + 1)(1 - 4)(1 - 7)}$$

$$= \frac{1}{36}(x^3 - 10x^2 + 17x + 28).$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = \frac{(x + 1)(x - 1)(x - 7)}{(4 + 1)(4 - 1)(4 - 7)}$$

$$= -\frac{1}{45}(x^3 - 7x^2 - x + 7).$$

$$l_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{(x + 1)(x - 1)(x - 4)}{(7 + 1)(7 - 1)(7 - 4)}$$

$$= \frac{1}{144}(x^3 - 4x^2 - x + 4).$$

Note that we need not compute $l_1(x)$ since $f(x_1) = 0$.

The Lagrange interpolation polynomial is given by

$$P_3(x) = l_0(x)\, f(x_0) + l_1(x)\, f(x_1) + l_2(x)\, f(x_2) + l_3(x)\, f(x_3)$$

$$= -\frac{1}{80}(x^3 - 12x^2 + 39x - 28)(-2) - \frac{1}{45}(x^3 - 7x^2 - x + 7)(63) + \frac{1}{144}(x^3 - 4x^2 - x + 4)(342)$$

$$= \left(\frac{1}{40} - \frac{7}{5} + \frac{171}{72}\right)x^3 + \left(-\frac{3}{10} + \frac{49}{5} - \frac{171}{18}\right)x^2 + \left(\frac{39}{40} + \frac{7}{5} - \frac{171}{72}\right)x + \left(-\frac{7}{10} - \frac{49}{5} + \frac{171}{8}\right)$$

$$= x^3 - 1.$$

Hence, $f(5) = P_3(5) = 5^3 - 1 = 124.$