Chapter Two

Matrix Algebra

1. Transpose

If A is square matrix Then

I. $(A')' = A$

II. $(A + B)' = A' + B'$

III. If $A'A = AA' \Rightarrow$ Then A is a symmetric matrix, for instance:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$$

IV. If $AA' = 0 \Rightarrow A = 0$

V. $(AB)' = B'A'$

2. Multiplication

I. $AI = IA = A$, what is I matrix?

II. $A0 = 0A = 0$, what is 0 matrix?

III. In general, $AB \neq BA$

But $AB = BA$ if :

- $A = B$ or $B = A$
- A or B is identity matrix
- If A or B is $zero$ Matrix
- If $A = B^{-1}$ or $B = A^{-1}$
- If A or B is diagonal matrix, for instance

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

IV. If D_1 and D_2 are two diagonal matrices of all the **same order** then

$D_1 D_2 = D_2 D_1$

3. Determinants

Def^n: For any square matrix $A_{n.n}$,then the determinant of A ($|A|$) is defined by:

$$|\boldsymbol{A}| = \sum a_{ij} A_{ij}$$

Where A_{ij} is the cofactor of a_{ij} which is equal to $A_{ij} = (-1)^{i+j} \times \text{ minor}$

The minor of element a_{ij} is the determinant of the sub matrix A obtained by deleting the ith row and the jth column of A.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Arrow Method to Find Determinants:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$|\boldsymbol{A}| = (a_{11}.a_{22}.a_{33}) + (a_{12}.a_{23}.a_{31}.) + (a_{13}.a_{21}.a_{32}) - (a_{13}.a_{22}.a_{31}) - (a_{11}.a_{23}.a_{32}) - (a_{12}.a_{21}.a_{33})$$

Theorems about the properties of determinants:

1.The Determinates of a diagonal matrix or identity matrix is the product of diagonal elements.

2. Let A be $(n * n)$ matrix, then B obtained from A by multiply row (or column) of A by a scalar C, then $|B| = C|A|$

3.If B obtained from A by interchanging two rows or columns, Then $|B| = -|A|$

4.If a row or column of a square matrix is zero, then the determinant is zero.

5.If two rows or columns in A are similar, Then $|A| = 0$

6.If A has an inverse, then $|A^{-1}| = |A|^{-1} = \frac{1}{|A|}$

7.If A and B have determinants and in the same order, Then $|AB| = |A||B|$

4. Matrix Inverse

$$A^{-1} = \frac{adj(A)}{\det(A)} = \frac{\left[Cof(a_{ij})\right]'}{|A|} = \frac{\left[(-1)^{(i+j)} \times M_{ij}\right]'}{|A|} \quad \text{if} \quad |A| \neq 0$$

where, $M_{ij} = minor(a_{ij})$

Example: find A^{-1} for the matrix A

$$A = \begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}$$

Solution:

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$Adj(A) = Adj \begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -4 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 2 \\ 4 & 5 \end{vmatrix} = (2 * 5) - (2 * 4) = 2$$

$$A^{-1} = \frac{\begin{bmatrix} 5 & -2 \\ -4 & 2 \end{bmatrix}}{2}$$

$$A^{-1} = \begin{bmatrix} 2.5 & -1 \\ -2 & 1 \end{bmatrix}$$

- $(A^{-1})^{-1} = A$
- If A and B have Inverse and have the same order Then

  $(AB)^{-1} = B^{-1}A^{-1}$
- If K is $non\ zero$ scalar and A has an inverse then

  $(KA)^{-1} = \frac{1}{K}A^{-1}$

5. Orthogonal Matrix

Def[n]: The square matrix A is an Orthogonal if: $A^{-1} = A'$. An example of this kind of matrices is as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A^{-1} = A^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

- If A is an Orthogonal matrix then A exists and is Orthogonal

  Proof:

  $A^{-1} = A'$

  $(A^{-1})^{-1} = (A')^{-1}$

  $A = (A^{-1})^{-1}$

  $A = A$
- If A and B are two Orthogonal matrix and have the same order, then $(A.B)$ is an Orthogonal.

  Proof:

  $A^{-1} = A'\ , B^{-1} = B'$

  $(AB)^{-1} = (AB)'$

  $(AB)^{-1} = B^{-1}A^{-1} \qquad \because A, B\ are\ Orthogonal$

  $(AB)' = B'A'$

  $\therefore B^{-1}A^{-1} \quad = B'A'$
- The Determinant of an Orthogonal matrix is either (+1) or (-1)

6. Idempotent Matrix

If A is square matrix of order n, then is called Idempotent matrix if:

$A^2 = A.A = A$

Example: $B = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$ Show that B is idempotent matrix?

Solution:

$B.B = B$

$$B.B = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} . \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

$$B.B = \begin{bmatrix} \frac{16}{25}+\frac{4}{25} & -\frac{8}{25}-\frac{2}{25} \\ -\frac{8}{25}-\frac{2}{25} & \frac{4}{25}+\frac{1}{25} \end{bmatrix}$$

$$B.B = \begin{bmatrix} 20/25 & -10/25 \\ -10/25 & 5/25 \end{bmatrix}$$

$$B.B = B = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

7. **Trace of Matrix**

If $A = (a_{ij})$ is a square matrix of order n, then the trace of A is:

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \; ; \; Tr(A) = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

- If A and B are two matrices of order n and let C_1 and C_2 be two Scalar, then:
  $tr(C_1A + C_2B) = C_1tr(A) + C_2tr(B)$

- If A and B are two matrices of order n such that (AB) is defined a square matrix, then:
  $tr(AB) = tr(BA)$
- Let A a square matrix of order n , and let C is non-singular matrix $(|C| \neq 0)$, then: $tr(C^{-1}AC) = tr(A)$
  Proof:
  $tr(C^{-1}AC) = tr(AC^{-1}C) = tr(A)$
  And if C is an Orthogonal matrix then
  $tr(C'AC) = tr(AC'C)$
  $= tr(AC^{-1}C)$
  $= tr(AI) = tr(A)$

Example: Let $\underline{X} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, show that $\left(\frac{XX'}{X'X}\right)^2$ $is\ an$ Idempotent matrix

Solution: $\underline{X} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\underline{X}' = [1 \quad 2 \quad -1]$

$$XX' = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = (1 + 4 + 1) = 6$$

$$\frac{XX'}{X'X} = \frac{\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}}{6} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{-1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix}$$

$$\left(\frac{XX'}{X'X}\right)^2 = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{-1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{-1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{-1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix} \therefore \text{ is idempotent}$$

HW1

Q1/ If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11} and A_{22} are non-singular matrix ($|A_{11}| \neq 0, |A_{22}| \neq 0$), find a matrix $C = \begin{bmatrix} I & 0 \\ C_{21} & I \end{bmatrix}$ such that $CA = \begin{bmatrix} A_{11} & A_{12} \\ 0 & B \end{bmatrix}$

Use the Result to show that $|A| = |A_{11}||A_{22} - A_{21}(A_{11})^{-1}A_{12}|$

Q2/ Find Determinant, Inverse, Eigenvalues and eigenvectors of the following:

$A = \begin{bmatrix} 7 & 6 \\ 6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & 4 \\ 2 & -1 & 2 \\ 4 & 2 & 2 \end{bmatrix}$,

Q3/ Find value of K for the following matrix:

$$A = \begin{bmatrix} 1 & -2 & k \\ k & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$