Chapter Three

SPSS Syntax

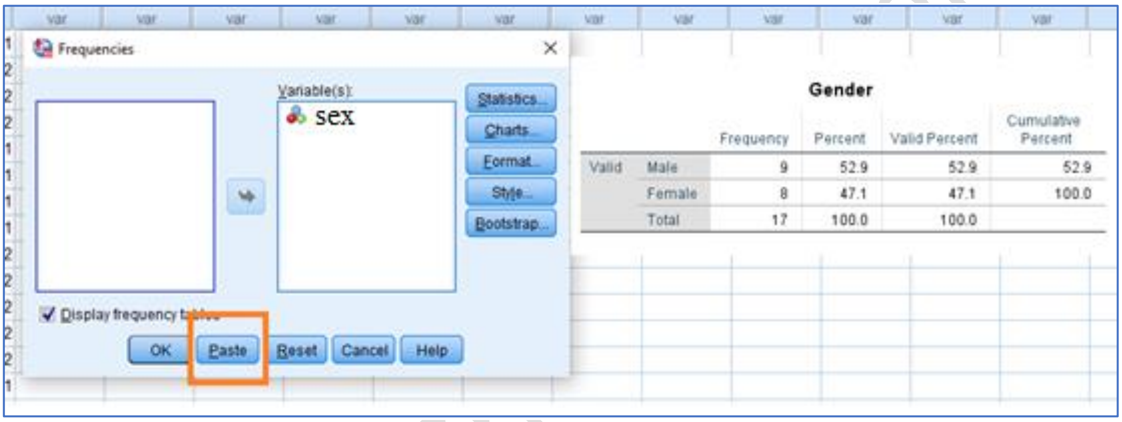
<u>SPSS *syntax*</u>

It is a programming language that is unique to SPSS. It allows you to write commands that run SPSS procedures, rather than using the graphical user interface (GUI).

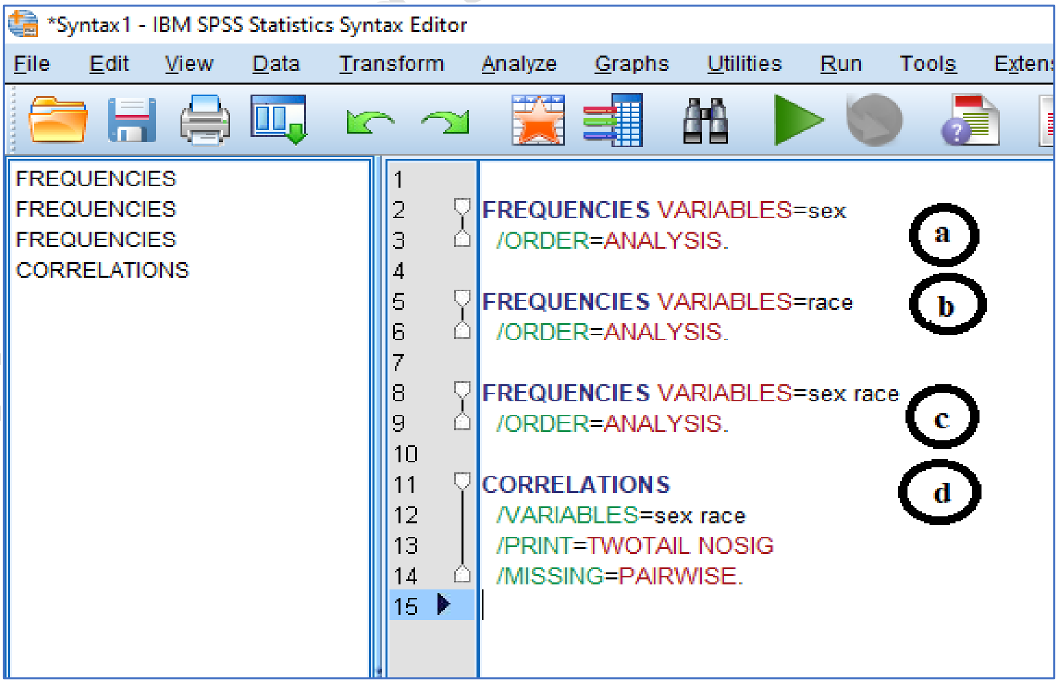
Example1: For the following data (survey_sample.sav), find the following using GUI and syntax (paste) methods.

a. Frequency distribution for SEX variable;
b. Frequency distribution for RACE variable;
c. Frequency distribution for SEX and RACE variables;
d. Correlation between SEX and RACE variables.

Solution: Using GUI method



Using syntax (paste) method:



Example2: For the following data (survey_sample.sav), find frequency distribution for SEX variable using syntax method.

Solution:

**********Frequency***************.**

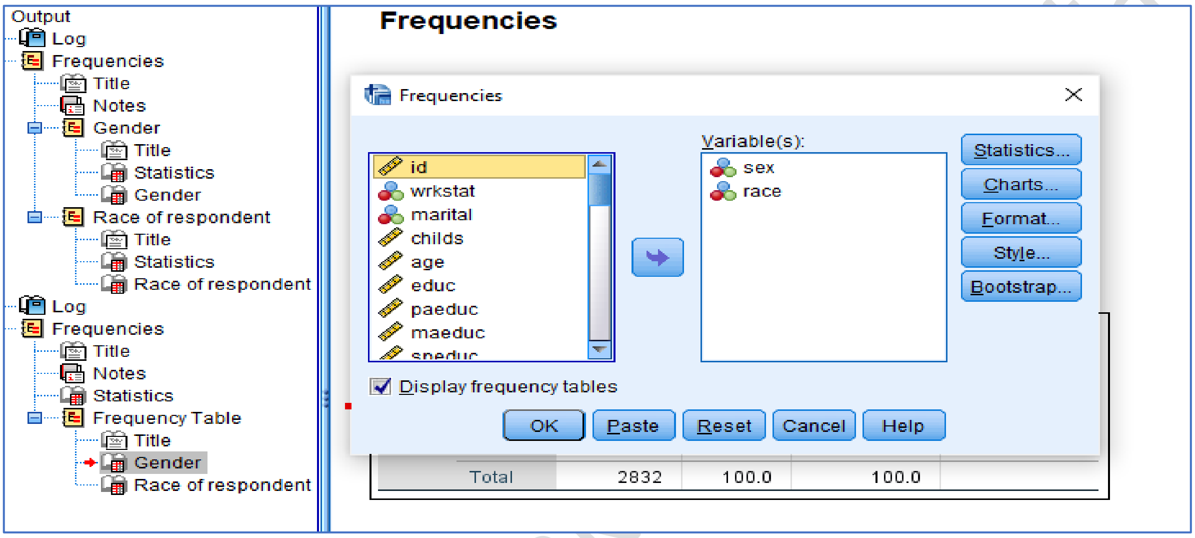
FREQUENCIES VARIABLES = sex.

*The * symbol means what?*

The . (dot) symbol means what?

Example3: For the following data (survey_sample.sav), find frequency distribution for SEX and RACE variables using GUI and syntax methods.

Solution: Using GUI method



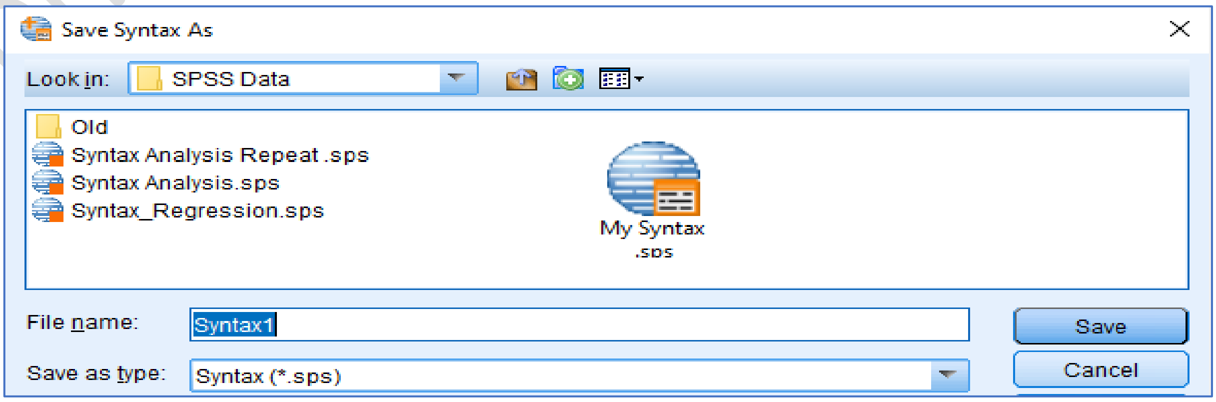
Using syntax method:

frequencies variables=sex race
 /order=analysis.
Or
frequencies variables=sex race
 /order=variable.

The command ORDER refers to what?

The / symbol means what?

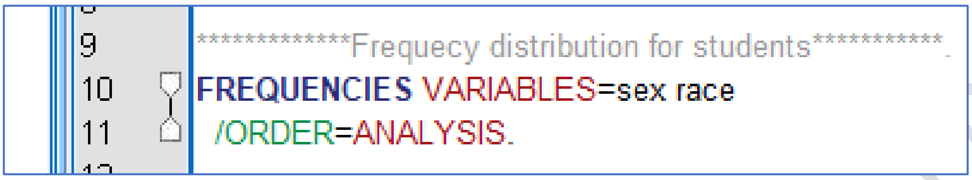
Remark: When saving a syntax file, its extension and icon are different (see below figure).



Color-Coding

SPSS uses color and bolding to indicate the roles of the words in the syntax. The colors are as follows:

Dark blue/purple	Procedure names; execution statements
Green	Statements associated with the given procedure
Dark red/orange	Option keywords
Gray	Comments
Black	Variable names; other text



Variable Labels and Value Labels

The labeling of one or more variables will be accomplished as follows:

VARIABLE LABELS
sex "Gender of Respondent"
age “Age of my family”
race “Race of people in Kurdistan’.

The value labeling of one or more variables will be accomplished as follows:

VALUE LABELS
/sex
0 "Female"
1 "Male"
/race
1 “White”
2 “Black”
3 “Brown“
4 “Other”
/Marital
1 “Married”
2 “Widowed”
3 “Divorced”.

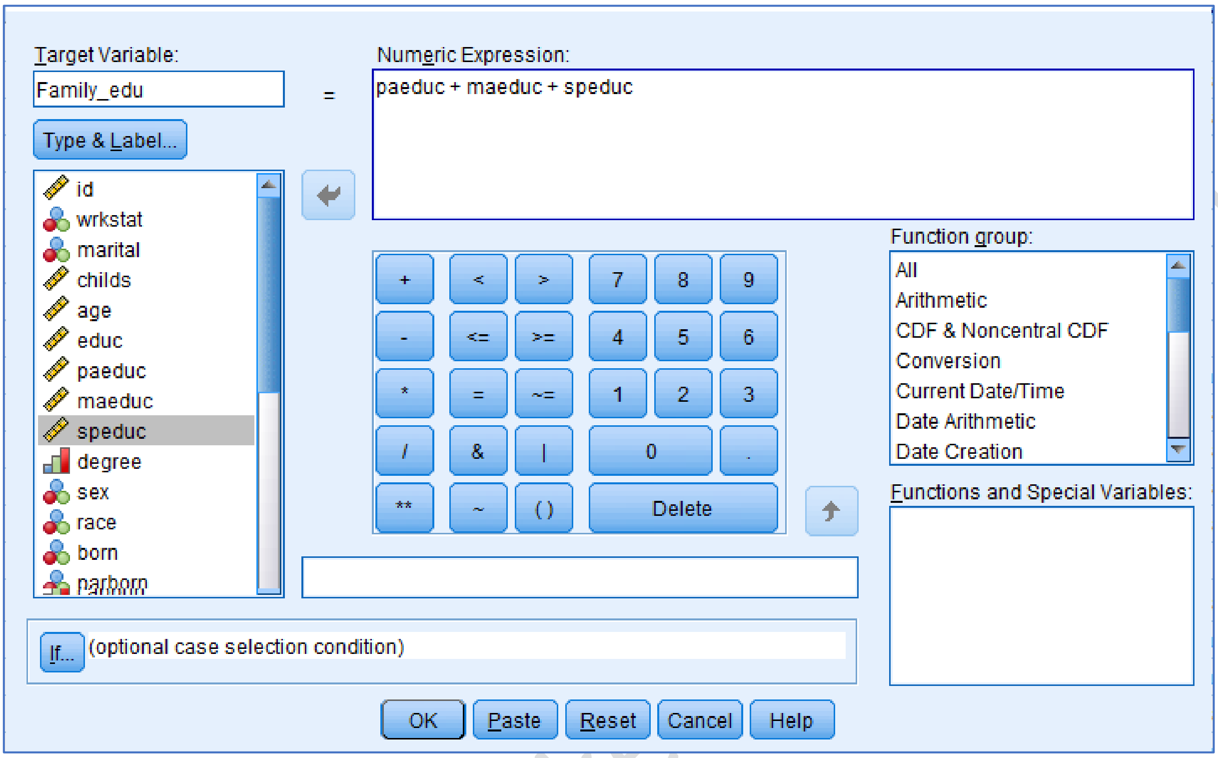
Computing Variables using Syntax

The general form of the syntax for computing a new (numeric) variable is:

Compute y = (x1+x2) / 5.
Execute.

Example4: For the following data (survey_sample.sav), compute new variable from adding the values of the highest year school completed by father, mother,and spouse (**paeduc, maeduc, speduc**) using GUI and syntax methods.

Solution: Using GUI method



Using Syntax method:

```
COMPUTE Family_edu=paeduc + maeduc + speduc.
EXECUTE.
```

Remark: The following table describes the arithmetic operators, arithmetic functions, and logical operations.

Arithmetic operations	Arithmetic functions	Comparisons and logical operations
+ Addition	ABS (expr)	EQ or = Equal to
- Subtraction	RND (expr)	NE or ~= Not equal to
* Multiplication	TRUNC (expr)	GE or >= Greater than or equal to
/ Division	MOD (expr, divid)	GT or > Greater than
** Exponentiation	SQRT (expr)	LE or <= Less than or equal to
	LN (expr)	LT or < Less than

Homework1: Find the mean, variance, log, square root, and cubic of the new variable (Family_edu) in example 4.

Matrix Operations:

To start a Matrix session, the first command must be: **Matrix.**
To end a Matrix session, the final command must be: **End Matrix.**
At any time, to display a vector or matrix or results of some computation in the output:
Print vector_matrix_name.
To create a vector or matrix, use the compute command. Vectors and matrices are enclosed in braces {}. The elements of each row are separated by **commas**, and rows are separated by **semicolons**. For example: **Compute name= {a1, a2, a3, a4, a5} or**
Compute name= {a1; a2; a3; a4; a5}

Example5: Write the following Matrices in SPSS syntax

$$X = [1 \quad 2 \quad 3 \quad 4], Y = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$$

Solution:

```
***** X is a 1 X 4 matrix********.
Matrix.
compute x = {1, 2, 3, 4}.
print x
/ title “Matrix X”.
End Matrix.

***** Y is a 5 X 1 matrix********.
Matrix.
compute y = {7; 8;9;10;11}.
print y
/ title "Matrix Y".
End Matrix.

***** A is a 3 X 3 matrix********.
Matrix.
compute A = {1,2,3;2,4,6;3,6,1}.
print A
/ title "Matrix A".
End Matrix.
```

Example6: Write the following Matrix operations for matrix A and B in SPSS syntax

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

a.) A + B b.) A – B c.) A*B d.) A^2

Solution:

a.)
```
***** A + B ********.
Matrix.
```

```
compute A = {1,2,3;2,4,6;3,6,1}.
compute B = {1,0,0;0,2,0;0,0,3}.
compute Z = A + B.
print A / title "Matrix A".
print B / title "Matrix B".
print Z / title "Matrix A + B".
End Matrix.
```

b.) & c.) & d.)

```
***** A - B & A*B & A^2********.
Matrix.
compute A = {1,2,3;2,4,6;3,6,1}.
compute B = {1,0,0;0,2,0;0,0,3}.
compute Z = A - B.
compute M = A * B.
compute K = A * A.
print A / title "Matrix A".
print B / title "Matrix B".
print Z / title "Matrix A - B".
print M / title "Matrix A * B".
print K / title "Matrix A^2".
End Matrix.
```

Example7: Using SPSS syntax, find A', |A|, and A^{-1} .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

Solution:

```
***** Transpose, Determinant, & Inverse********.
Matrix.
compute A = {1,2,3;0,4,0;3,6,1}.
compute Trans_A =transpos (A).
compute det_A = det(A).
compute halg_A = inv (A).
print Trans_A / title "Transpose of Matrix A".
print det_A / title " Determinant of Matrix A".
print halg_A / title "Inverse of Matrix A ".
End Matrix.
```

Homework2: Find for the following matrix: A', det(A), A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 3 & 6 & 1 \end{bmatrix}$$

Opening SPSS Data Files and Variables:

When we select File⇨Open⇨Data, SPSS issues a “GET” command to open an SPSS-formatted file and load it into SPSS. For example, the following opens and loads the file named survey_sample.sav:

```
GET FILE = "C:\Program Files\IBM\SPSS\Statistics\26\Samples\English\survey_sample.sav".
```

Example8: Open survey_sample.sav data, find a matrix (X) consists of respondent, father, mother, and spouse education (**educ, paeduc, maeduc, speduc**) using syntax methods. Then find $X'X$.

Solution:

```
GET FILE = "C:\Program Files\IBM\SPSS\Statistics\26\Samples\English\survey_sample.sav".
Matrix.
 /* the following creates a matrix with the n rows and  p=4 columns.
get x
 /variables = educ, paeduc, maeduc, speduc
 /missing=accept
 /sysmis=omit.
* /sysmis=15.
compute z=transpos(x)*x.
print x / title "Matrix X".
print z / title "Matrix X'X".
End Matrix.
```

Remark: in case we have missing values, they must be **omitted or accepted**, while when we have system missing values, they must be **omitted** or **changing** them to a number such as 15 or 20 or any other numbers.

Example9: Using survey_sample.sav data, find a matrix (X) consists of father, mother, and spouse education (**paeduc, maeduc, speduc**), and a matrix (Y) consists of respondents' education (**educ**) using syntax methods. Then find $(X'X)^{-1}X'Y$.

Solution:

```
Matrix.
 /* the following creates a matrix with the n rows and  p=3 columns.
get x
 /variables = paeduc, maeduc, speduc
 /missing=accept
 /sysmis=omit.
 /* the following creates a matrix with the n rows and  p=1 columns.
get y
/variables = educ
 /missing=accept
 /sysmis=omit.
compute z=transpos(x)*x.
compute inv_z=inv(z).
compute xtrans_y=transpos(x)*y.
compute B=inv_z*xtrans_y.
print x / title "Matrix X".
print z / title "Matrix X'X".
print xtrans_y / title "Matrix X'Y".
print inv_z / title "Inverse of Matrix (X'X)".
print B / title "Matrix (X'X)-1X'Y".
End Matrix.
```

Chapter Four

Multivariate Normal Distribution

1. **Univariate:** When we have one variable in a function $(p = 1)$
   p: number of variables
   Then the p.d.f of X is: $X \sim N(\mu, \sigma^2)$ is given by: $f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

2. **Bivariate:** When we have two variables in a function $(p = 2)$

   $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

   Then the J.P.d.f of $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left\{\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\right\}$ is given by:

$$f(x_1, x_2; \mu_1, \mu_2, \sigma_{11}, \sigma_{22}) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}^{-1} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}\right)$$

   Or

$$f\left(\underline{X}; \underline{\mu}, \Sigma\right) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\underline{X}-\underline{\mu}\right)' \Sigma^{-1}\left(\underline{X}-\underline{\mu}\right)\right) \quad -\infty < x_1 < \infty, -\infty < x_2 < \infty$$

3. **Multivariate:**
   Where we have P-variables in function $(p = p)$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, \quad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}_{p*p}$$

   Where Σ is square, non-singular and symmetric matrix and $(p * p)$ dimensional.
   Then, the j.p.d.f of $\underline{X} \sim N(\underline{\mu}, \Sigma)$ is given by:

$$f(x_1, x_2, \dots, x_p, \mu_1, \mu_2, \dots, \mu_p, \sigma_{11}, \sigma_{22}, \dots, \sigma_{pp}) =$$

$$\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 & \cdots & x_p-\mu_p \end{bmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix}^{-1} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \vdots \\ x_p-\mu_p \end{bmatrix}\right)$$

   Or

$$f\left(\underline{X}; \underline{\mu}, \Sigma\right) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\underline{X}-\underline{\mu}\right)' \Sigma^{-1}\left(\underline{X}-\underline{\mu}\right)\right)$$

Example: Find the j.p.d.f of Bivariate Normal dist^{n} $(p = 2)$

$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

Solution: $f\left(\underline{X}; \underline{\mu}, \Sigma\right) = \frac{1}{(2\pi)^{\frac{2}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\underline{X}-\underline{\mu}\right)' \Sigma^{-1}\left(\underline{X}-\underline{\mu}\right)\right)$

$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

$\because \rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2} \Rightarrow \quad \sigma_{12} = \rho_{12}\sigma_1\sigma_2$

$\Sigma = \begin{bmatrix} \sigma_{11} & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_{22} \end{bmatrix}$

$$\Sigma^{-1} = \frac{adj\begin{bmatrix} \sigma_{11} & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_{22} \end{bmatrix}}{\begin{vmatrix} \sigma_{11} & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_{22} \end{vmatrix}}$$

$$adj(\Sigma) = \begin{bmatrix} \sigma_{22} & -\rho_{12}\sigma_1\sigma_2 \\ -\rho_{12}\sigma_1\sigma_2 & \sigma_{11} \end{bmatrix} \;,\; |\Sigma| = \sigma_{11}\sigma_{22} - \rho_{12}{}^2\sigma_{11}\sigma_{22} = \sigma_{11}\sigma_{22}(1-\rho_{12}{}^2)$$

$$\Sigma^{-1} = \frac{1}{\sigma_{11}\sigma_{22}(1-\rho_{12}{}^2)}\begin{bmatrix} \sigma_{22} & -\rho_{12}\sigma_1\sigma_2 \\ -\rho_{12}\sigma_1\sigma_2 & \sigma_{11} \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{1-\rho_{12}{}^2}\begin{bmatrix} \frac{\sigma_{22}}{\sigma_{11}\sigma_{22}} & \frac{-\rho_{12}\sigma_1\sigma_2}{\sigma_{11}\sigma_{22}} \\ \frac{-\rho_{12}\sigma_1\sigma_2}{\sigma_{11}\sigma_{22}} & \frac{\sigma_{11}}{\sigma_{11}\sigma_{22}} \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{1-\rho_{12}{}^2}\begin{bmatrix} \sigma_{11}{}^{-1} & \frac{-\rho_{12}}{\sigma_1\sigma_2} \\ \frac{-\rho_{12}}{\sigma_1\sigma_2} & \sigma_{22}{}^{-1} \end{bmatrix}$$

Then the Quadratic form of $\underline{X}$ is:

$$Q(\underline{X}) = \left(\underline{X} - \underline{\mu}\right)' \Sigma^{-1}\left(\underline{X} - \underline{\mu}\right)$$

$$Q(\underline{X}) = [x_1 - \mu_1 \quad x_2 - \mu_2]\frac{1}{1-\rho_{12}{}^2}\begin{bmatrix} \sigma_{11}{}^{-1} & \frac{-\rho_{12}}{\sigma_1\sigma_2} \\ \frac{-\rho_{12}}{\sigma_1\sigma_2} & \sigma_{22}{}^{-1} \end{bmatrix}\begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$Q(\underline{X}) = \frac{1}{1-\rho_{12}{}^2}\left[\frac{x_1-\mu_1}{\sigma_{11}} - \frac{\rho_{12}(x_2-\mu_2)}{\sigma_1\sigma_2} \quad -\frac{\rho_{12}(x_1-\mu_1)}{\sigma_1\sigma_2} + \frac{x_2-\mu_2}{\sigma_{22}}\right]\begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$Q(\underline{X}) = \frac{1}{1-\rho_{12}{}^2}\left[(x_1-\mu_1)\left(\frac{x_1-\mu_1}{\sigma_{11}} - \frac{\rho_{12}(x_2-\mu_2)}{\sigma_1\sigma_2}\right) + (x_2-\mu_2)\left(\frac{-\rho_{12}(x_1-\mu_1)}{\sigma_1\sigma_2} + \frac{x_2-\mu_2}{\sigma_{22}}\right)\right]$$

$$Q(\underline{X}) = \frac{1}{1-\rho_{12}{}^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} - \frac{\rho_{12}(x_2-\mu_2)(x_1-\mu_1)}{\sigma_1\sigma_2} - \frac{\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]$$

$$Q(\underline{X}) = \frac{1}{1-\rho_{12}{}^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} - 2\frac{\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]$$

$$|\Sigma| = \sigma_{11}\sigma_{22}(1-\rho_{12}{}^2)$$

$$|\Sigma|^{\frac{1}{2}} = \sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}{}^2)} = \sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}$$

Then the j.p.d.f of $\underline{X}$ is

$$f\left(\underline{X};\underline{\mu},\Sigma\right) = \frac{1}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}}\exp\left(-\frac{1}{2}\frac{1}{1-\rho_{12}{}^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} - 2\frac{\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]\right)$$

Or

$$f\left(\underline{X},\underline{\mu},\Sigma\right) = \frac{1}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}}\exp\left(-\frac{1}{2}Q(\underline{X})\right)$$

Example: Write the j.p.d.f of Bivariate Normal distribution when:

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;,\; \underline{\mu} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \;,\; \Sigma = \begin{bmatrix} 9 & 16 \\ 16 & 64 \end{bmatrix} \;,\; \rho = 0.667$$

Solution: The j.p.d.f. of bivariate normal distribution could be written as follows:

$$f\left(\underline{X};\underline{\mu},\Sigma\right) = \frac{1}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}}\exp\left(-\frac{1}{2}\frac{1}{1-\rho_{12}{}^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} - 2\frac{\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]\right)$$

$$f\left(\underline{X};\underline{\mu},\Sigma\right) = \frac{1}{2\pi\,(3)(8)\sqrt{1-(0.667)^2}}\exp\left(-\frac{1}{2}\frac{1}{1-(0.667)^2}\left[\frac{(x_1-5)^2}{9} - 2\frac{0.667(x1-5)(x_2-10)}{(3)(8)} + \frac{(x_2-10)^2}{64}\right]\right)$$

$$f\left(\underline{X};\underline{\mu},\Sigma\right) = \frac{1}{48\pi\sqrt{1-(0.667)^2}}\exp\left(-\frac{1}{2}\frac{1}{1-(0.667)^2}\left[\frac{(x_1-5)^2}{9} - 2\frac{0.667(x1-5)(x_2-10)}{24} + \frac{(x_2-10)^2}{64}\right]\right)$$

$$f\left(\underline{X};\underline{\mu},\Sigma\right) = \frac{1}{13.32\pi}\exp\left(-\frac{1}{1.11}\left[\frac{(x_1-5)^2}{9} - \frac{0.667(x1-5)(x_2-10)}{12} + \frac{(x_2-10)^2}{64}\right]\right)$$

Then complete the operations inside the quadratic form: واتا کۆکردنەوەو کەمکردنەوە تەواو بکە

Comparison between Univariate and Multivariate Normal distribution:

1. The variable X has been transferred to vector of variables $\underline{X}$: $X \to \underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}$
2. The mean of the variable X is μ has been transferred to the vector of means:
$E(X) = \mu \to \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$
3. The variance of the variable X is σ^2 has been transferred to the Var-Cov. matrix and it's squared, non-singular, and symmetric matrix of order $(p \times p)$:
$V(x) = \sigma^2 \to Var - Cov(x) = \sum, \sigma^2 \to \sum,\ \sigma = |\sum|^{\frac{1}{2}}$
4. The square $\left(\frac{X-\mu}{\sigma}\right)^2$ has been transferred to the quadratic form $\left(\underline{X} - \underline{\mu}\right)' \Sigma^{-1} \left(\underline{X} - \underline{\mu}\right)$ in both case (Univariate and Multivariate) we must get a scalar $\left(\frac{X-\mu}{\sigma}\right)^2 \to \left(\underline{X} - \underline{\mu}\right)' \Sigma^{-1} \left(\underline{X} - \underline{\mu}\right)$
5. The p.d.f of X

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

The j.p.d.f of $\underline{X}$

$$f\left(\underline{X}; \underline{\mu}, \Sigma\right) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\underline{X} - \underline{\mu}\right)' \Sigma^{-1} \left(\underline{X} - \underline{\mu}\right)\right)$$

==

4. Quadratic Form

Def^n: If we have p variables $(X_1, X_2, \dots, X_p)$ or $(\underline{X})$ and A is $(p \times p)$ symmetric matrix where:

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix},\ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{pmatrix}$$

Then the $Q(\underline{X})$ is called the quadratic form in $\underline{X}$ which is a function of the following form:

$Q(\underline{X}) = \underline{X}' A \underline{X}$

$$Q(\underline{X}) = [X_1 \quad X_2 \quad \cdots \quad X_p] \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{pmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}$$

$$Q(\underline{X}) = \sum_{i=1}^{p} \sum_{j=1}^{p} X_i a_{ij} X_j$$

Example: If A is a symmetric matrix by (2×2) dimension. Find Quadratic form.

Solution: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \underline{X}' = [x_1 \quad x_2]$

$Q(\underline{X}) = \underline{X}' A \underline{X} = [x_1 \quad x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [a_{11}x_1 + a_{21}x_2 \quad a_{12}x_1 + a_{22}x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q(\underline{X}) = [a_{11}{x_1}^2 + a_{21}x_1x_2 + a_{12}x_1x_2 + a_{22}{x_2}^2] = a_{11}{x_1}^2 + 2a_{21}x_1x_2 + a_{22}{x_2}^2$

Example: Let A $= \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ and $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. Find Quadratic form $Q(\underline{X})$

Solution: $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\underline{X}' = [x_1 \quad x_2]$

$Q(\underline{X}) = \underline{X}'A\underline{X} = [x_1 \quad x_2]\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [2x_1 + x_2 \quad x_1 + 4x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q(\underline{X}) = 2{x_1}^2 + x_1x_2 + x_1x_2 + 4{x_2}^2 = 2{x_1}^2 + 4{x_2}^2 + 2x_1x_2$

Example: If A is a symmetric matrix by (3×3) dimension. Find Quadratic form.

Solution: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\underline{X}' = [x_1 \quad x_2 \quad x_3]$

$Q(\underline{X}) = \underline{X}'A\underline{X} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$Q(\underline{X}) = a_{11}{x_1}^2 + a_{22}{x_2}^2 + a_{33}{x_3}^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$

Example: Find Quadratic form of matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\underline{X}' = [x_1 \quad x_2 \quad x_3]$

$Q(\underline{X}) = \underline{X}'A\underline{X} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2{x_1}^2 - 6{x_2}^2 + 9{x_3}^2$

Example: Find Quadratic form of matrix $A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 7 & 1 \\ 3 & 1 & 8 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 7 & 1 \\ 3 & 1 & 8 \end{bmatrix}$, $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\underline{X}' = [x_1 \quad x_2 \quad x_3]$

$Q(\underline{X}) = \underline{X}'A\underline{X} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 5 & 2 & 3 \\ 2 & 7 & 1 \\ 3 & 1 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$Q(\underline{X}) = 5{x_1}^2 + 7{x_2}^2 + 8{x_3}^2 + 4x_1x_2 + 6x_1x_3 + 2x_2x_3$

Example: If the quadratic form $\underline{X}'A\underline{X} = 2{x_1}^2 + 4{x_3}^2 + 2x_1x_2 + 6x_2x_3$, find the matrix A

Solution: $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 4 \end{bmatrix}$

Classification of Quadratic Form

1. **Positive definite (p.d.)**

   The quadratic form of $\underline{X}$ $Q(\underline{X})$ is called p.d. if $\underline{X}'A\underline{X} > 0$ for all $\underline{X} \neq 0$

   Example: Let $p = 2$ then $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, prove that the quadratic form of $\underline{X}$, $Q(\underline{X})$ is p.d.

   Solution: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{X}' = [x_1 \quad x_2]$

   $Q(\underline{X}) = \underline{X}'A\underline{X} = [x_1 \quad x_2]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = {x_1}^2 + {x_2}^2 > 0$ why?

   $\therefore Q(\underline{X})$ is p.d.

   Note that for any real vector $\underline{X} \neq 0$, that $Q(\underline{X})$ will be positive, because the square of any number is positive, the coefficients of the squared terms are positive and the sum of positive numbers is always positive.

Example: Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$ prove the $Q(\underline{X})$ is p.d.?

Solution: $A = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$ $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{X}' = [x_1 \quad x_2]$

$Q(\underline{X}) = \underline{X}'A\underline{X} = [x_1 \quad x_2]\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2{x_1}^2 + 4{x_2}^2 - 2x_1x_2$

The first and second terms are clearly positive. But with $|x_1| > |x_2|$, $|2x_1^2| > |2x_1x_2|$, so that the first term is more positive than the third term, and so the whole expression is positive. The same thing if $|x_1| < |x_2|$.

$\therefore Q(\underline{X}) > 0 \Rightarrow \quad Q(\underline{X})$ is p.d.

2. **Positive semi-definite (p.s.d)**

   The quadratic form is p.s.d is $\underline{X}'A\underline{X} \geq 0$ for all $\underline{X} \neq 0$

   Example: Let $p = 2$ then $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, prove that the quadratic form of $\underline{X}$, $Q(\underline{X})$ is p.s.d

   Solution: $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{X}' = [x_1 \quad x_2]$

   $Q(\underline{X}) = \underline{X}'A\underline{X} = [x_1 \quad x_2]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = {x_1}^2 + {x_2}^2 - 2x_1x_2 \geq 0$ why?

   where: ${x_1}^2 + {x_2}^2 - 2x_1x_2 = (x_1 - x_2)^2 \geq 0$

   $\therefore Q(\underline{X})$ is p.s.d

3. **Negative and Negative semi-definite (n.d. & n.s.d)**

   Negative definite and negative semi-definite quadratic forms are similarly defined. Meaning that: The quadratic form is n.d. if $\underline{X}'A\underline{X} < 0$ for all $\underline{X} \neq 0$

   The quadratic form is n.s.d if $\underline{X}'A\underline{X} \leq 0$ for all $\underline{X} \neq 0$

Another Method: Eigenvalues to determine Classification of the quadratic form

The basic equation is A$\underline{X}$ =λ$\underline{X}$.

We may find λ = 2 or $\frac{1}{2}$ or -1 or 1. Most 2 by 2 matrices have two eigenvector directions and two eigenvalues. We will show that $\boldsymbol{det(A - \lambda I) = |A - \lambda I| = 0}$.

This section will explain how to compute the x's and λ's. Let us use $\boldsymbol{|A - \lambda I| = 0}$ to find the eigenvalues and eigenvectors.

Classification of the quadratic form

1. Positive definite (p.d.)

The quadratic form of $Q(\underline{X})$ is called p.d. if $\lambda_1, \lambda_2, \ldots \ldots, \lambda_n > 0$ for all $\lambda \neq 0$.

2. Positive Simi-definite (p.s.d)

The quadratic form is p.s.d if $\lambda_1, \lambda_2, \ldots \ldots, \lambda_n \geq 0$ for all $\lambda \neq 0$

Example: Let $p = 2$ then, A = $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, prove that the quadratic form $Q(\underline{X})$ is p.d.

Solution : $|A - \lambda I| = 0$

$\Rightarrow \left|\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$

$$\Rightarrow \left|\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4-\lambda) = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 4) = 0$$

Then, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$

$\therefore\ \lambda_1 = 1$ and $\lambda_2 = 4 > 0$

$\therefore\ Q(\underline{X})$ is p.d.

Example: Let $p = 3$ then, A = $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, prove that the quadratic form $Q(\underline{X})$ is p.s.d.

Solution: $det(A - \lambda I) = 0$

$$\left|\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\Rightarrow \left|\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^3 - 2 - 3(2-\lambda) = 0$$

$$-\lambda^3 + 6\lambda^2 - 12\lambda + 8 - 2 - 6 + 3\lambda = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 9) = 0$$

Either $\lambda_1 = 0\ or\ \ \lambda^2 - 6\lambda + 9 = 0$

$\Rightarrow \lambda_2, \lambda_3 = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)} = 3$ or $(\lambda - 3)^2 = 0$

$\therefore\ \lambda_1 = 0$ and $\lambda_2\ and\ \lambda_3 = 3 \geq 0$

$\therefore\ Q(\underline{X})$ is p.s.d

3. Negative definite and Negative Simi-definite (n.s.d) and (n.d)

Negative definite and negative semi-definite quadratic forms are similarly defined, meaning that:

The quadratic form is n.d. if $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots\dots, \boldsymbol{\lambda_n} < 0$ for all $\lambda \neq 0$

The quadratic form is n.s.d. if $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots\dots, \boldsymbol{\lambda_n} \leq 0$ for all $\lambda \neq 0$

Example: If $A=\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, *prove that* $Q(\underline{X})$ is n.d.

Solution: $|A-\lambda I|=0$

$$\left|\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$(-2-\lambda)^3 = 0$

The eigenvalues are λ_1, λ_2 *and* $\lambda_3 = -2$

$\therefore$ $Q(\underline{X})$ is n.d.

Example: If A = $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, *prove that* $Q(\underline{X})$ is n.s.d.

Solution: $|A-\lambda I|=0$

$$\left|\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$(-1-\lambda)(-1-\lambda)-1=0$

$\lambda^2+2\lambda=0$

$\lambda(\lambda+2)=0$

The eigenvalues are $\lambda_1=-2$ and $\lambda_2=0$

$\therefore$ $Q(\underline{X})$ is n.s.d.

H.W.:

A. If $A=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, show that $Q(\underline{X})$ is n.d.

B. If $A=\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, prove that $Q(\underline{X})$ is n.d.

C. If $A=\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$, prove that $Q(\underline{X})$ is n.s.d

D. Determine the classification of quadratic form $Q(\underline{X}), if\ A=\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix}$.