CHAPTER ONE

Vector analysis

<u>The Unit vector:</u>

The Unit vector in three direction of the vector $\vec{a}$ is defined as:

$$a_A = \frac{\vec{A}}{|A|}$$

$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
a_x	a_y	a_z

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ .... Cartesian coordinate

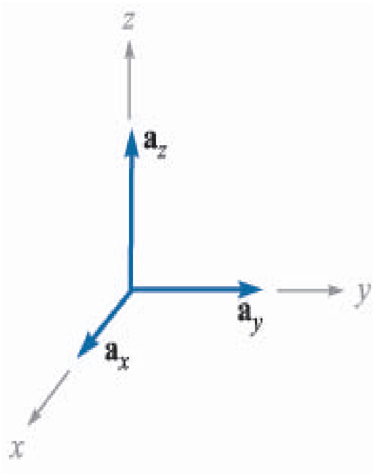
$$|\vec{A}| = \sqrt{A.A} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$a_A = \frac{A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$



<u>Vector Algebra:</u>

1- Vector may be add or subtracted, $\vec{A}$, $\vec{B}$ are tow vectors

$$\vec{A} \mp \vec{B} = \left(A_x\hat{\imath} \mp A_y\hat{\jmath} \mp A_z\hat{k}\right) \mp \left(B_x\hat{\imath} \mp B_y\hat{\jmath} \mp B_z\hat{k}\right)$$
$$= (A_x \mp B_x)\hat{\imath} \mp \left(A_y \mp B_y\right)\hat{\jmath} \mp (A_z \mp B_z)\hat{k}$$

2- The associative distributive and commutative law apply:

$$\vec{A} + \left(\vec{B} + \vec{C}\right) = \left(\vec{A} + \vec{B}\right) + \vec{C}$$
$$k\left(\vec{A} + \vec{B}\right) = k\vec{A} + k\vec{B}$$
$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

3- The dot product:

$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$

$\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$

$$\vec{A}.\vec{B} = AB\cos\theta$$

$$= \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right).\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

$$= A_xB_x + A_yB_y + A_zB_z$$

4- The cross product:

$\vec{A} \times \vec{B}$

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

$$\vec{A} \times \vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) \times \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

Right hand rule

$\hat{\imath} \times \hat{\jmath} = \hat{k}$	$\hat{\jmath} \times \hat{k} = \hat{\imath}$	$\hat{k} \times \hat{\imath} = \hat{\jmath}$
$\hat{\jmath} \times \hat{\imath} = -\hat{k}$	$\hat{k} \times \hat{\jmath} = -\hat{\imath}$	$\hat{\imath} \times \hat{k} = -\hat{\jmath}$

$$\vec{A} \times \vec{B} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$= \left(A_yB_z - A_zB_y\right)\hat{\imath} - (A_xB_z - A_zB_x)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$$

The Del operator:

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

This operator on scalar and vector function

Operation on scalar functions:

If (u) is the scalar function (is a potential)

∇u gradient of u

$u = u_x + u_y + u_z$

$$du = du_x + du_y + du_z$$

$$= \left(\frac{\partial u_x}{\partial x}\right)dx + \left(\frac{\partial u_y}{\partial y}\right)dy + \left(\frac{\partial u_z}{\partial z}\right)dz$$

$$du = \left(\frac{\partial u_x}{\partial x}\hat{\imath} + \frac{\partial u_y}{\partial y}\hat{\jmath} + \frac{\partial u_z}{\partial z}\hat{k}\right).\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= (\nabla u).dl$$

$$\nabla u = \frac{\partial u_x}{\partial x}\hat{\imath} + \frac{\partial u_y}{\partial y}\hat{\jmath} + \frac{\partial u_z}{\partial z}\hat{k}$$

$$dl = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$$

Example: Find $\nabla|\vec{r}|$

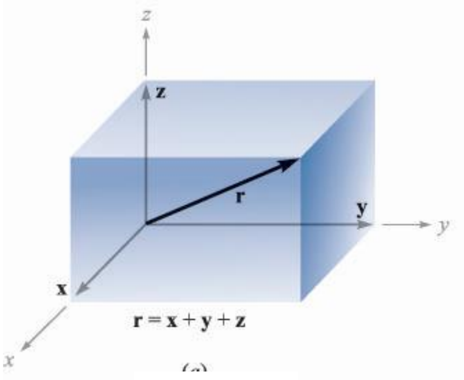
$\vec{r} = \mathrm{x}\hat{\imath} + y\hat{\jmath} + z\hat{k}$ $\qquad$ $|\vec{r}| = \sqrt{rr} = \sqrt{x^2 + y^2 + z^2}$

$$\nabla|\vec{r}| = \frac{\partial|\vec{r}|}{\partial x}\hat{\imath} + \frac{\partial|\vec{r}|}{\partial y}\hat{\jmath} + \frac{\partial|\vec{r}|}{\partial z}\hat{k}$$

$$= \frac{\frac{1}{2}(2x)}{\sqrt{x^2 + y^2 + z^2}}\hat{\imath} + \frac{\frac{1}{2}(2y)}{\sqrt{x^2 + y^2 + z^2}}\hat{\jmath} + \frac{\frac{1}{2}(2z)}{\sqrt{x^2 + y^2 + z^2}}\hat{k}$$

$$= \frac{\mathrm{x}\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\vec{r}}{|\vec{r}|} = r$$

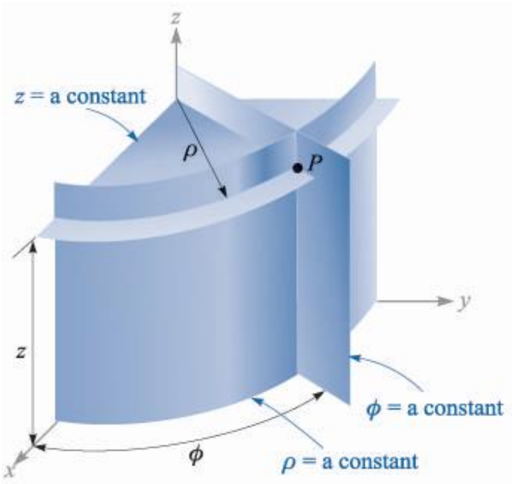
1- The Cartesian coordinate system:

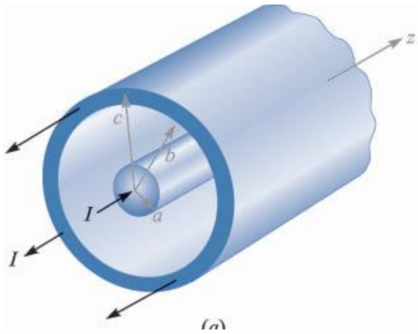


$$dl = dx a_x + dy a_y + dz a_z$$
$$dV = dx . dy . dz$$
$$ds = (dydz) a_x \; \ldots \ldots .. in\ x\ direction$$
$$ds = (dxdz) a_y \; \ldots \ldots .. in\ y\ direction$$
$$ds = (dxdy) a_z \; \ldots \ldots .. in\ z\ direction$$

2- The Cylindrical coordinate system:

$$dl = dr a_r + r d\varphi a_\varphi + dz a_z$$
$$dV = r dr\, d\varphi\, dz$$
$$ds = (r\, d\varphi\, dz) a_r \; \ldots \ldots .. in\ r\ direction$$
$$ds = (dr dz) a_\varphi \; \ldots \ldots .. in\ \varphi\ direction$$
$$ds = (r dr\, d\varphi) a_z \; \ldots \ldots .. in\ z\ direction$$

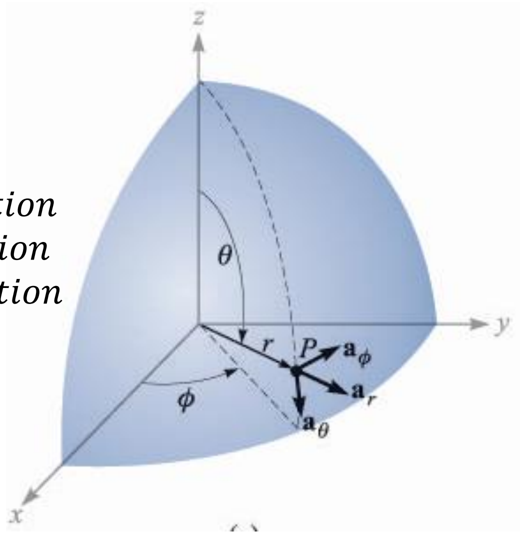


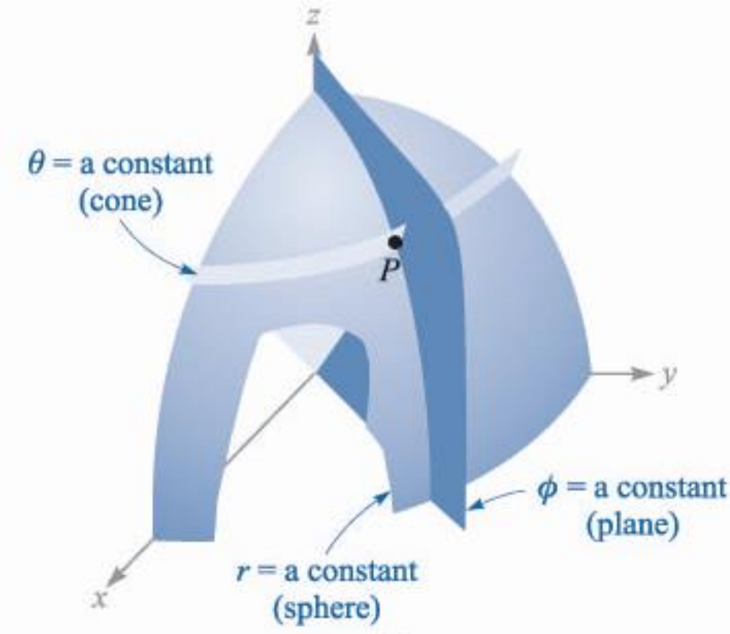


3- The Spherical coordinate system:

$$dl = dr a_r + r d\theta a_\theta + r \sin\theta\, d\varphi\, a_\varphi$$
$$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$ds = (r^2 \sin\theta\, d\theta\, d\varphi) a_r \; \ldots \ldots .. in\ r\ direction$$
$$ds = (r \sin\theta\, dr\, d\varphi) a_\theta \; \ldots \ldots .. in\ \theta\ direction$$
$$ds = (r dr\, d\theta) a_\varphi \; \ldots \ldots .. in\ \varphi\ direction$$




z
θ = a constant
(cone)
P
y
ϕ = a constant
(plane)
r = a constant
(sphere)
x

Operation on vector

$$\nabla A \begin{cases} \nabla \cdot A \ \ Divergence\ (flux) \ \ \ldots\ldots \ \ \int A \cdot ds \\ \nabla \times A \ \ \ Curl \ \ \ \ldots\ldots\ldots\ldots\ldots \ \ \int A \cdot dl \end{cases} \qquad \begin{pmatrix} \nabla{:}operation \\ A{:}vector\ function \end{pmatrix}$$

$\nabla \cdot A$ Divergence:

The divergence of a vector is the total flux of a vector per unit volume shrink to a point.

$$\nabla \cdot A = \lim_{v \to 0} \frac{1}{v} \oint A \cdot n da$$

In general:

$$\nabla \cdot A = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(A_1 h_2 h_3) + \frac{\partial}{\partial q_2}(A_2 h_1 h_3) + \frac{\partial}{\partial q_3}(A_3 h_1 h_2)\right]$$

$dl = dx a_x + dy a_y + dz a_z \quad \rightarrow \quad h_1 = h_2 = h_3 = 1$

$dl = dr a_r + r d\varphi a_\varphi + dz a_z \quad \rightarrow \quad h_1 = 1 \ ,h_2 = r \ ,h_3 = 1$

$dl = dr a_r + r d\theta a_\theta + r \sin\theta \, d\varphi \, a_\varphi \quad \rightarrow \quad h_1 = 1 \ ,h_2 = r \ ,h_3 = r\sin\theta$

	Cartesian	Cylindrical	Spherical
∂q_1	dx	dr	dr
∂q_2	dy	$d\varphi$	$d\theta$
∂q_3	dz	dz	$d\varphi$

1- Divergence in Cartesian coordinate :

$$\nabla \cdot A = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

2- Divergence in Cylindrical coordinate :

$$\nabla \cdot A = \frac{1}{r}\left[\frac{\partial}{\partial r}(A_r r) + \frac{\partial}{\partial \varphi}(A_\varphi) + \frac{\partial}{\partial z}(A_z r)\right]$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(A_r r) + \frac{1}{r}\frac{\partial}{\partial \varphi}(A_\varphi) + \frac{\partial}{\partial z}(A_z)$$

3- Divergence in Spherical coordinate :

$$\nabla \cdot A = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(A_r r^2 \sin\theta) + \frac{\partial}{\partial \theta}(A_\theta r \sin\theta) + \frac{\partial}{\partial \varphi}(A_\varphi r)\right]$$

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(A_\varphi)$$

Divergence theorem:

$$\oint_s A \cdot n da = \int_v (\nabla \cdot A) dV$$

The integral of the divergence of a vector over a volume (V) is equal to the flux integral of the normal component of the vector over the surface boundary the volume (V).

Example:

The region ($r \le a$) in spherical coordinates has an Electric intensity, $E = \frac{\rho r}{3\epsilon} a_r = E_r$ which $E = E_r a_r + E_\theta a_\theta + E_\varphi a_\varphi$. Examine of both sides of divergence theorem.

$$\oint_s A \cdot n da = \int_v (\nabla \cdot A) dV$$

$$n = a_r$$

$$\oint_s E \cdot n da = \iint \left(\frac{\rho r}{3\epsilon}\right) a_r \cdot (r^2 \sin\theta \, d\theta \, d\varphi) a_r = \frac{\rho r^3}{3\epsilon} \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi =$$

$$\frac{\rho r^3}{3\epsilon} |-\cos\theta|_0^\pi |\varphi|_0^{2\pi} = \frac{\rho r^3}{3\epsilon}\left[\frac{-(-1-1)}{2\pi}\right] = \frac{4\pi\rho r^3}{3\epsilon} \qquad \text{L.H.S}$$

But

$$\text{R.H.S} = \int_v (\nabla \cdot A) dV$$

$$= \int (\nabla \cdot E) \cdot r^2 \sin\theta \, dr d\theta d\varphi = \iiint (\nabla \cdot E) \cdot r^2 \sin\theta \, dr d\theta d\varphi$$

$E_r = \frac{\rho r}{3\epsilon} a_r$ and for spherical $\left[\nabla \cdot E = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\rho r}{3\epsilon}\right) = \frac{\rho}{\epsilon}$

$$\therefore \int_v \int (\nabla \cdot E) \cdot dV = \int \frac{\rho}{\epsilon} \cdot r^2 \sin\theta \, dr d\theta d\varphi$$

$$= \frac{\rho}{\epsilon} \int_0^r r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi = \frac{\rho}{\epsilon}\frac{4}{3}\pi r^3 \qquad \text{R.H.S}$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

Del operator Operation cross vector $\boldsymbol{\nabla \times A \equiv curl}$

$$\nabla \times A = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \equiv curl\ A$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

In general

$$\nabla \times A = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\hat{\imath} & h_2\hat{\jmath} & h_3\hat{k} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
Cartesian	a_x	a_y	a_z
Cylindrical	a_r	a_φ	a_z
Spherical	a_r	a_θ	a_φ

1- in Cartesian coordinate :

$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

2- In Cylindrical coordinate :

$$\nabla \times A = \frac{1}{r} \begin{vmatrix} a_r & r a_\varphi & a_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & r A_\varphi & A_z \end{vmatrix}$$

$$= \frac{1}{r}\left(\frac{\partial A_z}{\partial \varphi} - \frac{\partial (r A_\varphi)}{\partial z}\right) a_r - \frac{1}{r}\left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right)(r a_\varphi) + \frac{1}{r}\left(\frac{\partial (r A_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right) a_z$$

$$= \left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right) a_r - \left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right) a_\varphi + \frac{1}{r}\left(\frac{\partial (r A_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right) a_z$$

3- In Spherical coordinate :

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & ra_\theta & r\sin\theta\, a_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta\, A_\varphi \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta}\left[\left(\frac{\partial\left(r\sin\theta\, A_\varphi\right)}{\partial\theta} - \frac{\partial(rA_\theta)}{\partial\varphi}\right)a_r - \left(\frac{\partial\left(r\sin\theta\, A_\varphi\right)}{\partial r} - \frac{\partial A_r}{\partial\varphi}\right)(ra_\theta) + \left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\left(r\sin\theta\, a_\varphi\right)\right]$$

$$= \frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta A_\varphi\right)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)a_r - \frac{1}{r}\left(\frac{\partial\left(rA_\varphi\right)}{\partial r} - \frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi}\right)a_\theta + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)a_\varphi$$

Stokes's theorem:

$$\oint A \cdot dl = \int_s (\nabla \times A) nda$$

The line integral of a vector around any close path is equal the surface integral of its curl.

$$\oint A \cdot dl = 0 \qquad A\ is\ conservative$$

$$\therefore \int (\nabla \times A) \cdot nda = 0$$

$$\nabla \times A = 0 \quad if \quad A = \nabla f \quad f - is\ a\ scalar\ function$$

$$\nabla \times \nabla f = 0 \qquad \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \ , \qquad \nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right)\hat{\imath} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{\jmath} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)\hat{k} = 0$$

$$\therefore A = \nabla \mathrm{f}$$

$$A \equiv gradint\ of\ a\ scalar\ function\ \nabla f$$

The Laplace equation

1- For Cartesian

$$\nabla \times \nabla f = 0$$

$$Now\ we\ examine\ \ \nabla \cdot \nabla f$$

$$\nabla \cdot \nabla f = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \nabla^2 f \quad laplace\ of\ (f)$$

$$\nabla^2 f = Laplace\ of\ f \qquad\qquad \nabla^2 = Laplace\ of\ operator$$

2- For cylindrical:

$$\nabla^2 f = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r}\frac{\partial f}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right)\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplace of f in cylinder

3- For spherical coordinate:

$$\nabla^2 f = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \varphi}\right)\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

Laplace of f in spherical

Problem:

Q1/ For the field $\left(D = \frac{a}{4\pi r^2} a_r\right)$. Find $(\nabla \cdot D)$.

Q2/ Given $(A = r \sin\varphi\ a_r + 2z^2\ a_z)$. Find $(\nabla \cdot A)$.

Q3/ Given $\left(A = 5x^2\ (\sin\frac{\pi x}{2})\ a_x\right)$. Find $(\nabla \cdot A)\ at\ x = 1$.

Q4/ Given that $\left(D = \frac{5r^2}{4} a_r\right)$ in the spherical coordinates, evaluate both sides of divergence theorem for the volume enclosed by $r = 4m, \theta = \frac{\pi}{4}, \varphi = 2\pi$

Q5/ Given the general vector field $[A = (y\cos ax)a_x + (y + e^x)a_z]$. Find $(\nabla \times A)$.

Q6/ Given the vector field $(A = 5r\sin\varphi\ a_z)$ in cylindrical coordinate. Find $(\nabla \times A)$ at $(2, \pi, 0)$.

Q7/ Given the vector field $(A = 10\sin\theta\ a_\theta)$ in spherical coordinate. Find $(\nabla \times A)$ at $\left(2, \frac{3\pi}{2}, 0\right)$.